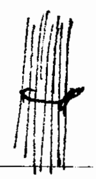
Introduction to Cosmological Models: Part IV   May 1988

Loose ends from part III: Active Euler Angles, Frame Lagrange Derivative
What is a fiber bundle?   Why bother?
Kinds of Fiber bundles, familiar examples: tensor bundles
Local trivializations, Horizontal and Vertical subspaces and Connections.
Structural group, Sections
Bundle of frames
U(1) bundle and electromagnetism
Nonabelian generalization
Bundle metric, higher-dimensional Einstein equations, Weyl transformations
End

This part gives an impression of how fiber bundles work using the tangent and cotangent bundle examples of part III and a circle bundle of electromagnetism and the bundle of frames. These are then placed in the context of higher dimensional spacetimes and unified field theories.

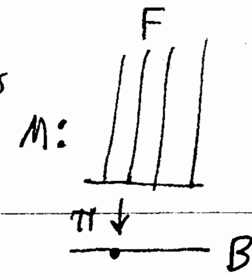
This part was written "on the fly" without notes relying on memory so they could easily stand being ~~re~~ rewritten someday.   Maybe.



wheat tied together in a bundle

bundle of fiberoptics tubes

Fiber bundle:

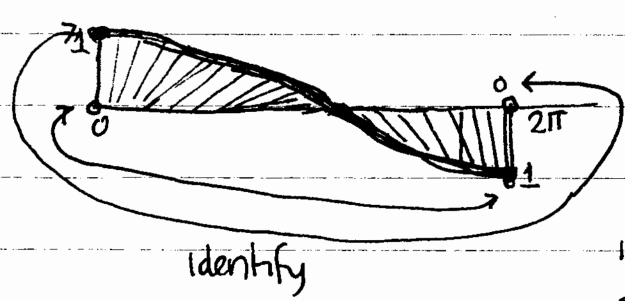


FIBER BUNDLES

A fiber bundle is a manifold M which is locally diffeomorphic to $B \times F$ where F is a fixed manifold (the fiber) usually a vector space or Lie group and B is a manifold (called the base manifold), in the same way that a manifold itself is locally diffeomorphic to R^N. "Locally" refers to the open cylinder in M "above" an open submanifold of the base B. The natural projection $\pi : M \to B$ projects each fiber down to the base point over which it is sithing.

One can think of making a fiber bundle by attaching a copy of the fiber space F to each point of the base in a smooth way, but this can be done in the simplest way as a global product manifold leading to a product fiber bundle or in more interesting ways with nontrivial topologies like the construction of the Mobius strip.

MOBIUS STRIP: This is a fiber bundle with base S^1 (circle) and fiber $\{x \in R \mid 0 \le x \le 1\}$, but the line segments are attached in such a way that there is a global twist.

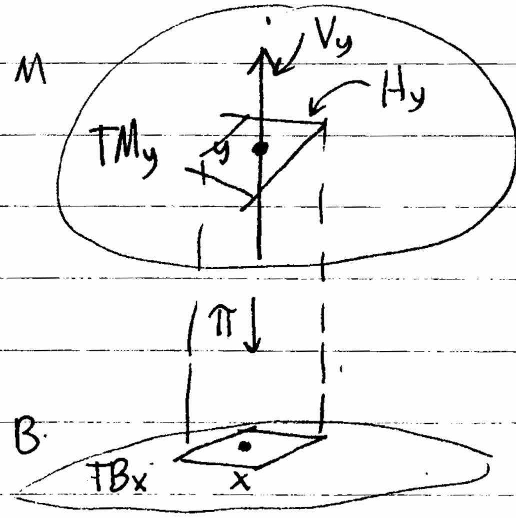


identify with a twist, get Mobius strip
without twist, get wedding band. (ring)

in either case, the piece of the strip above any open interval of $[0, 2\pi]$ is just a product of the open interval and the line segment.

WHY FIBER BUNDLES?

All higher dimensional theories involve fiber bundles at some point with spacetime as the base space and some "internal space" as the fiber. In the cases of most interest to us, the fiber is ~~itself~~ either a group or acted on "transitively" by a group (i.e., the fiber consists of a single orbit).



The whole tangent space at a point of the bundle has a natural "vertical" subspace along the fiber directions, but the complementary "horizontal" subspace needed to make a direct sum decomposition has to be selected, ie it will be determined by some field.

The horizontal subspace will be isomorphic to the tangent space at the base point.

When the fiber has a natural metric (Riemannian for our purposes), and the base also (Lorentzian, say spacetime), they can be used to put a metric on the whole bundle by using the fiber metric along the fiber (V_y) and the base metric "pulled up" to the horizontal space using the codifferential of π and declaring the two subspaces orthogonal.

Thus the "internal geometry" associated with the group and the "external geometry" associated with the spacetime can be incorporated in a single geometry, together with an additional field which picks out the horizontal subspace. [The group G which acts on the fibers is the "gauge group", while the "gauge field" or "connection" picks out the horizontal subspace.]

Kinds of ~~Vect~~ Fiber bundles

In principle the fiber F can be any kind of space. We are mostly interested in the cases where it is a real or complex vector space ($\rightarrow$ "vector bundles") or a Lie group ($\rightarrow$ "principal bundles"), or perhaps even a homogeneous space G/H. We need complex vector spaces to discuss complex scalar fields and spinor fields, typically arising from quantum mechanics (wavefunctions ...).

Familiar examples of bundles associated with the differential structure of a manifold:

vector bundles: *tangent bundle, cotangent bundle, tensor bundles*

principal bundle: *bundle of frames*

We have already introduced the tangent and cotangent bundles as the familiar velocity and momentum phase spaces over an n-dimensional manifold M considered as the configuration space of a classical dynamical system. TM (respectively T^*M) is just the collection of all the tangent (cotangent) spaces at all points of M. The natural projection π associates a tangent (cotangent) vector to a point of M with the point itself. All the fibers are isomorphic to the vector space $\mathbb{R}^n$. A local frame on M maps each fiber onto $\mathbb{R}^n$ be expressing tangent vectors (cotangent vectors) in components:

If $\{e_a\}$ is a frame on $U \subset M$, with dual frame $\{\omega^a\}$,

then $\quad X \in TM_x \subset TM$ is mapped to $\phi(X) = (x, X^a) \in M \times \mathbb{R}^n$

where $\quad X^a = \omega^a(x)(X)$.

Such a local diffeomorphism from the bundle to $B \times F$ (here M is the base space B) is called a "local trivialization" of the bundle. (A "trivial bundle" is a global product $B \times F$.) In the same way that a local coordinate chart on a manifold maps a piece of the manifold onto $\mathbb{R}^n$ where we can do ordinary calculus, a local trivialization of a fiber bundle gives us an explicit product manifold representation of the part of the bundle over some region of the base space, allowing us to work explicitly in terms of the differential structure of B and F (by taking local coordinates on each).

For example, a local coordinate chart $\{x^i\}$ on M gives us both a local trivialization of the tangent (cotangent) bundle as above by taking components, as well as natural coordinate charts on the factor manifolds (itself and $\mathbb{R}^n$) of the product $B \times F$, namely $\{q^i, \dot q^i\}$ on TM and $\{q^i, p_i\}$ on T^*M.

Now we can talk about vector fields on TM easily. For example, consider the vector field $X = \dot q^i \frac{\partial}{\partial q^i} + F(q)^i \frac{\partial}{\partial \dot q^i}$ on TM. An integral curve of this vector field, sloppily written $(q^i(t), \dot q^i(t))$ will satisfy:

$$\frac{d}{dt} q^i(t) = \dot q^i \,, \quad \frac{d}{dt} \dot q^i(t) = F(q)^i \,,$$

i.e., the trajectory of a particle on M under the influence of a force field with components $F(x)^i$ on M, lifted up to TM.

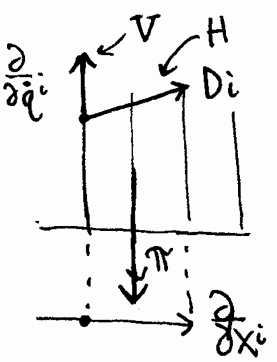
Or consider the integral curves of the vector field

$$\bar X = X^i(q) \left(\frac{\partial}{\partial q^i} - \Gamma^j{}_{ik}(q)\, \dot q^k \frac{\partial}{\partial \dot q^i} \right) \equiv X^i(q)\, D_i$$

on TM, where $\Gamma^j{}_{ik}(x)$ are the components of a connection on M:

$$\underbrace{\frac{d}{dt} q^i = X^i(q)}_{\text{integral curve of } X \text{ on } M} \qquad \underbrace{\frac{d}{dt} \dot q^i = -\Gamma^i{}_{jk}(q) X^j(q)\, \dot q^k}_{\text{let } \dot q^i = Y^i \,\to\, \frac{dY^i}{dt} + \Gamma^i{}_{jk} X^j Y^k = (\nabla_X Y)^i = 0}$$

This tells you how to parallel transport tangent vectors along curves in M.



The vector fields $\frac{\partial}{\partial \dot q^i}$ are tangent to the fiber (tangent space to M) while the vector fields D_i span a subspace isomorphic to the tangent space on the base, splitting the full tangent space naturally into a vertical space along the fibers and a complementary horizontal space which projects down to the base tangent space.

The horizontal spaces (a distribution on the bundle) have the following meaning. Pick a tangent vector Y at x and move it along a curve in such a way that the path of (x, Y) in the bundle above the curve always moves in the direction of the horizontal space. Then Y is parallel transported along the curve. The horizontal space tells you how to move to nearby fibers above a curve in M, i.e. it provides a "connection" between them. Thus a covariant derivative on M becomes part of the geometry on the tangent bundle.

These remarks are only intended to "whet your appetite" since there is no time to go into detail — that would take a course in itself.

In the same way that a manifold is defined globally by a covering by overlapping local coordinate charts which map pieces of the manifold onto $\mathbb{R}^n$, a bundle is defined globally by a covering of the base manifold by overlapping open sets $U_{(i)}$ (as in a coordinate covering) but with local trivialization maps

$$\phi_{(i)} : \underbrace{\pi^{-1}(U_{(i)})}_{\text{fibers above } U_{(i)}} \longrightarrow B \times F .$$

This tells us globally how the fibers fit together to make the bundle as in the Mobius strip / wedding ring example. Locally they are identical.

This topological structure is very interesting but NO TIME so we will consider at most a single local trivialization or a pair of overlapping local trivializations.

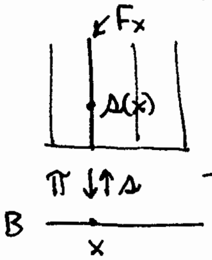
For example suppose we have two local frames $\{e_a\}$ and $\{e'_a\}$ defined on a common region of M. If $X \in TM_x$ is expressed in these frames $X = X^a e_a = X'^a e'_a$, one has

$$X \begin{cases} \phi(x) = (x, X^a) \\ \phi'(x) = (x, X'^a) \end{cases} \downarrow\ X'^a = A^a{}_b X^b \quad \text{if} \quad e'_a = A^{-1\,a}{}_b\, e_b$$

so the different images in $\mathbb{R}^n$ are related by a linear transformation, or "gauge transformation". Gauge transformations relate the different copies of the fiber F resulting from different local trivialization maps. The group involved ($GL(n,\mathbb{R})$ in this example) is called the "<u>structural group of the bundle</u>".

The existence of the structural group is the additional piece of information we need to make a vector bundle, i.e., a fiber bundle in which the linear structure of the fibers is encoded in the bundle geometry. Not only do we need a covering of the base space by overlapping local trivializations, but the representative fibers in each (copies of F in $B\times F$) should be related by <u>linear transformations</u>. Different kinds of structure for the Fiber F (vector space, affine space, group, homogeneous space) lead to different structural groups which encode that structure in the bundle itself.

<u>SECTIONS</u> An important concept associated with a fiber bundle which helps further understand the relationship between different local trivializations is a "section" of the fiber bundle. This is just a smooth choose of one element from each fiber, i.e., a fiber valued function s on the base space B (from each point $x \in B$ to its <u>own</u> fiber $\pi^{-1}(x) \equiv F_x$).



$\pi \circ s =$ identity on B

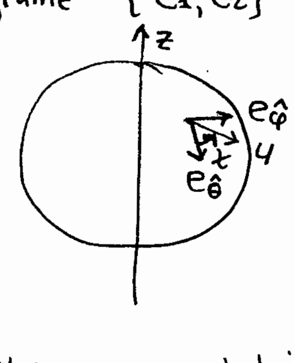
In other words: $x \mapsto s(x) \in \pi^{-1}(x) \equiv F_x$

For example, a section of the tangent bundle TM of a manifold is a smooth choice of tangent vector at each point of M, i.e., a vector field on M.

A section of the cotangent bundle T^*M is a smooth choice of covector, i.e., a 1-form field on M. Similarly a section of any tensor bundle is the corresponding tensor field. A global section may not exist if the bundle is nontrivial, in which case we must speak of local sections.

For example, suppose we consider the bundle of unit tangent vectors over S^2, whose fibers correspond to $F = S^1$ (which parametrizes the space of directions at each point on a 2-dim manifold). Since every vector field on S^2 must vanish at least at one point, there are no globally defined unit vector fields on S^2 and hence no global sections of this bundle.

A local trivialization is provided by a local orthonormal frame $\{\hat{e}_1, \hat{e}_2\}$ on S^2, like the one $\{e_{\hat\theta}, e_{\hat\varphi}\}$ obtained by normalizing the coordinate derivatives of spherical coordinates which cover S^2 minus the two poles



$$u_x = \cos\chi_x\, \hat{e}_1 + \sin\chi_x\, \hat{e}_2 .$$

$u_x \mapsto (x, \chi) \in S^2 \times S^1$ for any such frame is the local trivialization.

Different local trivializations will relate the fibers S^1 by a translation $\chi \mapsto \chi + \Lambda(x)$, i.e. the structural group consists of rotations of the circle. Spherical coordinates on S^2, together with a coordinate χ defined with respect to $\{e_{\hat\theta}, e_{\hat\varphi}\}$, provide coordinates on the bundle addapted to this local trivialization induced by spherical coordinates.

Returning to our tangent bundle example, a local trivialization is provided by a local frame which is itself a collection of n sections of TM which are linearly independent at each point of M, i.e., a each point in the base of a vector bundle we have to identify a basis in the fiber over the point, and then the linearity encoded in the bundle by the structural group makes the linear structure of the different local trivializations that come out of such choices of bases compatible.

This leads us to the second kind of bundle that interests us, a principal fiber bundle where the fiber is a group. In our continuing example we have the bundle of frames over M:

$$\mathcal{F}(M) = \{ (x, \underline{e}) \mid x \in M,\ e = \{e_a\} = \text{basis of } TM_x \}.$$

What we have called a local frame on M is a local section of this bundle. Here a local trivialization is obtained simply by choosing a local section, i.e., a local frame, and expressing all other local frames in terms of it:

Fix $\{\epsilon_a\}$ a local frame, let $e_a(x) = e^b{}_a(x)\,\epsilon_b$ represent an arbitrary local frame, so we have:

$$(x, e) \longmapsto (x, e^b{}_a(x)) \in M \times \underbrace{GL(n,\mathbb{R})}_{F}$$

The fibers are all diffeomorphic to $GL(n,\mathbb{R})$.
Notice that the "identity section" $x \mapsto s(x) = (x, \delta^b{}_a)$ in this local trivialization corresponds to the fixed local frame $\{\epsilon_a\}$ used to get the local trivialization.

If we take another local section $\epsilon'_b = \epsilon_b A^b{}_a(x) \Leftrightarrow \epsilon_a = \epsilon'_b A^{-1b}{}_a$ or the section $s(x) = (x, A^b{}_a(x))$ in terms of the given local trivialization, then

$$e_a = e^b{}_a \epsilon_b = e^c{}_a (\epsilon'_b A^{-1b}{}_c) = (A^{-1b}{}_c e^c{}_a)\epsilon'_b = e'^b{}_a\,\epsilon'_b$$

means the new local trivialization is $(x, e) \mapsto (x, e'^b{}_a)$ with

$$e'^b{}_a = A^{-1b}{}_c e^c{}_a\,,$$ an inverse left translation on $GL(n,\mathbb{R})$.

Now the "identity section" corresponds to $\{\epsilon'_a\}$. In other words for a principle bundle, a local trivialization is equivalent to a choice of identity element on each fiber, and the structural group action is just left (or right) translation on the fiber representative which is a copy of the structural group. The choice of left or right translation action is a matter of convention. Usually a "left action" is chosen.

↑ first I put right, see below

[A left (respectively right) action of a group G on a manifold M $x \mapsto f_a(x)$ satisfies $f_{a_1} \circ f_{a_2} = f_{a_1 a_2}$ $(f_{a_1} \circ f_{a_2} = f_{a_2 a_1})$. Inverse left translation on a group $f_a = L_{a^{-1}} = (L_a)^{-1}$ is a right action, for example. Hence in our frame bundle example we have a right action of $GL(n,\mathbb{R})$ on the fibers. Right translation is a right action of G on itself too. Left translation is a left action.]

Principal bundles in general are not only locally trivial to $B \times G$ for some Lie group G which coincides with the structural group, but they must also be equipped with a "right action" of G on the fibers which in any local trivialization reduces either to inverse left translation as above or simply right translation.

wrong as deduced below

The inverse left translation in our example came from the passive point of view we adopted. Right translations in the local trivialization correspond to the active point of view. Namely, if we change every frame by the active linear transformation $e_a \to e_b A^b{}_a$,
then in the local trivialization: $e^b{}_a \to e^b{}_c A^c{}_a$, just right translation.
so we get a right action. Now which is the action which interests us? (This is "live" deduction since all my notes are in America.)
For which case do constant linear transformations make sense? These correspond to a real action of $GL(n,\mathbb{R})$ on the bundle as a transformation group. Answer, clearly these right translations, since the inverse left translations corresponding to a change of local trivialization are in general always dependent on the base point — i.e., the structural group action is only defined on the individual fibers but this new right action is a real action on the whole bundle.

The right translations are generated by the left invariant vector fields; since these represent a real action on the bundle, these local trivialization fields must represent geometrically defined fields on the bundle which are mapped onto the left invariant vector fields in any local trivialization. This is not true of the right invariant vector fields.

Note that $\mathcal{F}(S^2)$ admits no global section since S^2 admits no globally nonzero vector field.

The tangent bundle, cotangent bundle, and higher rank tensor bundles, as well as the bundle of frames. all have fibers whose points are associated in some way with the differential structure of the base manifold. They are said to be "soldered to" the base manifold. [The structural group of a higher rank tensor bundle is the corresponding tensor representation of $GL(n\mathbb{R})$.) One can also consider vector and principal bundles where the fibers are unrelated to the base manifold. This is the case with the simple "gauge theories" of electromagnetism, electroweak theory, the gluon theory of the strong nuclear force, and in Grand Unified TheorieS (or GUTS) which unify the electroweak and strong nuclear forces. We will describe electromagnetism in this language.

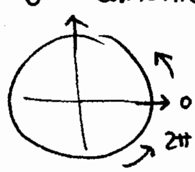
A section will be associated with a "choice of gauge" and the structural group transformations relating different sections or their corresponding local trivializations will be called a "gauge transformation".

Consider a $U(1)$ principal fiber bundle; first some details:

$U(1) \doteq \{ \underline{S} = e^{i\theta} \mid \theta \in \mathbb{R} \} \subset GL(1,\mathbb{C})$ ← (identify 1×1 complex matrix with just a complex number)

$u(1) = \{ i\theta \mid \theta \in \mathbb{R} \}$, basis: $\underline{E_1} = i$, $e_1 = \tilde{e}_1 = \partial/\partial\theta$ bi-invariant $\omega^1 = \tilde{\omega}^1 = d\theta$

θ = canonical coordinate (of first or second kind — no difference in 1-dimension)

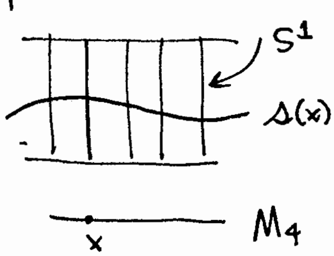


manifold of $U(1)$ is S^1 in complex plane, just a circle.

sometimes a $U(1)$ bundle is called a circle bundle since fibers are circles.

identity of $U(1)$ corresponds to $\theta = 0$.

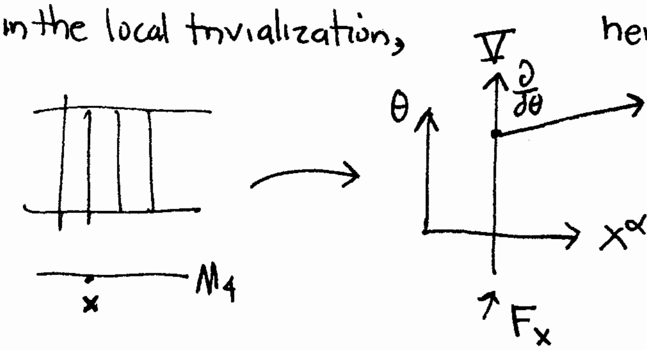
Now a $U(1)$ bundle over spacetime:



Plus we have a right action of $U(1)$ on this bundle

To get a local trivialization we pick a section $s(x)$. Then if $p \in \pi^{-1}(x)$ is a point in the fiber over x, we associate the point

$p \mapsto (x, U) \in M_4 \times U(1)$ where $p = R_U\, s(x)$ defines the unique "right translation" from $s(x)$ to p. This local trivialization (which can obviously be generalized to nonabelian groups where left and right no longer coincide) maps the given section $s(x)$ to the "identity ~~sect~~ section" in the local trivialization, here represented in coordinates $\{x^\alpha, \theta\}$



$\to H? \to D_\alpha = \frac{\partial}{\partial x^\alpha} - A_\alpha(x)\frac{\partial}{\partial\theta}$

Changing the local trivialization leads to point-dependent $U(1)$ transformations on spacetime of the fibers ($\to$ the structural group). On the other hand translations in θ which are independent of spacetime point represent the right action we need on a principal bundle.

There is no natural choice of subspace of the bundle tangent space complementary to the vertical subspace tangent to the fiber so we can introduce ~~an~~ a choice by defining the vector fields $D_\alpha = \frac{\partial}{\partial x^\alpha} - A_\alpha(x)\frac{\partial}{\partial\theta}$ in <u>this local trivialization</u> to pick out the horizontal subspace. Note that any vector field on spacetime $X = X^\alpha \frac{\partial}{\partial x^\alpha}$ can be "<u>lifted</u>" to a "<u>horizontal vector field</u>" on the bundle $\overline{X} = X^\alpha D_\alpha$. [Now we are no longer distinguishing between x^α as a coordinate on the base and in the local trivialization as we did for the tangent and cotangent bundle where we used $q^i = \Pi_* x^i$.]

So what? The frame $\{ e = \frac{\partial}{\partial\theta}, D_\alpha = \frac{\partial}{\partial x^\alpha} - A_\alpha \frac{\partial}{\partial\theta} \}$ is adapted to this direct sum decomposition of the bundle tangent space. The dual frame is $\{ dx^\alpha, \overline{\omega} = d\theta + A_\alpha dx^\alpha = \omega + A_\alpha dx^\alpha \}$.

Suppose we introduce a complex scalar field $\psi(x)$ on spacetime. In many situations (if it represents a wavefunction, for example), the phase of $\psi(x)$ is not physical and all functions related by a phase factor are considered equivalent, or "gauge equivalent".

This is where $U(1)$ comes in — it is the group of phase transformations of complex valued fields.

Define a field $\Psi(x,\theta) = e^{i\theta}\psi(x)$ on the bundle in this local trivialization, representing all the possible gauge equivalent values of $\psi(x)$ on spacetime. Choosing a section of the bundle produces a new value of the function related to $\psi(x)$ by a $U(1)$ gauge transformation

In this local trivialization using coordinates, a section is

$$x \mapsto s(x) \sim (x^\alpha, \theta = \Lambda(x)).$$

Since the section is a map from the base up to the bundle we can "pull back" (ie "pull down" in this context) functions and covariant ~~covariant tangent~~ fields on the bundle to the base spacetime.

So $\quad s_*\Psi \equiv \Psi \circ s = e^{i\Lambda}\psi \qquad$ (just substitute in $\theta = \Lambda(x)$)

This is a gauge transformation of ψ, which is associated with the original section used to get the local trivialization.

What about the fields D_α on the bundle? And the 1-form $W = d\theta + A_\alpha dx^\alpha$ which defines the distribution of horizontal spaces as the subspaces it evaluates to zero on? These define the gauge potential and gauge covariant derivative on the bundle, sections of which lead to the spacetime gauge potential and gauge covariant derivative in each choice of gauge.

Define $\quad D_\alpha \Psi = \left(\frac{\partial}{\partial x^\alpha} - A_\alpha \frac{\partial}{\partial \theta}\right)(e^{i\theta}\psi) = e^{i\theta}\underbrace{(\partial_\alpha \psi - iA_\alpha \psi)}_{\equiv \overset{G}{\nabla}_\alpha \psi} \qquad$ gauge covariant derivative of ψ

and $D\Psi = D_\alpha \Psi\, dx^\alpha = e^{i\theta} \underbrace{\overset{G}{\nabla}_\alpha \psi\, dx^\alpha}_{\overset{G}{D}_A \psi} \qquad$ gauge covariant differential of ψ with respect to $A = A_\alpha dx^\alpha$

on the bundle and on spacetime using the local trivialization.

Now use another section to pull it down to spacetime, ie, change the gauge:

$$s^*(D\Psi) = e^{i\Lambda}\overset{G}{D}\psi \underset{\uparrow \text{verify.}}{=} \overset{G}{D}_{A+d\Lambda}(\underbrace{e^{i\Lambda}\psi}_{\text{transformed } \psi})$$

transformed gauge derivative.

$$s^* W = d\Lambda + A_\alpha dx^\alpha = A + d\Lambda .$$

In other words the gauge potential A transforms by adding the differential of Λ and the new gauge covariant derivative or differential with respect to the transformed potential is exactly the gauge transformation of the old gauge covariant derivative or differential.

In other words, the "gauge covariant" derivative is really gauge covariant. The gauge potential picks out the horizontal subspaces in the ~~potent~~ bundle and provides it with a "connection", i.e. a parallel transport of complex valued functions on the bundle which behave like Ψ, namely under constant translations in θ representing a constant $U(1)$ transformation of the bundle, they are multiplied by that $U(1)$ matrix:

$$\theta \to \theta + \theta_1 \quad , \theta_1 \text{ constant.}$$
$$\Psi \to e^{i(\theta+\theta_1)}\psi = e^{i\theta_1}\Psi \,.$$

These fields represent the gauge equivalent scalar fields on the base. Any curve in spacetime can be lifted to a curve in the bundle and ~~then~~ evaluating Ψ along that curve in the bundle it can be pulled down by an appropriate section to yield a parallelly transported function along the curve in spacetime (vanishing gauge covariant differential along the curve).

To sum up, a $U(1)$ bundle over spacetime just attaches a circle to each point of spacetime representing all possible phase transformations of a complex field at that point. A connection on this potential provides a gauge potential. Complex fields on the bundle which transform by $U(1)$ under the "right action" that comes with the principal bundle correspond to fields on spacetime which undergo phase transformations. The gauge potential provides a gauge covariant derivative for such fields.

EXCEPT MOTIVATED STUDENTS SKIP THIS:

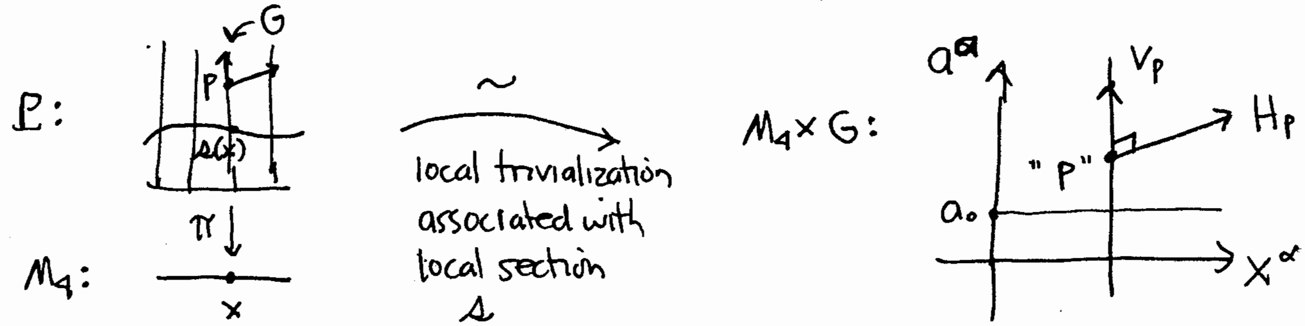
What happens for a nonabelian group? Just to give an idea without going into great detail:

Lie group G; points a; identity a_0; Lie algebra $\mathfrak{g}$; basis $\{e_a\}$; dual basis $\{\omega^a\}$; right invariant basis: $\{\tilde{e}_a\}$; dual basis $\{\tilde{\omega}^a\}$:

$$[e_a, e_b] = C^c{}_{ab}\, e_c \qquad [e_a, \tilde{e}_b] = 0 \qquad [\tilde{e}_a, \tilde{e}_b] = -\, C^c{}_{ab}\, \tilde{e}_c$$

$$\tilde{\omega}^a = R^a{}_b\, \omega^b \ , \qquad R^a{}_b \text{ adjoint matrix.}$$

Now consider a principal bundle P with group G over the base spacetime M_4,



$$p \sim (x^\alpha, a^a)$$
$$s(x) \sim (x^\alpha, a_0^a)$$

In the local trivialization:

The vertical space V_p is tangent to G; one can use either $\{e_a\}$ (left) or $\{\tilde{e}_a\}$ (right) as a basis of this subspace, with dual bases $\{\omega^a\}$ and $\{\tilde{\omega}^a\}$. The left invariant vector fields have a geometrical meaning independent of the local trivialization since they generate the right translations which represents the right action of G on the fiber bundle which is assumed to exist. However, the right invariant vector fields, though related by spacetime dependent adjoint transformations in different local trivializations, are useful in a fixed local trivialization since they are invariant under the right action of G.

The horizontal space H_p is determined by a connection. One "lifts" the coordinate derivatives from spacetime to the bundle by defining

$$D_\alpha = \frac{\partial}{\partial x^\alpha} - A^a{}_\alpha(x)\, \tilde{e}_a = \frac{\partial}{\partial x^\alpha} - R^{-1a}{}_b\, A^b{}_\alpha(x)\, e_a$$

Note that $[e_a, D_\alpha] = 0$, ie, these are invariant under the right action of G

Now choose $\{\partial_\alpha, e_a\}$ or $\{D_\alpha, \tilde{e}_a\}$ as a frame on $M_4 \times G$.
The dual bases are respectively $\{dx^\alpha, W^a\}$ and $\{dx^\alpha, \widetilde{W}^a\}$ where

$$\widetilde{W}^a = \tilde{\omega}^a + A^a{}_\alpha(x)\,dx^\alpha \quad , \quad W^a = R^{-1a}{}_b\, \widetilde{W}^b .$$

The second frame has the advantage of being invariant under the right action of G but the disadvantage that ~~the same~~ it depends on the local trivialization.
The first frame is independent of the local trivialization.

The right action of G on the bundle $\quad p \mapsto \mathcal{R}_{a_1}(p)$
becomes in the local trivialization: $(x, a) \mapsto (x, a a_1) = (x, \underbrace{R_{a_1}}_{\text{right translation by } a_1}(a))$

The adjoint matrix transforms in a simple way; let $a_1 \in G$ be a fixed element:

$$\underline{R}(a) \mapsto \underline{R}(\underline{a a_1}) = \underline{R}(a)\underline{R}(a_1) \,, \quad \text{or supressing } a: \ \underline{R} \to \underline{R}\,\underline{R}(a_1)$$

hence $\quad (\mathcal{R}_{a_1})_* W^a = [(\mathcal{R}_{a_1})_* (R^{-1a}{}_b)]\, \widetilde{W}^b = [R^{-1a}{}_b(a_1) R^{-1b}{}_c]\, \widetilde{W}^b$

(pullback of p-forms)

$$= R^{-1a}{}_b(a_1)\, W^b$$

(invariant)

so the 1-forms W^a undergo an inverse adjoint transformation.

These 1-forms define the distribution of horizontal spaces as the subspaces annihilated by $\{W^a\}$.

Suppose we put a metric on the bundle by declaring H_p orthogonal to V_p, lifting up the spacetime metric to H_p, and putting a right invariant metric along the fibers on V_p: $\quad \dim G = r$

$${}^{(4+r)}g = g_{\alpha\beta}(x)\, dx^\alpha \otimes dx^\beta + g_{ab}(x)\, \widetilde{W}^a \otimes \widetilde{W}^b \quad \left(\text{using invariant second frame}\right)$$

$$(\mathcal{R}_{a_1})_*\, {}^{(4+r)}g = {}^{(4+r)}g \qquad \text{metric invariant under right action of } G$$

This is a Lorentz metric if (g_{ab}) is positive-definite. The generators $\{e_a\}$ of the right translations are Killing vector fields of this metric.

Suppose one imposes the Einstein equations on this fiber metric. One may collapse the theory to the quotient space of the right action of G because of the invariance, namely to a theory on the base space spacetime M_4 in a particular choice of gauge involving the fields

$$g_{\alpha\beta} \text{ (gravity)}, \qquad A^a{}_\alpha \text{ (gauge potentials)}, \qquad g_{ab} \text{ (scalar fields)}.$$

In the special case of a compact group G, taking g_{ab} to be a multiple of the bi-invariant cartan-killing metric (constant components), one obtains a generalization of the Einstein-Maxwell equations with r interacting photon fields $A^a{}_\alpha dx^\alpha$ instead of one, but with a cosmological constant which creates problems.

On the other hand one can allow ~~these~~ above fields which parametrize the bundle metric to depend also on the fiber coordinates, breaking the fiber symmetry. Then one cannot quotient out the theory to obtain an effective theory on spacetime. Now one must interpret the bundle itself as a higher dimensional spacetime and treat all spatial dimensions on an equal footing. The original symmetric theory can be thought of as a lower energy limit — a lowest mode in a harmonic expansion of the parametrizing fields along the fibers. For example for the 5-dimensional Einstein-Maxwell theory, the fibers are circles and one can perform a Fourier expansion — the lowest harmonic is the constant term in the expansion.

Historically, symmetric theories were considered which allowed "dimensional reduction" to 4 spacetime dimensions but the course of events led to these truly higher dimensional theories. Today spacetime is "nearly" four-dimensional, and very high energies are necessary to probe the extra dimensions which are assumed to be compact with very small radius of curvature. To understand how this comes about, one is led to study cosmological models representing possible models of the very early universe when all the dimensions were comparable.

A general feature of these models is that the length scale associated with the internal dimensions (fiber dimensions) affects the scale of the external dimensions ("usual" 4-spacetime dimensions).

In the frame $\{D_\alpha, \tilde{e}_a\}$, the metric has block diagonal form

$$({}^{(4+r)}g_{AB}) = \left(\begin{array}{c|c} {}^{(4)}g_{\alpha\beta} & 0 \\ \hline 0 & {}^{(D)}g_{ab} \end{array}\right)$$

Let $r = D$ be the fiber dimension.
attach superscripts to distinguish subblocks.

$${}^{(4+D)}g = |\det({}^{(4+D)}g_{AB})| = {}^{(4)}g \; {}^{(D)}g \,.$$

The question is, why identify ${}^{(4)}g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta$ as the 4-dimensional metric and not some multiple of it which depends on the fiber metric?
In fact no one tells you how to identify the 4-dimensional metric.

The usual Einstein action has Lagrangian

$$L \sim {}^{(4+D)}g^{1/2}\; {}^{(4+D)}R = \underbrace{{}^{(4)}g^{1/2}\, {}^{(D)}g^{1/2}\, ({}^{(4)}R}_{} + {}^{(D)}R + \dots)$$

4-dim Einstein sector involves Brans Dicke like scalar field factor of ${}^{(D)}g^{1/2}$.

One can absorb the Brans-Dicke-like factor into the 4-dimensional metric by a conformal transformation (called a <u>Weyl transformation</u> in this context)

$$\left.\begin{aligned} {}^{(4)}g_{\alpha\beta} &= e^{-2\psi}\, {}^{(4)}\bar{g}_{\alpha\beta} \\ {}^{(4)}g^{1/2} &= e^{-4\psi}\, {}^{(4)}\bar{g}^{1/2} \\ {}^{(4)}R_{\alpha\beta} &= {}^{(4)}\bar{R}_{\alpha\beta} + \underbrace{\dots}_{\text{gradient of } \psi \text{ terms}} \end{aligned}\right\} \qquad \underbrace{{}^{(4)}g^{1/2}}_{e^{-4\psi}(\,)}\, \underbrace{{}^{(4)}g^{\alpha\beta}}_{e^{+2\psi}(\,)}\, \underbrace{{}^{(4)}R_{\alpha\beta}}_{(\bar{R}+\dots)} = e^{-2\psi}\, {}^{(4)}\bar{R}_{\alpha\beta} + \dots$$

So

$${}^{(D)}g^{1/2}\, {}^{(4)}g^{1/2}\, {}^{(4)}R \approx \underbrace{e^{-2\psi}\, {}^{(D)}g^{1/2}}_{\text{choose } = 1} \left[{}^{(4)}\bar{g}^{1/2}\, {}^{(4)}\bar{R} + \dots \right]$$

$$e^{2\psi} = ({}^{(D)}g^{1/2}) \equiv \mathcal{R}^D$$

$$\cancel{e^{\psi} = \mathcal{R}^{+D} \text{ if } \mathcal{R}^D = {}^{(D)}g^{1/2}}$$

$\mathcal{R}$ = average scale factor of internal space

${}^{(4)}\bar{g}_{\alpha\beta} = \mathcal{R}^{D}\, {}^{(4)}g_{\alpha\beta}$ Thus one must rescale the 4-dimensional fields by a power of the internal radius.

Of course actual unified theories are much more complicated with anticommuting fields and supersymmetries, etc., but the fiber bundle model is intimately related to all of them.

If you were to undertake a study of the things I have only given you a vague impression of, you would be interested in:

Y. M. Cho, Physics Letters B199, 358 (1987) *Effective Couplings in Kaluza-Klein Unification*
and the references therein (be careful, he uses the frame $\{\frac{\partial}{\partial x^\alpha}, e_a\}$ instead of $\{D_\alpha, e_a\}$ so the metric looks more complicated).

Going back to our January study of DeSitter space and higher dimensional generalizations, one can consider generalizing the Lifshitz perturbation analysis to higher dimensions as in

P.F. González-Díaz, Physical Review D36, 3651 (1987), *Perdurance of higher-dimensional de Sitter space.*

[see P.F. González-Díaz, Phys. Lett B191, 267 (1987) for ~~DeSitter~~ Schwarzschild De Sitter and Schwarzschild in higher dimensions]

Unfortunately, there is no text which compromises between the classical and modern approaches ~~is an~~ so that physicists can easily learn the necessary mathematics. You have to struggle if you want to understand this stuff.