INTRODUCTION TO COSMOLOGICAL MODELS: PART III

Another glance at differential geometry:

classical mechanics and phase spaces

Additional useful facts about matrix groups:

canonical parametrizations, invariant field computation, Adjoint group, $SO(3,\mathbb{R})$ and $SU(2)$

Rigid Body dynamics

Part III studies in more detail the geometry of Lie groups and group invariant metrics and applications to the rigid body problem to bridge the new ideas with an elementary physics style perspective.

ANOTHER GLANCE AT DIFFERENTIAL GEOMETRY

In order to consider dynamics on manifolds, one needs a little more knowledge of differential geometry to understand what you may already know from a physics perspective.

Suppose $\{x^a\}$ are local coordinates on some manifold M which is the configuration space for some classical mechanical system. The velocity and momentum phase spaces are important in formulating the dynamics and are just classical names for the tangent and cotangent bundles.

Let TM_x be the tangent space at x and let TM_x^* be its dual space, the cotangent space. Then the tangent bundle is the set of all tangent spaces $TM = \{TM_x \mid x \in M\} = \{(x, X) \mid X \in TM_x\}$ and the cotangent bundle TM^* is similarly defined $TM^* = \{(x, \sigma) \mid \sigma \in TM_x^*\}$.

These spaces have natural coordinate systems extending the coordinates $\{x^a\}$ on M which correspond to expressing tangent vectors and covectors in the coordinate frame $\{\partial/\partial x^a\}$ with dual frame $\{dx^a\}$.

$\{q^a, \dot{q}^a\}$ on TM (velocity phase space):
$$q^a(x, X) = x^a(x)$$
$$\dot{q}^a(x, X) = dx^a(X) = X^a$$
$$(\text{if } X = X^a \, \partial/\partial x^a|_x)$$

$\{q^a, p_a\}$ on TM^* (momentum phase space):
$$q^a(x, \sigma) = x^a(x)$$
$$p_a(x, \sigma) = \sigma(\partial/\partial x^a) = \sigma_a$$
$$(\text{if } \sigma = \sigma_a \, dx^a|_x)$$

These two spaces may be related to each other if one has a way to relate tangent and cotangent vectors. This can be done with a metric or thinking classically mechanically, using a Lagrangian (in practice these are often equivalent).

Suppose $g = g_{ab} dx^a \otimes dx^b$ is a metric on M. Then index raising and lowering relates tangent and cotangent vectors and

$$\{q^a, \dot{q}^a\} \to \{q^a, p_a = g_{ab}\dot{q}^b\} \quad \text{with inverse} \quad \{q^a, p_a\} \to \{q^a, \dot{q}^a = g^{ab} p_b\}$$

provides an invertible map between the velocity and momentum phase spaces which enables us to think of them as two different realizations of the same abstract phase space. This same correspondence may be made using a Lagrangian. Let $T = \frac{1}{2} g_{ab} \dot{q}^a \dot{q}^b$ be the kinetic energy function associated with the metric; it is a function on TM.

If U is any function on M thought of as a potential, the Lagrangian $L = T - U$ describes motion in the geometry of the metric subject to the force field $F = -dU = -\frac{\partial U}{\partial x^a} dx^a \equiv F_a dx^a$ which is invariantly described by a 1-form. If $U = 0$, then the Lagrangian reduces to the kinetic energy and describes the geodesics of the metric. [equations of motion: $0 = -\frac{\delta L}{\delta q^a} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) - \frac{\partial L}{\partial q^a}$.]

One defines "the momenta" in the usual way:

$p_a = \frac{\partial L}{\partial \dot{q}^a} = \frac{\partial T}{\partial \dot{q}^a} = g_{ab} \dot{q}^b$. This relation between velocities and momenta establishes an identification map between the two phase spaces which amounts to "lowering the index."

The Hamiltonian is introduced as

$H = p_a \dot{q}^a - L$ re-expressed as a function of $\{q^a, p_a\}$, ie, a function on TM^*.

In our example $H = \frac{1}{2} g^{ab} p_a p_b + U = T + U$, where here T is considered as a function on TM^*.

By introducing the Poisson Bracket of functions on TM^*

$\{f, h\}_p = \frac{\partial f}{\partial q^a} \frac{\partial h}{\partial p_a} - \frac{\partial h}{\partial q^a} \frac{\partial f}{\partial p_a}$ so that the only nonzero Poisson brackets among the canonical coordinates $\{q^a, p_a\}$ (here brackets mean set notation) are $\{q^a, p_b\}_p = \delta^a{}_b$, one obtains the Hamiltonian equations of motion

$$\frac{d}{dt} q^a = \{q^a, H\}_p, \qquad \frac{d}{dt} p_a = \{p_a, H\}_p .$$

(we may be sloppy and put $\dot{q}^a = \frac{dq^a}{dt}$, $\dot{p}_a = \frac{dp_a}{dt}$)

A trajectory $c(t)$ in M which is a solution of the equations of motion produces curves in both bundles (phase spaces). For example, letting $c^a(t) = x^a(c(t))$ and $c'(t) = \frac{dc^a(t)}{dt} \frac{\partial}{\partial x^a}\Big|_{c(t)}$, then $\left(c^a(t), \frac{dc^a(t)}{dt}\right)$ is the coordinate representation of the curve in the velocity phase space, while $\left(c^a(t), g_{ab}(c(t)) \frac{dc^b(t)}{dt}\right)$ is the one for the momentum phase space.

In some problems it is important to use a noncoordinate frame on M. This can be extended to the phase spaces by taking components of tangent and cotangent vectors with respect to the noncoordinate frame instead of a coordinate frame.

Suppose $\{e_a\}$ is a local frame, with $e_a = e^b{}_a \frac{\partial}{\partial x^b}$ when expressed in local coordinates. Let $\{\omega^a\}$ be the dual frame, with $\omega^a = \omega^a{}_b dx^b$. The duality condition $\delta^a{}_b = \omega^a(e_b) = \omega^a{}_c e^c{}_b$ means that their component matrices are inverse matrices. The amount by which a frame fails to be a coordinate frame is characterized by the "structure functions" $C^a{}_{bc}$ for the frame:

$$[e_a, e_b] = C^c{}_{ab} e_c \quad \rightarrow \quad C^c{}_{ab} = \omega^c([e_a, e_b]).$$

[The Lie brackets of the basis vectors can be expressed in the frame — the expansion coefficients are the structure functions.]

If we use $\{i, j, k, ..\}$ for coordinate indices then

$$X^a = \omega^a{}_i X^i \qquad \sigma_a = \sigma_i e^i{}_a$$
$$X^i = e^i{}_a X^a \qquad \sigma_i = \sigma_a \omega^a{}_i$$

are the transformations between coordinate and frame components. Similarly we can introduce new coordinates on the tangent and cotangent bundles:

$\{q^a, \dot\omega^a\}$ on TM with $\dot\omega^a \equiv \omega^a{}_i \dot q^i$ (generalized velocities)

$\{q^a, P_a\}$ on TM^* with $P_a \equiv e^i{}_a p_i$ (generalized momenta)

Then if $g = g_{ab}\,\omega^a \otimes \omega^b = g_{ij}\, dx^i \otimes dx^j$, $T = \frac{1}{2} g_{ab} \dot\omega^a \dot\omega^b$ on TM and $T = \frac{1}{2} g^{ab} P_a P_b$ on TM^*, with $P_a = \frac{\partial T}{\partial \dot\omega^a} = g_{ab}\dot\omega^b$ giving the new representation of the correspondence between the phase spaces. For a curve $c(t)$ in M leading to a curve in the phase spaces, note that $\dot\omega^a(c(t)) = \omega^a{}_i(c(t)) \frac{d}{dt} c^i(t)$ is not a total time derivative; the dot "$\cdot$" in fact refers to a natural lifting operation in general which coincides with the time derivative when evaluated in this way for a coordinate frame. [For example the kinetic energy is half the lift of the metric on TM and half the lift of the contravariant metric on TM^* — any vector field $X = X^i \partial/\partial x^i = X^a e_a$ lifts to a "moment function" $X^i p_i = X^a P_a$ or momentum on TM^*.]

The coordinates $\{q^i, p_i\}$ on TM^* are called "canonical coordinates" and are characterized by their special Poisson brackets. The coordinates $\{q^i, P_a\}$ do not satisfy the same brackets.

EXERCISE. Using the expansion $P_a = e^i{}_a p_i$ and the Poisson brackets of the canonical coordinates, show that

$$\{q^i, P_a\}_P = e^i{}_a \qquad \{P_a, P_b\}_P = -C^c{}_{ab} P_c \, .$$

Hint: You need to use the coordinate formula for the Lie bracket:

$$[e_a, e_b]^i = e^j{}_a \partial_j e^i{}_b - e^j{}_b \partial_j e^i{}_a .$$

EXERCISE. The coordinates $\{q^i, \dot\omega^a\}$ are "noncanonical coordinates" on TM since the Lagrange derivative used to define the Lagrange equations takes a noncanonical form when re-expressed in these coordinates. Show that the frame components of this derivative $\delta L/\delta \dot\omega^a = e^i{}_a \, \delta L/\delta q^i$ satisfy $\delta L/\delta \dot\omega^a = e_a L - (\partial L/\partial \dot\omega^a)^\cdot + (\partial L/\partial \dot\omega^c) C^c{}_{ab} \dot\omega^b$, where $e_a L$ means the derivative of $L(q^i, \dot\omega^a)$ holding $\dot\omega^a$ fixed, i.e. $e_a L = e^i{}_a \left(\dfrac{\partial L(q^j, \dot q^j)}{\partial q^i} - \dfrac{\partial L(q^j, \dot\omega^c)}{\partial \dot\omega^b} (\partial_i \omega^b{}_k) \dot q^k \right)$

Hint: You need to use duality to evaluate derivatives of $\omega^a{}_i$ to those of $e^i{}_a$: $\omega^b{}_i e^i{}_a = \delta^b{}_a \rightarrow \omega^b{}_{i,j} e^i{}_a + \omega^b{}_i e^i{}_{a,j} = 0$ and then use the Lie brackets of $\{e_a\}$. This is a bit involved, but it is just product and chain rules involving the transformation $\dot\omega^a = \omega^a{}_i \dot q^i$.

EXERCISE. Show that if $X = X^i \partial_i$, $Y = Y^i \partial_i$ then

$$\{X^i p_i, Y^j p_j\}_P = -[X, Y]^i p_i .$$

Thus if $[X, Y] = 0$, these "momentum functions" commute.

Suppose we have a Lie group G. Then we have a basis $\{e_a\}$ of the Lie algebra $\mathfrak{g}$ of left invariant vector fields on G which is a global frame on G. There is also a corresponding right invariant frame $\{\tilde{e}_a\}$. Each may be used to obtain "noncanonical coordinates" on the tangent and cotangent bundles by the introduction of generalized velocities and momenta.

Recall $\quad [e_a, e_b] = C^c{}_{ab}\, e_c \qquad [\tilde{e}_a, \tilde{e}_b] = -C^c{}_{ab}\, \tilde{e}_c \qquad [e_a, \tilde{e}_b] = 0.$

Note that now the structure functions are <u>constants</u>, and are therefore called structure constants.

If $\{q^a\}$ are local coordinates on G, then being sloppy (not changing the symbol for these when considered as functions on TG or TG^*) one has canonical coordinates $\{q^a, \dot{q}^a\}$ on TG and $\{q^a, p_a\}$ on TG^*.

Using the left invariant frame, with structure functions $C^a{}_{bc}$:

$$\dot{\omega}^a = \omega^a{}_i\, \dot{q}^i \,, \quad P_a = e^i{}_a\, p_i \,, \qquad \{P_a, P_b\}_p = -C^c{}_{ab} P_c$$

Using the right invariant frame, with structure functions $-C^a{}_{bc}$:

$$\dot{\tilde{\omega}}^a = \tilde{\omega}^a{}_i\, \dot{q}^i \,, \quad \tilde{P}_a = \tilde{e}^i{}_a\, p_i \,, \qquad \{\tilde{P}_a, \tilde{P}_b\}_p = C^c{}_{ab} \tilde{P}_c .$$

and finally since $[e_a, \tilde{e}_b] = 0$: $\qquad \{P_a, \tilde{P}_b\} = 0.$

Finally, suppose $c(t)$ is a parametrized curve in G with tangent $c'(t)$. In local coordinates $c^a(t) = q^a(c(t))$, $c'(t) = \dot{c}^a(t) \frac{\partial}{\partial q^a}\Big|_{c(t)}$, where here the dot means d/dt. Then $(c(t), c'(t))$ is a curve in the tangent bundle; in canonical coordinates: $(c^a(t), \dot{c}^a(t))$.

One may introduce generalized velocities associated with the left or right invariant frames:

$$\dot{\omega}^a(c(t)) = \omega^a{}_b(c(t))\, \dot{c}^b(t) \,, \qquad \dot{\tilde{\omega}}^a(c(t)) = \tilde{\omega}^a{}_b(c(t))\, \dot{c}^b(t).$$

ANSWER TO EXERCISE ON PAGE DG4:

FRAME LAGRANGE DERIVATIVE

$\{e_a\}$ on M; $[e_a, e_b] = C^c{}_{ab} e_c$, $\omega^a(e_b) = \delta^a_b$

$$\frac{\delta L}{\delta q^i} = \frac{\partial L}{\partial q^i} - \left(\frac{\partial L}{\partial \dot q^i}\right)^{\cdot}$$

Note $L = L(q^i, \dot q^i)$ defines partial derivatives.

Now introduce generalized velocities $\dot\omega^a = \omega^a{}_i \dot q^i$, $L = L(q^i, \dot\omega^a)$ defines new partial derivatives.

$$e_a L = \underbrace{e_a L\big|_{\dot\omega}}_{\text{held fixed}} + \frac{\partial L}{\partial \dot\omega^c} \underbrace{e_a \dot\omega^c}_{(e_a \omega^c{}_i)\dot q^i = (e_a \omega^c{}_i) e^i{}_b \dot\omega^b}$$

$$\frac{\delta L}{\delta \dot\omega^a} \equiv e^i{}_a \frac{\delta L}{\delta q^i} = e^i{}_a\left(\frac{\partial L}{\partial q^i} - \left(\frac{\partial L}{\partial \dot q^i}\right)^{\cdot}\right) = e_a L - e^i{}_a \left(\frac{\partial L}{\partial \dot q^i}\right)^{\cdot}$$

$$e^i{}_a \frac{\partial L}{\partial \dot q^i} = e^i{}_a\left(\frac{\partial L}{\partial \dot\omega^c}\omega^c{}_i\right)^{\cdot} = e^i{}_a \frac{\partial L}{\partial \dot\omega^c}(\omega^c{}_i)^{\cdot} + \underbrace{e^i{}_a \left(\frac{\partial L}{\partial \dot\omega^c}\right)^{\cdot} \omega^c{}_i}_{\left(\frac{\partial L}{\partial \dot\omega^a}\right)^{\cdot}}$$

δ^c_a

$$(\omega^c{}_i)^{\cdot} = \omega^c{}_{i,j} \dot q^j$$
$$\omega^c{}_{i,j} e^j{}_b \dot\omega^b = (e_b \omega^c{}_i)\dot\omega^b$$

$$\frac{\delta L}{\delta \dot\omega^a} = e_a L\big|_{\dot\omega} + \frac{\partial L}{\partial \dot\omega^c}\underbrace{\left[\underbrace{(e_a \omega^c{}_i) e^i{}_b}_{-\omega^c{}_i e_a e^i{}_b} - \underbrace{(e_b \omega^c{}_i) e^i{}_a}_{-\omega^c{}_i e_b e^i{}_a}\right]}_{-\omega^c{}_i [e_a, e_b]^i = -C^c{}_{ab}}\dot\omega^b$$

$$-\omega^c{}_i \underbrace{[e_a, e_b]^i}_{C^d{}_{ab} e^i_d}, \quad \omega^c{}_i e^i_d = \delta^c_d$$

$$\boxed{\frac{\delta L}{\delta \dot\omega^a} = e_a L\big|_{\dot\omega} - \left(\frac{\partial L}{\partial \dot\omega^c}\right) C^c{}_{ab}\dot\omega^b}$$

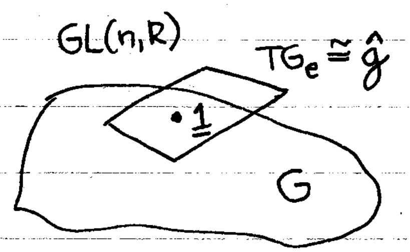
ADDITIONAL USEFUL FACTS ABOUT MATRIX GROUPS

Suppose $\{\underline{E}_a\}_{a=1,\ldots,r}$ is a basis of a Lie subalgebra $\hat{g}$ of the Lie algebra $gl(n,R)$ of $n\times n$ matrices, i.e., $[\underline{E}_a, \underline{E}_b] = C^c{}_{ab}\underline{E}_c$.
Exponentiation of this Lie algebra (all products of all exponentials of elements of $\hat{g}$) leads to a matrix subgroup G of $GL(n,R)$. Locally near the identity one can write for a general matrix in the subgroup:

$$\underline{S} = e^{x^a \underline{E}_a}$$ a parametrization corresponding to "canonical coordinates of the first kind" on G

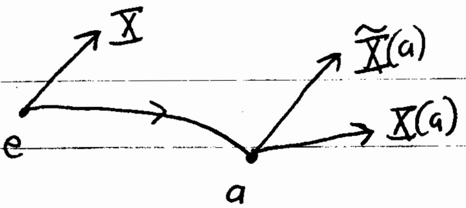
or $$\underline{S} = e^{x^1\underline{E}_1} e^{x^2\underline{E}_2} \cdots e^{x^r\underline{E}_r}$$ a parametrization corresponding to "canonical coordinates of the second kind" on G,

or other types of exponential parametrizations like Euler angle coordinates on $SO(3,R)$ or even non exponential parametrizations.
$\underline{S}$ is a matrix-valued function on the manifold G.



G is a submanifold of $GL(n,R)$, itself an open submanifold of the vector space $gl(n,R)$ of $n\times n$ matrices. As a subgroup, it must pass through the identity matrix $\underline{1}$, which is the identity e of G. Since one may always identify tangent vectors to a vector space manifold with vectors in the original vector space in a natural way, the tangent space TG_e at the identity of G may be identified with a vector subspace of $gl(n,R)$. This subspace is $\hat{g}$, a Lie subalgebra of $gl(n,R)$.

[If $\underline{S}(x(t))$ is a curve in G parametrized by specifying the "parameters" x^a as functions of t, then $\frac{d}{dt}\underline{S}(x(t))$ is the matrix tangent which corresponds to the tangent vector $\frac{dx^a(t)}{dt}\frac{\partial}{\partial x^a}\Big|_{x(t)}$ in the corresponding local coordinates.]



To each tangent vector $\mathbb{X} \in TG_e$ at the identity corresponds a unique left invariant vector field $\mathbb{X}$ and a unique right invariant vector field $\tilde{\mathbb{X}}$; just translate $\mathbb{X}$ from e to a using the differential of the appropriate translation:

$$\mathbb{X}(a) = d(L_a)(e)\,\mathbb{X} \qquad\qquad \tilde{\mathbb{X}}(a) = d(R_a)(e)\,\mathbb{X}$$

[[Note: any differentiable map $h: M \to N$ has the differential $dh(x): TM_x \to TN_{h(x)}$ which "pushes forward" tangent vectors by $\mathbb{X} \in TM_x \mapsto \mathbb{Y} = dh(x)\mathbb{X} \in TN_{f(x)}$. In local coordinates $\{x^i\}$ on M and $\{y^\alpha\}$ on N, $y^\alpha = h^\alpha(x)$ and $\mathbb{Y}^\alpha = \dfrac{\partial y^\alpha(x)}{\partial x^i}\mathbb{X}^i$

For a 1-1 map, one can push a whole vector field $\mathbb{X}$ from M to $h(M)$ in this way. For a diffeomorphism $h: M \to M$, one obtains a new vector field $h\mathbb{X}$ on M by pushing forward the value of $\mathbb{X}$ from $h^{-1}(x)$ to x: $\quad (h\mathbb{X})(x) = dh(h^{-1}(x))\,\mathbb{X}(h^{-1}(x))$.

This is called "dragging along" the vector field by h.

The chain rule says that the differential of a composition of maps is the composition of the differentials $\quad d(h_1 \circ h_2) = dh_1 \circ dh_2$, neglecting arguments. This is enough to derive the left or right invariance of the above vector fields.

$$\mathbb{X}(L_{b^{-1}}(a)) \overset{\text{def}}{=} \mathbb{X}(b^{-1}a) \overset{\text{def}}{=} dL_{b^{-1}a}(e)\,\mathbb{X} \overset{\substack{\text{composition of left translations}\\ \downarrow}}{=} d(L_{b^{-1}} \circ L_a)(e)\,\mathbb{X}$$

$$\underset{\text{chain rule}}{=} dL_{b^{-1}}(a)\, dL_a(e)\,\mathbb{X} \underset{\text{def}}{=} dL_{b^{-1}}(a)\,\mathbb{X}(a)$$

Now pushing forward by dL_b undoes $dL_{b^{-1}}$ and gives back $\mathbb{X}(a)$:

$$\underbrace{(L_b\mathbb{X})}_{\text{dragged along field}}(a) \overset{\text{def}}{=} dL_b(L_{b^{-1}}(a))\,\mathbb{X}(\underbrace{L_{b^{-1}}(a)}) \overset{\text{(substitute)}}{=} \underbrace{dL_b(L_{b^{-1}}(a))\, dL_{b^{-1}}(a)}_{d(\underbrace{L_b \circ L_{b^{-1}}})(a) = d(\underbrace{L_e}_{\text{identity}})(a)}\,\mathbb{X}(a)$$

$$= \mathbb{X}(a), \quad \text{i.e. } L_b\mathbb{X} = \mathbb{X},$$

so the field is invariant under dragging along by any left translation.]]

So on our matrix group G, we can identify the matrix basis $\{\underline{E}_a\} \subset \hat{g}$ with a basis $\{e_a\}$ of the tangent space at the identity TG_e and and then left translate it over the group to obtain a basis $\{e_a\} \subset g$ of left invariant vector fields (= left invariant frame on G) and right translate it to get a right invariant frame $\{\tilde{e}_a\}$. Here "g" denotes "the Lie algebra of G."

One can show that $\{e_a\}$ have the same Lie brackets as the matrices $\{\underline{E}_a\}$ have commutators:

$$[e_a, e_b] = C^c{}_{ab}\, e_c$$

and hence from previous notes (Part II):

$$[\tilde{e}_a, \tilde{e}_b] = -C^c{}_{ab}\, e_c \qquad \text{and} \qquad [e_a, \tilde{e}_b] = 0.$$

How do we evaluate these fields in local coordinates?
Note that the matrix $\underline{S}^{-1} d\underline{S}$ of 1-forms is invariant under left translation $S \to S_1 S$, where $S_1 \in G$ is some fixed point

$$\underline{S}^{-1} d\underline{S} \to (\underline{S_1 S})^{-1} d(\underline{S_1 S}) = \underline{S}^{-1} \underline{S_1}^{-1} \underline{S_1} d\underline{S} = \underline{S}^{-1} \underline{1}\, d\underline{S} = \underline{S}^{-1} d\underline{S}.$$

Similarly $d\underline{S}\, \underline{S}^{-1}$ is right invariant.

<u>VERY USEFUL FACT</u> $\qquad \underline{S}^{-1} d\underline{S} = \omega^a \underline{E}_a \,, \quad d\underline{S}\, \underline{S}^{-1} = \tilde{\omega}^a \underline{E}_a$

where $\{\omega^a\}$ and $\{\tilde{\omega}^a\}$ are the dual bases of respectively left and right invariant 1-forms. Given any parametrization of $\underline{S}$ by local coordinates $\{x^a\}$, simply compute $\underline{S}^{-1} d\underline{S}$ and $d\underline{S}\, \underline{S}^{-1}$ and read off the invariant 1-forms as the coefficients of the matrices $\underline{E}_a$. One may then construct the invariant frame vectors using the duality relations (equivalent to inverting the matrix of coordinate components of the 1-forms).

SECOND VERY USEFUL FACT

Conjugating the basis matrices by $\underline{S}$ leads to a nonsingular linear transformation:

$$\underline{S}\,\underline{E}_a\underline{S}^{-1} = \underline{E}_b \underline{R}^b{}_a$$

which defines a second matrix valued function on G (namely $\underline{R}$) which is a matrix representation of the so called Adjoint group associated with G (actually the linear adjoint group). (The map $\underline{S} \to \underline{R}$ defined by this equation is a homomorphism between the two matrix groups.)

Conjugating the first very useful fact:

$$\left.\begin{array}{l}\underline{S}(\underline{S}^{-1}d\underline{S})\underline{S}^{-1} = \omega^b(\underline{S}\,\underline{E}_b\underline{S}^{-1}) = (R^a{}_b\,\omega^b)\underline{E}_a \\ \quad \| \ d\underline{S}\,\underline{S}^{-1} = \tilde{\omega}^a \underline{E}_a\end{array}\right\} \to \tilde{\omega}^a = R^a{}_b\,\omega^b.$$

The adjoint matrix maps the left invariant 1-forms onto the right invariant 1-forms, and consequently by duality $\quad \tilde{e}_a = e_b\, R^{-1\,b}{}_a$.

EXAMPLE ROTATION GROUP $SO(3,\mathbb{R})$

Matrix Lie algebra $so(3,\mathbb{R})$ = {antisymmetric 3x3 matrices}

Basis introduced in part II: $\{\underline{E}_a\} = \{J_1, J_2, J_3\} = \{J_{23}, J_{31}, J_{12}\}$

where $-(\underline{J}_a)^b{}_c = \epsilon_{abc} = C^a{}_{bc}$, $[J_a, J_b] = \epsilon_{abc} J_c$.

$J_{12} = J_3$ generates active rotations in the x^1-x^2 plane of R^3 or equivalently about the x^3 axis. (positive angle defined by right hand screw rule) (Note finite rotation $e^{\theta \underline{J}_3}$ is active rotation by angle θ about x^3 axis using this convention).

For this basis: $\quad \underline{S}\,\underline{E}_a\underline{S}^{-1} = \underline{E}_b S^b{}_a$, $\quad$ ie. $\underline{R} = \underline{S}$

(Adjoint group coincides with original matrix group.) This just says that the generator of rotation about a rotated axis is the rotation applied to the basis vector.

[Note that the Cartan-Killing inner product is

$\gamma_{ab} = C^d{}_{ca} C^e{}_{bd} = \epsilon_{dca}\, \epsilon_{ebd} = -2\,\delta_{ab}$, ie modulo a constant, the basis $\{\underline{E}_a\}$ is orthonormal. In fact, the adjoint matrix group leaves both the structure constants and Cartan-Killing inner product invariant: $R^a{}_d\, C^d{}_{fg}\, R^{-1f}{}_b\, R^{-1g}{}_c = C^a{}_{bc}$, $\gamma_{fg}\, R^{-1f}{}_a\, R^{-1g}{}_b = \gamma_{ab}$

so the matrix $\underline{R}$ has to be orthogonal.]

<u>Euler Angles</u> Many different conventions for Euler angles exist. Perhaps the best known is due to GOLDSTEIN (*Classical Mechanics*)

$$\underline{S} = e^{-\theta^2 \underline{E}_3}\, e^{-\theta^1 \underline{E}_1}\, e^{-\theta^3 \underline{E}_3} \qquad \text{with } (\theta^1, \theta^2, \theta^3) = (\theta, \psi, \phi).$$

I use "θ" rather than "x" since it is more suggestive of angles. The minus signs occur since he adapts the passive point of view of coordinate transformations which is inverse to the active point transformation point of view. Note that $\underline{E}_2$ is not needed in the parametrization due to the nonabelian nature of the group ($\underline{E}_2 = [\underline{E}_3, \underline{E}_1]$ is generated by products involving $\underline{E}_1$ and $\underline{E}_3$), but the parametrization is singular at the identity since $\theta^1 = 0$ leads to a 1-dimensional subspace rather than a 2-dimensional subspace as in canonical parametrizations. [This singularity exists on the whole subgroup $\{e^{\theta \underline{E}_3} \mid \theta \in \mathbb{R}\}$.]

An explicit coordinate representation of the invariant fields is obtained as indicated above. Let $\partial_a = \partial/\partial\theta^a$, $c_a = \cos\theta^a$, $s_a = \sin\theta^a$:

$$
\begin{aligned}
-\omega^1 &= c_3\, d\theta^1 + s_1 s_3\, d\theta^2 & -e_1 &= c_3\,\partial_1 + s_3\,[s_1^{-1}\partial_2 - \cot_1 \partial_3] \\
-\omega^2 &= -s_3\, d\theta^1 + s_1 c_3\, d\theta^2 & -e_2 &= s_3\,\partial_1 - c_3\,[s_1^{-1}\partial_2 - \cot_1 \partial_3] \\
-\omega^3 &= c_1\, d\theta^2 + d\theta^3 & -e_3 &= \partial_3
\end{aligned}
$$

$$
\begin{aligned}
-\tilde\omega^1 &= c_2\, d\theta^1 + s_1 s_2\, d\theta^3 & -\tilde e_1 &= c_2\,\partial_1 + s_2\,[s_1^{-1}\partial_3 - \cot_1 \partial_2] \\
-\tilde\omega^2 &= -s_2\, d\theta^1 + s_1 c_2\, d\theta^3 & -\hat e_2 &= -s_2\,\partial + c_2\,[s_1^{-1}\partial_3 - \cot_1 \partial_2] \\
-\tilde\omega^3 &= d\theta^2 + c_1\, d\theta^3 & -\hat e_3 &= \partial_2
\end{aligned}
$$

The overall minus sign comes from the passive point of view

Note that right multiplication by exponentials of $\underline{E}_3$ is equivalent to translation in θ^3, so the left invariant vector field which generates this right translation is just ∂_3. Ditto for $\tilde{e}_3$ and left translations.

Next suppose we lift the Euler angles to coordinates on the tangent ~~b~~ and cotangent bundles and use the invariant frames instead of the coordinate frame.

In particular the left and right generalized velocities are:

$$-\dot{\omega}^1 = c_3\dot{\theta}^1 + s_1 s_3 \dot{\theta}^2 \qquad -\dot{\tilde{\omega}}^1 = c_2\dot{\theta}^1 + s_1 s_2 \dot{\theta}^3$$

$$-\dot{\omega}^2 = s_3\dot{\theta}^1 - s_1 c_3 \dot{\theta}^2 \qquad -\dot{\tilde{\omega}}^2 = -s_2\dot{\theta}^1 + s_1 c_2 \dot{\theta}^3$$

$$-\dot{\omega}^3 = c_1\dot{\theta}^2 + \dot{\theta}^3 \qquad -\dot{\tilde{\omega}}^3 = \dot{\theta}^2 + c_1\dot{\theta}^3$$

With $(\theta^1, \theta^2, \theta^3) = (\theta, \psi, \phi)$, one can find these expressions in GOLDSTEIN as the components of the angular velocity of a rigid body (whose orientation is specified by the Euler angles) with respect to space-fixed and body fixed axes respectively. We will describe this in detail shortly.

EXAMPLE special unitary group in 2-dimensions:

$SU(2)$ = unit determinant 2X2 complex matrices satisfying $\underline{S}^\dagger \underline{S} = \underline{1}$ where $\underline{S}^\dagger = \bar{\underline{S}}^T$ (complex conjugate transpose). (equivalently $\underline{S}^\dagger = \underline{S}^{-1}$)

matrix Lie algebra $su(2)$ consists of anti-Hermitian matrices

basis $\underline{E}_a = -\frac{i}{2}\underline{\sigma}_a$ where $\{\sigma_a\} = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

are the standard (Hermitian) pauli matrices: $(\sigma_a{}^\dagger = \sigma_a)$

$$[E_a, E_b] = \epsilon_{abc} E_c, \quad \text{ie} \quad C^a{}_{bc} = \epsilon_{abc}.$$

Note that this is the same Lie algebra structure as the standard basis of $so(3,R)$; they are called isomorphic. In fact the adjoint relation

$$\underline{S}\underline{E}_a\underline{S}^{-1} = \underline{E}_b R^b{}_a$$

defines a 2 to 1 homomorphism from $SU(2)$ onto $SO(3,R)$ by associating the orthogonal matrix R with $\underline{S}$ (and $-\underline{S}$)

In fact using a parametrization corresponding to canonical coordinates of the first kind one finds $\underline{S} = e^{\theta^a \underline{E}_a} \mapsto \underline{R} = e^{\theta^a \underline{J}_a}$.

The coordinates θ^a are just cartesian coordinates on the Lie algebra (of either group) exponentiated to yield coordinates on the group.

Suppose we introduce spherical coordinates (θ, Θ, Φ) with respect to these cartesian coordinates on the Lie algebra:

$$\begin{aligned} \theta^1 &= \theta \cos\Theta \cos\Phi \\ \theta^2 &= \theta \cos\Theta \sin\Phi \\ \theta^3 &= \theta \sin\Theta \end{aligned} \qquad \text{or} \qquad \begin{aligned} &\theta^a = \theta\, n^a(\Theta, \Phi) \\ &\text{with } \delta_{ab} n^a n^b = 1 \end{aligned}$$

Then the rotation $\underline{R} = e^{\theta n^a \underline{J}_a}$ corresponds to an active rotation by angle θ about the direction n^a (right hand screw rule), so on $SO(3,R)$: $\theta \in [0, 2\pi)$, $\Theta \in [0,\pi)$, $\Phi \in [0, 2\pi)$.

On the other hand any unitary matrix can be expressed in the form

$S = a^0 \underline{1} + a^a (2\underline{E}_a)$ [Exercise: verify that $(a^0, a^a) \to (a^0, -a^a)$ leads to $\underline{S} \to \underline{S}^{-1}$, then it is easy to verify $\underline{S}^\dagger = \underline{S}^{-1}$ using the anti-Hermitian property of the basis.]

The unit determinant condition yields $1 = |\underline{S}| = (a^0)^2 + \delta_{ab} a^a a^b$.

Thus the manifold of $SU(2)$ is the 3-sphere S^3 sitting in R^4.

In fact one can evaluate the exponential $\underline{S} = e^{\theta n^a \underline{E}_a}$ using properties of the Pauli matrices (exercise) to obtain:

$$\underline{S} = \cos\tfrac{\theta}{2}\, \underline{1} + \sin\tfrac{\theta}{2}\, (2 n^a \underline{E}_a)$$

i.e., $a^0 = \cos\frac{\theta}{2}$, $a^a = (\sin\frac{\theta}{2}) n^a$.

Standard spherical coordinates on S^3 $\{\chi, \theta, \varphi\}$ are defined by $a^0 = \cos\chi$, $a^a = \sin\chi\, n^a(\theta, \varphi)$, leading to the identification $\{\frac{\theta}{2}, \Theta, \Phi\} = \{\chi, \theta, \varphi\}$.

Thus for $SU(2)$, $\theta \in [0, 4\pi)$. Since $\underline{S}(\theta) = -\underline{S}(\theta - 2\pi)$ for $\theta \in [2\pi, 4\pi)$, the southern hemisphere of S^3 reproduces the negatives of the matrices of the northern hemisphere. Pairs of antipodal points (i.e., $(\underline{S}, -\underline{S})$) are associated with the same rotation $\underline{R}$, so in fact $SO(3,R)$ is P^3, the sphere with antipodal points identified.

If we introduce the same Euler angle parametrization for $SU(2)$ that we did for $SO(3,R)$, then all the invariant fields will have exactly the same expressions: $\underline{S} = e^{-\theta^2 \underline{E}_3} e^{-\theta^1 \underline{E}_1} e^{-\theta^3 \underline{E}_3}$.
Recall $(\theta^1, \theta^2, \theta^3) = (\theta, \psi, \phi)$. See GOLDSTEIN for an interpretation of the Euler angles. θ and φ determine the orientation of the new polar axis, while ψ describes the rotation about the new polar axis: $\theta \in [0,\pi]$, $\varphi \in [0,2\pi)$ but $\psi \in [0,2\pi)$ for $SO(3,R)$ and $\psi \in [0,4\pi)$ for $SU(2)$.

One can also consider canonical coordinates of the second kind on these two groups. These are regular everywhere but at the south pole where all the 1-parameter subgroups intersect, in the case of $SU(2)$.

EXERCISE. Since calculations with $SU(2)$ or $SO(3,R)$ are rather involved, try working with the group of nonsingular real upper triangular 2x2 matrices: $G = \{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid ad \neq 0 \}$.

matrix Lie algebra basis:

$$\{\underline{E}_1, \underline{E}_2, \underline{E}_3\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \quad \text{or} \quad \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

$$\begin{aligned} [E_2, E_3] &= -E_3 \\ [E_3, E_1] &= -E_3 \\ [E_1, E_2] &= 0 \end{aligned} \qquad\qquad \begin{aligned} [E_2, E_3] &= 2E_3 \\ [E_3, E_1] &= 0 \\ [E_1, E_2] &= 0 \end{aligned}$$

This turns out to be Bianchi type III in the classification of 3-dimensional groups. Assume canonical coordinates of the second kind with respect to either basis and evaluate $\underline{S}$, $\underline{S}^{-1}$, $\underline{R}$ and all the invariant fields.

one last thing about SU(2) and SO(3,R)

S^3 has a natural geometry as a sphere in R^4. What does this have to do with its group structure as the manifold of SU(2)?
Recall the Cartan-Killing inner product we already evaluated for the basis $\{E_a\}$ with structure constants $C^a{}_{bc} = \epsilon_{abc}$:

$$\gamma_{ab} = C^f{}_{ag}\, C^g{}_{bf} = -2\,\delta_{ab}.$$

Suppose we introduce a left invariant metric on SU(2) corresponding to this innerproduct on the Lie algebra, scaled by a suitable constant:

$$g = g_{ab}\,\omega^a \otimes \omega^b \qquad g_{ab} = -\tfrac{1}{8}\gamma_{ab} = \tfrac{1}{4}\delta_{ab}.$$

Then since $\tilde{\omega}^a = R^a{}_b\,\omega^b$ with $\underline{R}$ orthogonal $(\delta_{fg} R^f{}_a R^g{}_b = \delta_{ab})$
one also has $g = g_{ab}\,\tilde{\omega}^a \otimes \tilde{\omega}^b$,
i.e. this metric is also right invariant, therefore bi-invariant.
For SU(2) there is no overlap between left and right translations so g is invariant under a $3+3=6$ dimensional isometry group (3 left, 3 right), which means it has maximal symmetry and is a space of constant curvature.
In fact with this choice of rescaling, it is exactly the metric of S^3 (unit radius $\rightarrow$ constant Gaussian curvature $k=1$)

This generalizes to all "semi-simple" groups — those for which the Cartan-Killing inner product is nondegenerate (nonzero determinant). As mentioned before, the adjoint matrix group leaves invariant the structure constant tensor $C^a{}_{bc}$ and all tensors derived from it like γ_{ab}, so any multiple of the Cartan-Killing metric on the group will be bi-invariant exactly as in the case of SU(2). For SU(2) and SO(3,R), this is associated with the "total angular momentum", say $J^2 = \delta^{ab} J_a J_b$, but enough for now. Note that this is the Euclidean metric on the Lie algebra (with respect to cartesian coordinates defined by the standard basis). The exponential map makes them coordinates of the first kind on the group & is a conformal transformation which compactifies the ~~radial~~ coordinate θ on the Lie algebra to a circle on the Lie group, leaving the 2-sphere geometry invariant.

Adjoint Matrix group

For matrix groups G with matrix Lie algebra $\hat{g}$ and matrix basis $\{\underline{E}_a\}$:

$$[\underline{E}_a, \underline{E}_b] = C^c{}_{ab}\underline{E}_c \quad \rightarrow \quad [e_a, e_b] = C^c{}_{ab} e_c \quad \text{for Lie algebra } g,$$

we introduced the adjoint matrix $\underline{R}$ by

$$\underline{S}\,\underline{E}_a\,\underline{S}^{-1} = \underline{E}_b \underline{R}^b{}_a \,. \quad \rightarrow \quad \tilde{\omega}^a = R^a{}_b \omega^b \,, \quad \tilde{e}_a = e_b R^{-1b}{}_a \,.$$

This matrix measures how left and right differ on G. In fact one can define $\underline{R}$ on any Lie group as the matrix function which relates the left and right invariant frames generated by a given basis of the tangent space at the identity. For $SO(3,\mathbb{R})$, $\underline{R} = \underline{S}$ so the group coincides with its adjoint group, but for an abelian group there is no distinction between left and right and one must have $\underline{R} = \underline{1}$.

EXERCISE: $\hat{g} = \{\text{diagonal } 3\times3 \text{ matrices}\}$, $G = \{\text{nonsingular diagonal } 3\times3 \text{ matrices}\}$.
$\{\underline{E}_a\} = \{\mathrm{diag}(1,0,0), \mathrm{diag}(0,1,0), \mathrm{diag}(0,0,1)\}$

$$e^{x^1\underline{E}_1} e^{x^2\underline{E}_2} e^{x^3\underline{E}_3} = e^{x^a\underline{E}_a} = \mathrm{diag}(e^{x_1}, e^{x_2}, e^{x_3})$$ gives canonical parametrization of component of G connected to the identity.

Compute: $\omega^a = \tilde{\omega}^a = dx^a$, $\quad e_a = \tilde{e}_a = \partial/\partial x^a$. Conclude $\underline{R} = \underline{1}$.

For any matrix group, if one takes a canonical parametrization of the first kind, one can easily derive

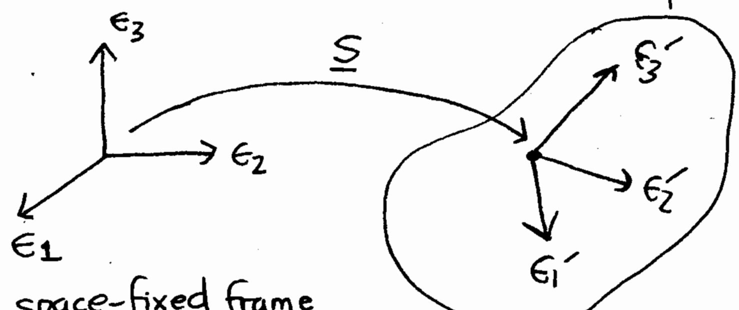
$$\underline{S} = e^{x^a \underline{E}_a} \rightarrow \underline{R} = e^{x^a \underline{k}_a}$$

where the adjoint matrices $(\underline{k}_a)^b{}_c = C^b{}_{ac}$ are defined in terms of the structure constants. The Jacobi identity may be rewritten in the form

$$[\underline{k}_a, \underline{k}_b] = C^c{}_{ab}\underline{k}_c \qquad \left[\text{This is the linearization of } R^a{}_d C^d{}_{fg} R^{-1f}{}_b R^{-1g}{}_c = C^a{}_{bc}\right]$$

which means that when $\{\underline{k}_a\}$ are linearly independent (as for $su(2)$ and $so(3,\mathbb{R})$) the generate an isomorphic matrix Lie algebra and the adjoint matrix group (which is generated by this Lie algebra) is locally isomorphic to the original group. Otherwise a lower dimensional group is obtained and there is some overlap between left and right, the extreme case being an abelian group where the adjoint matrix group is trivial.

RIGID BODY DYNAMICS (Goldstein passive point of view)



space-fixed frame coordinates X^a (left)

body-fixed frame coordinates X'^a (right)

Two (orthonormal) cartesian coordinate systems are used in this problem, related by a time dependent rotation matrix which is the dynamical variable.

passive $\begin{cases} \text{frame transformation:} & \epsilon'_a = \epsilon_b S^{-1b}{}_a \\ \text{coordinate transformation:} & X'^a = S^a{}_b X^b \end{cases}$

A point fixed in the body (X'^a = constant) undergoes the active rotation $X^a(t) = S^{-1a}{}_b(t) X'^b$, or $\underline{X}(t) = \underline{S}^{-1}(t)\underline{X}'$,

so $\frac{d}{dt}\underline{X}(t) = \frac{d}{dt}(\underline{S}^{-1}(t))\ \underline{X}'$. How do we evaluate this derivative?

EXERCISE: Use the formula $\underline{A}^{-1}\underline{A} = 1$ to derive the formula

$$\frac{d}{dt}\underline{A}^{-1} = -\underline{A}^{-1}\frac{d\underline{A}}{dt}\underline{A}^{-1} \quad \text{or} \quad d\underline{A}^{-1} = -\underline{A}^{-1}d\underline{A}\,\underline{A}^{-1}$$

$\underline{S}(t)$ is a trajectory in the configuration space $SO(3,R)$, with matrix tangent $\frac{d}{dt}\underline{S}(t) = \frac{\partial \underline{S}(t)}{\partial \theta^a}\dot{\theta}^a(t)$, if θ^a are local coordinates parametrizing $\underline{S}$. This corresponds to the tangent vector $V(t) = \dot{\theta}^a(t)\frac{\partial}{\partial\theta^a}\Big|_{\theta(t)}$ which represents the velocity of the system. Classically, θ^a are the position variables and $\dot{\theta}^a$ the velocities. We can introduce generalized velocities corresponding to taking components in the invariant frames on $SO(3,R)$ which correspond to the standard basis $\{\underline{J}_a\}$ of the matrix Lie algebra

$$\text{left:} \quad \dot{\omega}^a(t) = \omega^a{}_b(\theta(t))\,\dot{\theta}^b(t)$$
$$\text{right:} \quad \dot{\tilde{\omega}}^a(t) = \tilde{\omega}^a{}_b(\theta(t))\,\dot{\theta}^b(t)$$

We evaluated this in Euler angle coordinates above.

Recall that $\tilde{\omega}^a = R^a{}_b \omega^b$ so $\dot{\tilde{\omega}}^a = R^a{}_b \dot{\omega}^b$, but $\underline{R} = \underline{S}$ for this case so $\dot{\tilde{\omega}}^a = S^a{}_b \dot{\omega}^b$.

Now return to the time derivative computation:

$$\frac{d}{dt}\underline{x} = \left(\frac{d}{dt}\underline{S}^{-1}\right)\underline{x}' = -\underline{S}^{-1}\frac{d\underline{S}}{dt}\underbrace{\underline{S}^{-1}\underline{x}'}_{\underline{x}} = -\underline{S}^{-1}\frac{d\underline{S}}{dt}\underline{x} = -\dot{\omega}^a \underline{E}_c \underline{x}$$

$$\frac{d}{dt}x^a(t) = -(E_c)^a{}_b\, x^b(t)\dot{\omega}^c(t) = -\epsilon_{acb}\,\dot{\omega}^c(t) x^b(t) \quad \text{or} \quad \boxed{\frac{d}{dt}\underline{x} = [-\underline{\dot{\omega}}] \times \underline{x}}$$

Thus $-\dot{\omega}^a$ are the space-fixed components of the angular velocity of the body from our classical knowledge of rotation, and hence $-\dot{\tilde{\omega}}^a$ are the body-fixed components of the angular velocity of the body since they are correctly transformed by $\underline{S}$. One can now verify that the expressions we obtained for $-\dot{\omega}^a$ and $-\dot{\tilde{\omega}}^a$ in Euler angle coordinates are the expressions quoted by GOLDSTEIN in the text and in a problem at the end of the chapter on rigid body dynamics.

Notice that left translations on $SO(3,\mathbb{R})$ $\quad \underline{S} \to \underline{S}_1\underline{S}$

lead to $\quad x'^a \to (\underline{S}_1\underline{S})^a{}_b x^b = S_1{}^a{}_b x'^b$,

which correspond to fixed rotations of the body-fixed axes, while

right translations $\quad S \to SS_1$

lead to $\quad x^a \to (SS_1)^{-1a}{}_b x'^b = S_1^{-1a}{}_c \underbrace{S^c{}_b x'^b}_{x^c} = S_1^{-1a}{}_c x^c$

which correspond to fixed rotations of the space-fixed axes.

If the system is free, i.e., there are no applied forces, then a fixed rotation of the space-fixed axes is a symmetry of the system, but such a rotation of the body-fixed axes is equivalent to a time dependent rotation of the space-fixed axes

$$\left[\, S_1 S(t) = S(t) S_2(t), \text{ where } S_2(t) = S^{-1}(t) S_1 S(t) \,\right]$$

which is not a symmetry. Thus right translations on $SO(3\,\mathbb{R})$ are symmetries of the dynamics.

Suppose the body-fixed axes are chosen to be the principal axes of the body so that the component matrix of the moment of inertia tensor is diagonalized $(I_{ab}) = \mathrm{diag}(I_1, I_2, I_3)$. The kinetic energy function is

$T = \frac{1}{2} I_{ab} \dot{\tilde{\omega}}^a \dot{\tilde{\omega}}^b$, since $\dot{\tilde{\omega}}^a$ are the body-fixed components of the angular velocity. But this is just the kinetic energy function associated with the right invariant metric $\mathcal{L} = I_{ab} \tilde{\omega}^a \otimes \tilde{\omega}^b$ on $SO(3,R)$ — namely half the square of the velocity vector of the system (tangent vector of the trajectory in configuration space), and which alone describes geodesics of the metric $\mathcal{L}$.

Introduce the generalized momenta (denoted by $\mathcal{P}_a$ and $\tilde{\mathcal{P}}_a$ above) conjugate to $\dot{\omega}^a$ and $\dot{\tilde{\omega}}^a$: $\quad \tilde{L}_a = I_{ab} \dot{\tilde{\omega}}^b$, $\quad L_a = \tilde{L}_b \underbrace{S^b{}_a} = e^i{}_a p_i$
(index lowering with respect to the metric $\mathcal{L}$). $\qquad$ (since $\underline{R} = \underline{S}$)
Their Poisson brackets were already evaluated above:

$$\{\tilde{L}_a, \tilde{L}_b\}_p = -\epsilon_{abc} \tilde{L}_c \qquad \{L_a, L_b\}_p = \epsilon_{abc} L_c \qquad \{L_a, \tilde{L}_b\}_p = 0$$

Re-express the kinetic energy in terms of the momenta:

$$T = \tfrac{1}{2} I^{ab} \tilde{L}_a \tilde{L}_b, \qquad \text{where } (I^{ab}) = \mathrm{diag}(I_1^{-1}, I_2^{-1}, I_3^{-1}).$$

The free rigid body has zero potential and the solutions of the equations of motion are geodesics of the right invariant metric $\mathcal{L}$. When the diagonal values of (I_{ab}) are distinct, this metric has a Killing vector field basis $\{e_a\}$, generators of right translations. The inner products of these vectors with the velocity vector are constants of the motion (Part II); these are equivalent to the contraction of the momentum with those vectors, i.e, just the left (space-fixed) components of the angular momentum L_a. Note that these obviously commute (Poisson bracketwise) with the kinetic energy (since $\{L_a, \tilde{L}_b\}_p = 0$) so are conserved from the Hamiltonian point of view.

The body-fixed momenta $\tilde{L}_a$ are not individually conserved but the sum of their squares is: $\tilde{L}^2 = \delta^{ab} \tilde{L}_a \tilde{L}_b = \delta^{ab} L_a L_b = L^2$
since it equals the same expression for the space-fixed axes. This expression represents the squared length of the momentum of the system with respect to the Cartan-Killing metric on $SO(3,R)$.

The case in which exactly two diagonal values of the moment of inertia tensor coincide is referred to as a symmetric top (as opposed to an asymmetric top when all values are distinct). Suppose $I_1 = I_2 \neq I_3$. In this case the metric $\mathcal{I}$ acquires an additional linearly independent Killing vector field $\tilde{e}_3$ which generates rotations of $(\tilde{\omega}^1, \tilde{\omega}^2)$ and the corresponding momentum $\tilde{L}_3 = \tilde{e}^i{}_3 p_i = \mathcal{I}(\tilde{e}_3, v)$ is conserved either because 1) it represents the inner product of the Killing vector field with the tangent to the geodesic solution curve (with respect to the right invariant metric $\mathcal{I}$) or 2) since $\{\tilde{L}_3, T\}_p = 0$ as one easily computes.

The degenerate case $I_1 = I_2 = I_3$ describes a spherical top which isn't very interesting – the geometry is the natural spherical geometry on $SO(3,\mathbb{R})$ locally corresponding to S^3 ~~whose~~ geodesics are great circles. The symmetry group becomes 6-dimensional ($\mathcal{I}$ is bi-invariant) and the geometry has constant curvature.

Abraham & Marsden *Foundations of Mechanics* has a (very) mathematically sophistocated discussion of the rigid body but using the active point of view which identifies their matrix $\underline{A}(t)$ with our $S^{-1}(t)$ and has the effect of interchanging left and right.

Goldstein Euler angles active interpretation:

Suppose we take the same diagram in Goldstein but interpret the rotation actively, namely an active rotation of a point fixed in space with coordinates x^i to a new position with coordinates $x^{i'}$ with respect to the original axes. [In the passive point of view x'^i and x^i are different coordinates of the same physical point.]

$$x'^a = \mathcal{S}^a{}_b x^b \qquad \underline{\mathcal{S}} = S^{-1} = e^{\theta^3 E_3} e^{\theta^1 E_1} e^{\theta^2 E_3} \qquad (\theta^i) = (\theta, \psi, \phi)$$

First you rotate by ψ about 3rd axis, then by θ about 1st axis tilting the plane of the ~~final~~ rotation by ψ, and then by ϕ about the 3rd axis which rotates the line of nodes. [In quantum mechanics the 2nd rotation is instead about the 2nd axis with the result that (θ, ϕ) are the usual spherical coordinates of the image under this active rotation of the 3rd axis.]

Then a point initially at coordinates x^a will have coordinates

$x'^a(t) = \mathcal{S}^a{}_b(t)\, x^b$ with respect to the space fixed axes.

Then $\dot{x}' = \dot{\mathcal{S}} x = \underline{\dot{\mathcal{S}}\mathcal{S}^{-1}} x' \rightarrow \dot{x}'^a = -E^a{}_{cb}\,\omega^c x'^b = \epsilon_{abc}\, \omega^c x'^b$
$= \epsilon_{acb}[-\dot{\omega}^c]\, x'^b$

$-\dot{S}S = -E_a \dot{\omega}^a$

or $\rightarrow E_c\, \tilde{\dot{\omega}}^c_{active}\, x' \rightarrow E^a{}_{cb}\, \tilde{\dot{\omega}}^c_{active}\, x'^b = \epsilon_{acb}\, \tilde{\dot{\omega}}^c_{active}\, x'^b$

i.e. If instead we interpret $\underline{\mathcal{S}}$ as specifying the ~~state~~ configuration of the system, we obtain the expressions $\tilde{\dot{\omega}}^a_{active} = -\dot{\omega}^a$ for the spacefixed components of the angular velocity of the body

Group inversion on any group $(a \rightarrow a^{-1})$ interchanges left and right and multiplies all the fields by a minus sign. In the problem of the rigid body it is in fact the active interpretation which is more useful, and in new editions of Goldstein at a certain point he switches from ~~passive~~ to ~~passive~~ active but without explicit formulas involving Euler angles. The identification of the configuration space through $\underline{\mathcal{S}}$ instead of $\underline{S}$ leads the association of body fixed components with the left invariant frame and spacefixed components with the right invariant frame, as in Abraham and Marsden.* I should probably rewrite all of this in a clearer fashion. Next time.

* Foundations of Mechanics

ERROR CORRECTION: on pages MG5, MG6 precede all invariant fields by a minus sign and $-\tilde{e}_2 = -s_2\partial_1 + c_2[s_1^{-1}\partial_3 - \cot_1 \partial_2]$ change indicated sign on MG5.
On page RB2, equations $\frac{d}{dt}x^a(t) = -(\ \ ) = -(\ \ ) \Rightarrow \frac{d}{dt}\underline{x} = [-\underline{\omega}]\times \underline{x}$
so $-\dot{\omega}^a$ and $-\tilde{\dot{\omega}}^a$ are the components of the angular velocity