Introduction to Cosmological Models
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INTRODUCTION TO COSMOLOGICAL MODELS : PART II

Generating vector fields, Killing vector fields, Lie algebras, matrix groups

Lie group actions (transformation groups), orbits, quotient spaces, homogeneous spaces

Lie Derivatives, invariance

Part II is designed to cover the topic of symmetry groups, in particular, isometry groups (groups of motions of metrics). Metrics with transitive isometry groups (all points equivalent under the action of the group) are studied as homogeneous spaces useful as spatial sections of a spacetime or as fiber spaces in higher dimensional spacetimes or simply in ordinary gauge theories, both of which are important in current unification schemes. These applications will be covered in part III.

UPGRADING YOUR CLASSICAL KNOWLEDGE OF TENSOR CALCULUS

IDENTIFY:

contravariant vector field $X^\mu(x)$ with its directional derivative $X = X^\mu \frac{\partial}{\partial x^\mu}$

covariant vector field $\sigma_\mu(x)$ with the (in general) inexact differential $\sigma = \sigma_\mu dx^\mu$

In each case under a change of coordinates, the (contra/co)-variant vector indices transform inversely to the indices on the (derivatives/differentials) against which they are summed, and one obtains a "coordinate invariant object" called respectively a "vector field" and a "1-form" or "differential form of degree 1". [Remember, an "exact differential" of a function f is defined by $df = \frac{\partial f}{\partial x^\mu} dx^\mu$.]

The value of a vector field at a point $X(x)$ is called a "tangent vector" and is connected to the tangent of a curve through its action on functions. If $x^\mu = c^\mu(t)$ is a parametrized curve in local coordinates, then $\dot{x}^\mu = \frac{dc^\mu(t)}{dt}$ are the classical tangent vector components and $c'(t) = \dot{x}^\mu \frac{\partial}{\partial x^\mu}$ is the modern tangent vector. If f is any function:

$$c'(t) f = \frac{dc^\mu(t)}{dt} \frac{\partial f}{\partial x^\mu}(c(t)) = \frac{d}{dt} f(c(t))$$

the derivative of f by the tangent vector is just the derivative of f along the parametrized curve by the classical chain rule.

The connection between vector fields and 1-forms comes from the evaluation operation. One can "evaluate" a 1-form on a vector field obtaining a function by defining

$$\sigma(X) = \sigma_\mu X^\mu \quad \text{or equivalently setting} \quad dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu{}_\nu$$

and extending by linearity to arbitrary fields:

$$\underset{\uparrow \text{vector argument}}{\sigma(X)} = \sigma_\mu dx^\mu\left(X^\nu \frac{\partial}{\partial x^\nu}\right) = \sigma_\mu X^\nu dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \sigma_\mu X^\nu \delta^\mu{}_\nu = \sigma_\mu X^\mu.$$

In other words a 1-form is a linear function on vector fields.

UPGRADE PAGE TWO

Looking at things at a point x, we are identifying 1-forms at x with the elements of the "dual space" to the linear tangent space. Given any vector space V, with a basis $\{e_\alpha\}$ such that any vector can be expressed as $X = X^\alpha e_\alpha$, one can introduce the dual space V^* of real valued linear functions on V

$$X \mapsto \sigma(X) = \sigma(X^\alpha e_\alpha) = X^\alpha \sigma(e_\alpha) \equiv X^\alpha \sigma_\alpha .$$

A basis $\{\omega^\alpha\}$ of V^* "dual to" $\{e_\alpha\}$ satisfies by definition $\omega^\alpha(e_\beta) = \delta^\alpha{}_\beta$ so that $\omega^\alpha(X) = \omega^\alpha(X^\beta e_\beta) = X^\beta \delta^\alpha{}_\beta = X^\alpha$,

hence from above $\sigma(X) = X^\alpha \sigma_\alpha = \omega^\alpha(X)\sigma_\alpha = (\sigma_\alpha \omega^\alpha)(X)$,

ie $\boxed{\sigma = \sigma_\alpha \omega^\alpha \leftrightarrow \sigma_\alpha = \sigma(e_\alpha)}\boxed{X = X^\alpha e_\alpha \leftrightarrow X^\alpha = \omega^\alpha(X)}$

The coordinate derivatives $\partial_\mu = \partial/\partial x^\mu$ provide a basis of each tangent space, and the coordinate differentials provide the dual basis since $dx^\mu(\partial/\partial x^\nu) \equiv \delta^\mu{}_\nu$. One may also take a noncoordinate basis $e_\alpha = e^\mu{}_\alpha \partial_\mu$ of linearly independent vector fields (at each point x), with its dual basis $\omega^\alpha = \omega^\alpha{}_\mu dx^\mu$ of inexact differentials. ($\omega^\alpha(e_\beta) = \omega^\alpha{}_\mu e^\mu{}_\beta = \delta^\alpha{}_\beta$ makes the coordinate component matrices inverses.) $\{e_\alpha\}$ is called a "frame" and $\{\omega^\alpha\}$ the "dual frame."

The connection between the differentials & directional derivatives is given by

$$df(X) = Xf$$

value of exact differential on vector field = directional derivative of function

The tensor product operation, for example, $(X^\mu, \sigma_\mu) \mapsto S^\mu{}_\nu = X^\mu \sigma_\nu$ is represented in invariant notation by $\otimes$, namely

$$(X,\sigma) \mapsto X\otimes\sigma = (X^\mu \partial_\mu) \otimes (\sigma_\nu dx^\nu) = \underbrace{X^\mu \sigma_\nu}_{\text{components of (1,1) tensor field.}} \partial_\mu \otimes dx^\nu$$

$$= (X^\alpha \sigma_\beta)\, e_\alpha \otimes \omega^\beta \quad \text{in a noncoordinate frame.}$$

ETC.

KILLING VECTOR FIELDS (in general: GENERATING VECTOR FIELDS)

Until now we have only talked about groups of motions of various spaces. However, even more useful than the groups themselves are the Lie algebras of Killing vector fields ("infinitesimal motions" = generators of rotations and translations) which are said to generate these groups of motions. ~~They~~ They are important in yielding conserved quantities both in classical and quantum mechanics, for helping to solve differential equations by eigenfunction expansion, and for quantization, to name a few applications.

Suppose $x^\mu \to \bar{x}^\mu = f^\mu(x,a)$ is a <u>group of transformations</u> on a space M, where $\{a^a\}_{a=1,\dots,r}$ are r-independent ~~parameters~~ variables which parametrize the group.
Let a_0 correspond to the identity transformation: $f^\mu(x,a_0) = x^\mu$.

Consider an "<u>infinitesimal transformation</u>" corresponding to $a_0 + \delta a$, i.e. one which is very close to the identity transformation:

$$x^\mu \to \bar{x}^\mu = f^\mu(x, a_0 + \delta a) \approx \underbrace{f^\mu(x,a_0)}_{= x^\mu} + \underbrace{\left(\frac{\partial f^\mu}{\partial a^a}\right)(x,a_0)}_{\equiv\, \xi^\mu_a(x)} \delta a^a$$

i.e $x^\mu \to \bar{x}^\mu \approx x^\mu + \xi^\mu_a(x)\delta a^a = (1 + \delta a^a \xi_a)\, x^\mu$

where r first order differential operators $\{\xi_a\}$ are defined by $\xi_a = \xi^\mu_a(x)\dfrac{\partial}{\partial x^\mu}$ corresponding to the r vector fields with components $\{\xi^\mu_a\}$.
They are called the "<u>generating vector fields</u>" and when the group is a group of motions they are called "<u>Killing vector fields</u>".

WE AGREE TO IDENTIFY VECTOR FIELDS WITH THEIR CORRESPONDING FIRST ORDER DIFFERENTIAL OPERATORS.
The generating vector fields have the interpretation that under the infinitesimal transformation corresponding to $a_0 + \delta a$, all points of the space M move a distance $\delta x^\mu = \delta a^a \xi^\mu_a$ in the coordinates $\{x^\mu\}$.

Note: $\bar{x}^\mu \approx (1 + \delta a^a \xi_a)\, x^\mu \approx e^{\delta a^a \xi_a} x^\mu$

In fact the finite transformations of the group may be represented in the form:

$$\bar{x}^\mu \to \bar{x}^\mu = e^{\theta^a \xi_a} x^\mu \quad \text{, where } \{\theta^a\} \text{ are } r \text{ new parameters on the group.}$$

[This follows from a Taylor expansion of $f(x,a)$ about the identity a_0 using the fact that the transformations form a group.]

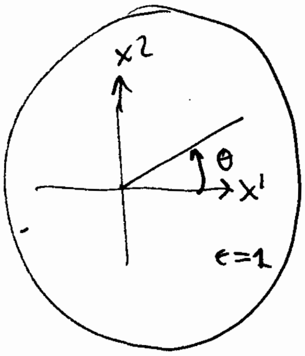
Simple example of transformation groups

rotations about origin of R^2, Euclidean plane, orthonormal cartesian coordinates (x^1, x^2).

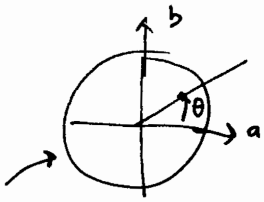
$$\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \to \underbrace{\begin{bmatrix} \epsilon a & -b \\ \epsilon b & a \end{bmatrix}}_{} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \cancel{f(x,)} \begin{bmatrix} \epsilon a x^1 - b x^2 \\ \epsilon b x^1 + a x^2 \end{bmatrix} := f(x, \theta)$$

$$O: \quad O^T 1\, O = O^T O = 1 \to O^T = O^{-1}$$

solution: $\begin{bmatrix} \epsilon a & \epsilon b \\ -b & a \end{bmatrix} \begin{bmatrix} \epsilon a & -b \\ \epsilon b & a \end{bmatrix} = \underset{1}{\overset{\parallel}{\epsilon^2}} \underbrace{(a^2+b^2)}_{1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$



M



G

S^1: group manifold.

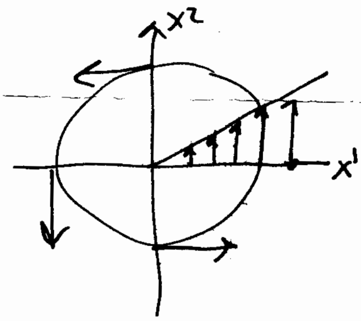
local coordinate: θ

$a = \cos\theta$
$b = \sin\theta$

$$\det O = \epsilon(a^2+b^2) = \epsilon < \begin{matrix} +1 \text{ (pure rotation)} \\ -1 \text{ (+ reflection)} \end{matrix}$$

Identity: $\epsilon = 1$, $\theta = 0 \to a = 1, b = 0$

$$\begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} = \frac{\partial f}{\partial \theta}(x, 0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$



$$\xi = \xi^1 \partial_1 + \xi_2 \partial_2 = -x^2 \partial_1 + x^1 \partial_2$$

$$\|\xi\|^2 = x^{12} + x^{22} = r^2$$

$$\xi \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{J} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

$$\begin{pmatrix} \bar{x}^1 \\ \bar{x}^2 \end{pmatrix} = e^{\theta \xi} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \underbrace{e^{\theta J}}_{O(\theta)} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

$$\epsilon = 1: \quad O(\theta_1) O(\theta_2) = O(\theta_1 + \theta_2)$$

$$\text{"}\theta_1 \cdot \theta_2\text{"} = \theta_1 + \theta_2 = \varphi(\theta_1, \theta_2).$$

3-space rotations $\quad G = SO(3R) \quad (a^1, a^2, a^3) =$ Euler angles

$(x^1 x^2 x^3)$ cartesian $E3$

$$x^\mu \to R^\mu{}_\nu(a) x^\nu = f^\mu(x, a)$$

$\underset{a_3}{\overset{\downarrow}{R_3}} = \underset{a_1}{\overset{\downarrow}{R_1}} \underset{a_2}{\overset{\downarrow}{R_2}}$ matrix mult., group of matrices ind. of action on $E3$

$a_3 = \varphi(a_1, a_2)$ comp law for rotations.

$d = p+q = \dim M^{p,q}$

EX's

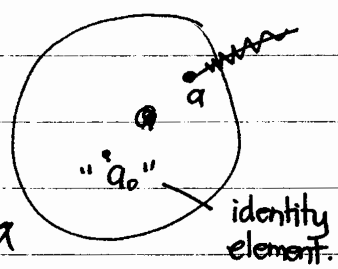
$M^{p,q}$: $\eta_{..} \sim \mathrm{diag}(\overbrace{1,\dots,1}^{p},\overbrace{-1,\dots,-1}^{q})$

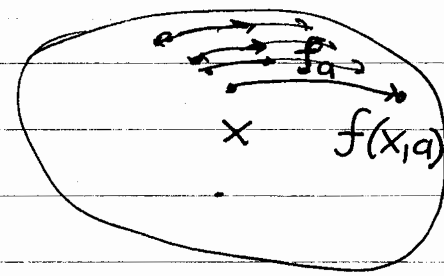
$O(p,q)$ $\quad O^{\gamma}{}_{\alpha}\,\eta_{\gamma\delta}\,O^{\delta}{}_{\beta} = \eta_{\alpha\beta}$ $\quad$ pseudo-orthogonal group.

$SO(p,q)$ $\quad$ $q=1 \rightarrow$ Lorentz group

$IO(p,q)$ inhomogeneous, add translations. $\quad$ $q=1$, Poincaré group.

G $\qquad\qquad\qquad\qquad$ M

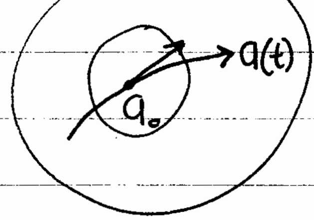




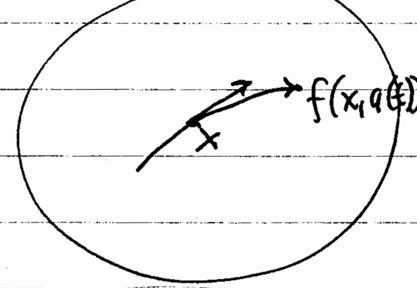
in local coords: $\quad x^{\mu} \longrightarrow \bar{x}^{\mu} = f^{\mu}(x,a)$ $\quad$ group of transformations

group parameters = local coords on group

$f(x, a_0) = x$ $\quad$ identity transformation.



"infinitesimal" transformations really means tangents to curves through identity.



group" $\quad x \rightarrow$ "$a \cdot x$" $= f(x,a)$

$a_2 \cdot (a_1 \cdot x) = (a_2 a_1) \cdot x$

$f_{a_2} \circ f_{a_1} = f_{a_2 a_1}$

$a_2 a_1 = \varphi(a_1, a_2)$

group
closure
identity
inverses
ass.

Example $\quad R^N$, translations: $x^\mu \to f^\mu(x,a) = x^\mu + a^\mu$

$$\xi^\mu{}_\nu = \left(\frac{\partial f^\mu}{\partial a^\nu}\right)(x,0) = \delta^\mu{}_\nu \qquad \xi_\nu = \delta^\mu{}_\nu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x^\nu}$$

The cartesian coordinate vector fields $\{\partial/\partial x^\mu\}$ generate the translations of R^N:

$$\bar{x}^\mu = e^{\theta^\mu \xi_\mu} x^\mu = \left(1 + \theta^\nu \frac{\partial}{\partial x^\nu} + \ldots\right) x^\mu = x^\mu + \theta^\mu + 0 .$$

FACT: The generating vector fields form a <u>Lie algebra</u>, i.e. a real r-dimensional vector space with basis $\{\xi_a\}$ which is closed under commutation, i.e. the commutators of the basis elements can be expressed as constant linear combinations of themselves:

$$[\xi_a, \xi_b] \equiv \xi_a\xi_b - \xi_b\xi_a = -C^c_{ab}\,\xi_c$$

$\uparrow$ <u>structure constants</u> of Lie algebra

Example

Let G be the group on which $\{a^a\}$ are coordinates ("the group manifold"). Then the multiplication function on the group tells us how to multiply two group elements: $\quad a_1 a_2 = \varphi(a_1, a_2)$.

G acts on itself by left multiplication as a transformation group:

$$a \mapsto L_{a_1}(a) = a_1 a = \varphi(a_1, a).$$

These are called left translations. Just as above we can introduce the vector fields which generate the left translations:

$$\tilde{e}_a = \left(\frac{\partial \varphi^b}{\partial a_1^a}\right)(a_0, a)\,\frac{\partial}{\partial a^b}$$

One finds the same commutation relations for these vector fields as for the $\{\xi_a\}$ introduced above: $\quad \boxed{[\tilde{e}_a, \tilde{e}_b] = -C^c_{ab}\,\tilde{e}_c}$

Similarly replacing left by right gives the right translations

$$a \mapsto R_{a_2}(a) = a a_2 = \varphi(a, a_2)$$

$$e_a = \left(\frac{\partial \varphi^b}{\partial a_2^a}\right)(a, a_0)\,\frac{\partial}{\partial a^b} \qquad \boxed{[e_a, e_b] = C^c_{ab}\,e_c}$$

Note the change of sign of the structure constants.

FACT. The vector fields $\{e_a\}$ turn out to be invariant under all left translations and are called left invariant. The Lie algebra $\mathfrak{g}$ spanned by $\{e_a\}$ is called <u>the Lie algebra</u> of the group G. Similarly, the Lie algebra spanned by $\{\tilde{e}_a\}$ is right invariant.

EXAMPLE For the translations on R^N, $\xi_\mu = \frac{\partial}{\partial x^\mu}$ so $[\xi_\mu, \xi_\nu] = 0$.
The structure constants vanish for an abelian group (commutative group).

EXAMPLE GROUPS OF LINEAR TRANSFORMATIONS

Consider the group $GL(n,\mathbb{R})$ of nonsingular linear transformations of $\mathbb{R}^N$:

$$x^\mu \to A^\mu{}_\nu x^\nu, \qquad A = (A^\mu{}_\nu) \in GL(N,\mathbb{R}) = \text{nonsingular } N\times N \text{ matrices}$$

The components $A^\mu{}_\nu$ are not independent since they satisfy the constraint $\det A \neq 0$.
In order to introduce generating vector fields we need to find independent variables which parametrize the group of transformations.

MATRIX EXPONENTIAL

Define $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ where $0! = 1$, $A^0 = 1$ (identity matrix) and $A \in gl(N,\mathbb{R})$ = space of $n\times n$ matrices.

Matrices of the form $e^{\theta A}$ satisfy

$$e^{\theta^1 A} e^{\theta^2 A} = e^{(\theta^1+\theta^2)A}, \qquad e^{0A} = 1,$$

and thus form a 1-dimensional abelian subgroup of $GL(N,\mathbb{R})$.
Another useful property is $\frac{d}{d\theta}(e^{\theta A}) = A e^{\theta A} = e^{\theta A} A$

$gl(N,\mathbb{R})$ is a Lie algebra since the commutator of any 2 matrices is again a matrix.

FACT The exponential of a matrix is nonsingular, ie points in $GL(N,\mathbb{R})$ can be (locally) represented in the form

$A = e^B$ where the components of B are independent.

FACT. Suppose $\hat{g}$ is a subspace of matrices which is closed under commutation. Then $\hat{g}$ is a Lie algebra and if $\{\hat{e}_a\}$ is a basis of $\hat{g}$, we have: $[\hat{e}_a, \hat{e}_b] = C^c_{ab}\hat{e}_c$

The set of matrices $e^{\theta^a \hat{e}_a}$ and products of such matrices form a subgroup of $GL(N,\mathbb{R})$. $\hat{g}$ is called the matrix Lie algebra of this subgroup G.

EXAMPLE The pseudo-orthogonal groups $SO(p,q)$ are subgroups of $GL(N,\mathbb{R})$. What is the matrix Lie algebra $so(p,q)$ of $SO(p,q)$?

If $L = e^{\theta A} \in SO(p,q)$ then

$$\eta = L^T \eta L = (e^{\theta A})^T \eta e^{\theta A}$$

$$\begin{cases} T = \text{matrix transpose} \\ \eta = ({}^{(p,q)}\eta_{\mu\nu}) \\ A = (A^\mu{}_\nu) \\ L = (L^\mu{}_\nu) \end{cases}$$

Taking $\frac{d}{d\theta}$ of this equation gives:

$$0 = (Ae^{\theta A})^T \eta e^{\theta A} + (e^{\theta A})^T \eta A e^{\theta A}$$

$$= (e^{\theta A})^T [A^T\eta + \eta A] e^{\theta A} \rightarrow \text{set } \theta = 0: \quad 0 = A^T\eta + \eta A$$

Expressing in components: $\quad 0 = \underbrace{A^\alpha{}_\mu \eta_{\alpha\nu}}_{\equiv A_{\nu\mu}} + \underbrace{\eta_{\mu\alpha} A^\alpha{}_\nu}_{\equiv A_{\mu\nu}} = A_{\nu\mu} + A_{\mu\nu} = 2A_{(\mu\nu)}$

symmetrization

Raise and lower indices with η:
$V^\mu \eta_{\mu\nu} \equiv V_\mu$, $V_\nu \eta^{\nu\mu} \equiv V^\mu$
where $(\eta^{\mu\nu}) = (\eta_{\mu\nu})^{-1} = (\eta_{\mu\nu})$

This says $so(p,q)$ consists of matrices $A = (A^\mu{}_\nu)$ whose fully covariant components $A_{\mu\nu}$ are antisymmetric:

$$A_{\mu\nu} = A_{[\mu\nu]} = \tfrac{1}{2}(A_{\mu\nu} - A_{\nu\mu}) = \tfrac{1}{2}\underbrace{(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu)}_{\equiv \delta^{\alpha\beta}_{\mu\nu}} A_{\alpha\beta}$$

antisymmetrization

The matrices $(J^{\alpha\beta})_{\mu\nu} = -\delta^{\alpha\beta}_{\mu\nu}$ with $\alpha < \beta$ are a basis of the space of all antisymmetric (covariant) matrices, hence raising the first index μ leads a basis of the Lie algebra $so(p,q)$:

$$(J^{\alpha\beta})^\mu{}_\nu = \eta^{\mu\delta}(J^{\alpha\beta})_{\delta\nu} = -\eta^{\mu\delta}(\delta^\alpha_\delta \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\delta) = -(\eta^{\mu\alpha}\delta^\beta_\nu - \eta^{\mu\beta}\delta^\alpha_\nu)$$

We could also use the matrices $J_{\alpha\beta}$ for $\alpha < \beta$:

$$(J_{\alpha\beta})^\mu{}_\nu = -(\delta^\mu_\alpha \eta_{\beta\nu} - \delta^\mu_\nu \eta_{\beta\nu})$$

and an arbitrary element of $so(p,q)$ can be written:

$$\theta = \sum_{\alpha<\beta} \theta^{\alpha\beta} J_{\alpha\beta} = \tfrac{1}{2}\theta^{\alpha\beta} J_{\alpha\beta}.$$

If (α,β) are both spacelike or both timelike indices, then $\eta_{\alpha\alpha} = \eta_{\beta\beta}$ and $J_{\alpha\beta}$ is still antisymmetric and generates a rotation in the $x^\alpha x^\beta$ plane.
If (α,β) consists of one spacelike and one timelike index, then $\eta_{\alpha\alpha} = -\eta_{\beta\beta}$ and $J_{\alpha\beta}$ is symmetric and generates a boost in the $x^\alpha x^\beta$ plane.

A general pseudo-orthogonal matrix (at least in a neighborhood of the identity matrix) can be parametrized by $L = e^{\frac{1}{2}\theta^{\alpha\beta} J_{\alpha\beta}}$ and hence a general inhomogeneous pseudo-orthogonal transformation of $M^{p,q}$ can be represented as:

$$x^\mu \to f^\mu(x;\theta,b) = L^\mu{}_\nu x^\nu + b^\mu = \left(e^{\frac{1}{2}\theta^{\alpha\beta}J_{\alpha\beta}}\right)^\mu{}_\nu x^\nu + b^\mu .$$

The Killing vector fields are now easily defined by identifying $\{a^a\}$ with $\{\theta^{\alpha\beta}\}_{\alpha<\beta} \cup \{b^\mu\}$

$$\mathcal{J}_{\alpha\beta} = \left(\frac{\partial f^\mu}{\partial \theta^{\alpha\beta}}\right)(x;0,0)\frac{\partial}{\partial x^\mu} = (J_{\alpha\beta})^\mu{}_\nu x^\nu \frac{\partial}{\partial x^\mu}$$

$$\mathcal{P}_\alpha = \left(\frac{\partial f^\mu}{\partial b^\alpha}\right)(x;0,0)\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x^\alpha}$$

The matrices $\{J_{\alpha\beta}\}$ span a Lie algebra so we can evaluate their commutation relations:

$$[J_{\alpha\beta}, J_{\gamma\delta}] = C^{\mu\nu}{}_{\alpha\beta\gamma\delta} J_{\mu\nu}$$

EXERCISE. Find the result: $C^{\mu\nu}{}_{\alpha\beta\gamma\delta} = -4\,\delta^{\mu}_{[\alpha}\eta_{\beta][\gamma}\delta^{\nu}_{\delta]}$

Then show the compact form holds: $[J^{\alpha\beta}, J_{\gamma\delta}] = -4\,\delta^{[\alpha}_{[\gamma} J^{\beta]}{}_{\delta]}$.

EXERCISE. Derive the identity

$$\left[A^\mu{}_\nu x^\nu \frac{\partial}{\partial x^\mu},\; B^\alpha{}_\beta x^\beta \frac{\partial}{\partial x^\alpha}\right] = -[A,B]^\mu{}_\nu x^\nu \frac{\partial}{\partial x^\mu}$$

It follows from this identity that:

A simple computation shows:

$$\boxed{\begin{aligned} [\mathcal{J}_{\alpha\beta}, \mathcal{J}_{\gamma\delta}] &= -C^{\mu\nu}{}_{\alpha\beta\gamma\delta}\mathcal{J}_{\mu\nu} \\ [\mathcal{J}^{\alpha\beta}, \mathcal{P}_\gamma] &= -(J^{\alpha\beta})^\delta{}_\gamma \mathcal{P}_\delta \end{aligned}}$$

For $ISO(3,\mathbb{R})$ acting on E^3, define $\mathcal{J}_i = \epsilon^{ijk}\mathcal{J}_{jk}$ and find

$$[\mathcal{J}_i, \mathcal{J}_j] = -\epsilon_{ijk}\mathcal{J}_k$$

$$[\mathcal{J}_i, \mathcal{P}_j] = -\epsilon_{ijk}\mathcal{P}_k .$$

This should be familiar.

$$\mathcal{J}_i = \epsilon_{ijk} x^j \frac{\partial}{\partial x^k}$$

Suppose $\{e_a\}$ is a basis of the Lie algebra $\mathfrak{g}$ of a group G
$$[e_a, e_b] = C^c{}_{ab} e_c .$$

Define $\quad \gamma_{ab} = C^c{}_{ad} C^d{}_{bc} = \gamma_{ba} \qquad$ (symmetric).

This provides a natural inner product on the Lie algebra:
$$\gamma_{ab} \equiv e_a \cdot e_b = \gamma(e_a, e_b) \quad , \quad \gamma(X^a e_a, Y^b e_b) = \gamma_{ab} X^a Y^b .$$

When $\det(\gamma_{ab}) \neq 0$ this inner product is nondegenerate.
The groups for which this is true are called <u>semisimple</u>.

Whether nondegenerate or not, γ is called the Cartan–Killing form on $\mathfrak{g}$.

<u>EXERCISE</u> $\quad SO(p,q)$ is a semisimple group.
Derive the result
$$\gamma_{\alpha\beta\gamma\delta} = C^{\mu\nu}{}_{\alpha\beta\,\rho\sigma} C^{\rho\sigma}{}_{\gamma\delta\,\mu\nu} = -2(N-2)\, \eta_{\alpha\mu}\, \eta_{\beta\nu}\, \delta^{\mu\nu}_{\gamma\delta}$$
or equivalently: $\quad \gamma^{\alpha\beta}{}_{\gamma\delta} = -2(N-2)\,\delta^{\alpha\beta}_{\gamma\delta}$.

This tells us that $\{ j_{\alpha\beta} \}_{\alpha<\beta} = \left\{ \dfrac{J_{\alpha\beta}}{\sqrt{2(N-2)}} \right\}_{\alpha<\beta}$ is an orthonormal basis of $so(p,q)$ and
$$j_{\alpha\beta} \cdot j_{\alpha\beta} = \begin{cases} -1 & \text{if } J_{\alpha\beta} \text{ generates a rotation } (\eta_{\alpha\alpha} = \eta_{\beta\beta}) \\ 1 & \text{if } J_{\alpha\beta} \text{ generates a boost } (\eta_{\alpha\alpha} = -\eta_{\beta\beta}) \end{cases}$$

For $SO(3,\mathbb{R})$ we get $\quad J_i \cdot J_j = -2\delta_{ij} = \gamma_{ij}$.
In fact γ is negative definite on all <u>compact</u> semisimple groups.
(so it is useful to change its sign).

The r vector fields $\{e_a\}$ on the group G may be used instead of the coordinate basis $\{ \frac{\partial}{\partial q^a} \}$ as a basis in which to express an arbitrary vector field on G. Such a basis is called a <u>frame</u>.

Since the $\{e_a\}$ are left invariant (invariant under all left transformations), a left invariant metric on the group can be obtained by specifying the inner products $g_{ab} = g(e_a, e_b)$ to be constants. Since the vector fields are unchanged by left translation, so are their lengths and relative angles, hence the geometry of the space is invariant.

By choosing $g_{ab} = \sigma \gamma_{ab}$ one obtains a natural left invariant metric on a semisimple group G (σ a nonzero constant, possibly <0) which turns out to be <u>bi-invariant</u>, i.e. also right invariant (and hence invariant under a $2r$-dimensional group of left and right translations). $\sigma = 1$ gives the Cartan–Killing metric on G, but it is convenient to choose $\sigma < 1$ on a compact group to make g positive definite.

THE ACTION OF A GROUP ON A SPACE

Suppose G is an abstract group (or even a matrix group) with a given multiplication law: $\quad a_1 a_2 = \varphi(a_1, a_2)$.

Next suppose this group acts on a manifold M as a group of transformations

$$x^\mu \to f^\mu(x, a) \equiv f_a^\mu(x)$$

In order for the transformations $\{f_a \mid a \in G\}$ to form a group, the product of two transformations must correspond to the product of the group elements:
$$x \to f_{a_1}(f_{a_2}(x)) = (f_{a_1} \circ f_{a_2})(x) = f_{a_1 a_2}(x)$$

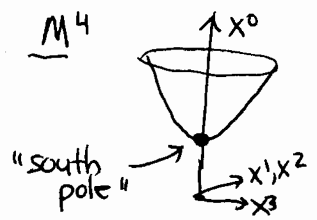
ie $\boxed{f_{a_1} \circ f_{a_2} = f_{a_1 a_2}}$ GROUP PROPERTY

Note: if $a_0 a = a a_0 = a$ (a_0 is the identity of G), then $f_{a_0} \circ f_a = f_{a_0 a} = f_a$ so f_{a_0} is the identity transformation.

Let $f_G(x) = \{ f_a(x) \mid a \in G \}$ = orbit of x = all points that can be reached from x under the group of transformations

Let $G_x = \{ a \in G \mid f_a(x) = x \}$ = isotropy group at x = subgroup of G which leaves x fixed

EXAMPLE The pseudospheres in $M^{p,q}$ are the orbits of the pseudo-orthogonal group $SO(p,q)$ acting on $M^{p,q}$. $SO(p,q)$ itself is the isotropy group at the origin of $M^{p,q}$ of the inhomogeneous group action $ISO(p,q)$.



The isotropy group at the "south pole" of H^3_- (future) is just $SO(3,R)$, the subgroup of the Lorentz group $SO(4,1)$ which leaves the x^0 axis fixed.

For the translations of R^N, the isotropy groups are all equal to $\{a_0\}$, ie. consist only of the identity transformation. R^N consists of a single orbit of the translation group.

COSETS Let $aG_x = \{ab \mid b \in G_x\}$ - set of all points in G obtainable from a under right translation by elements of G_x.

aG_x is called a <u>left coset</u> of the subgroup G_x

- Note that if $b \in G_x$ then $bG_x = G_x = a_0 G_x$; $a_0 G_x = G_x$ is called the identity coset
- Suppose $a_1 G_x = a_2 G_x$, then a_1 and a_2 lie in the same orbit under right translation by G_x and so differ from each other by right multiplication by some $b \in G_x$: $a_1 = a_2 b$
  $[\, a_2^{-1} a_1 G_x = G_x \to a_2^{-1} a_1 = b \in G_x \to a_1 = a_2 b \,]$

Let $G/G_x = \{aG_x \mid a \in G\}$ = <u>set of all left cosets</u> of G_x in G
= space of orbits of G under right translation by G_x

This space is useful for the following reason.

By definition, $b \in G_x \to f_b(x) = x$

(1) Then $f_{ab}(x) = f_a(f_b(x)) = f_a(x)$, i.e. all points in the coset aG_x move x to the same point $f_a(x)$.

(2) Conversely suppose $f_{a_1}(x) = f_{a_2}(x)$, i.e. both transformations move x to the same point. Then $f_{a_2^{-1}a_1}(x) = x \to a_2^{-1} a_1 \in G_x$
so $a_2^{-1} a_1 = b \to a_1 = a_2 b \to a_1 G_x = a_2 G_x$

In other words there is a 1-1 correspondence between the left cosets of G_x, namely the points of G/G_x , and the points of the orbit of x under the action of G. THIS IS IN FACT A DIFFEOMORPHISM :

$$\boxed{\varphi(aG_x) = f_a(x) \quad \text{defines a diffeomorphism } \varphi : G/G_x \to f_G(x)}$$

G acts on G/G_x by left multiplication as a transformation group :

$$aG_x \longrightarrow a_1(aG_x) = a_1 a G_x$$

This implies $\varphi(a_1 a G_x) = f_{a_1 a}(x) = f_{a_1}(f_a(x)) = f_{a_1}(\varphi(aG_x))$

$$\boxed{\text{In other words left multiplication by } G \text{ on } G/G_x \text{ corresponds to the action of } G \text{ on the orbit of } x.}$$

Note that if x_1, x_2 lie in the same orbit, then G_{x_1} and G_{x_2} are conjugate (therefore isomorphic) subgroups of G:

$$x_2 \in f_G(x_1) \rightarrow \quad x_2 = f_a(x_1) \text{ and } x_1 = f_{a^{-1}}(x_2).$$

$$\text{Now } b \in G_{x_1} \rightarrow \quad f_b(x_1) = x_1 \quad \text{or} \quad x_2 = f_a(x_1) = f_a(f_b(x_1))$$
$$= f_a(f_b(f_{a^{-1}}(x_2)))$$
$$= f_{aba^{-1}}(x_2)$$

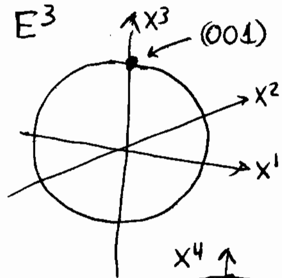
so $aba^{-1} \in G_{x_2}$, similarly if $b \in G_{x_2}$ then $a^{-1}ba \in G_{x_1}$,

i.e. $G_{x_2} = a\,G_{x_1} a^{-1}$!

So it is not important which point of the orbit we choose to make the correspondence between $f_G(x)$ and the coset space.

EXAMPLES

E^3

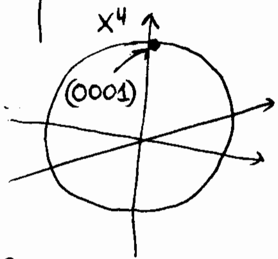


$$S^2 = f_{SO(3,R)}((0,0,1))$$

$$SO(3,R)_{(001)} = \text{rotations which leave } x^3\text{-axis fixed} = \text{rotations about } x^3$$
$$= SO(2,R)$$

$$S^2 \sim SO(3,R)/SO(2,R)$$

E^4

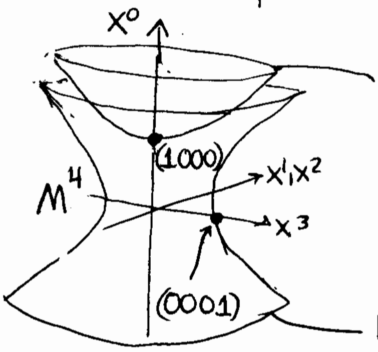


$$S^3 = f_{SO(4,R)}((0001))$$

$$SO(4,R)_{(0001)} = \text{rotations which leave } x^4\text{-axis fixed}$$
$$= \text{rotations of } (x^1x^2x^3) = SO(3,R)$$

$$S^3 \sim SO(4,R)/SO(3,R)$$

M^4



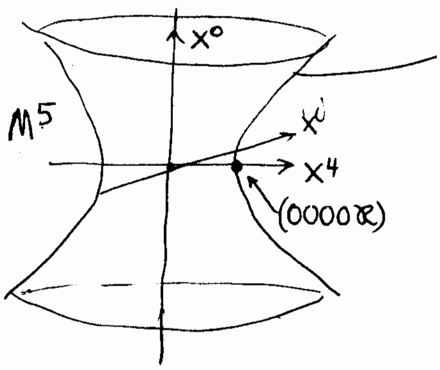
$$H^3_-(\text{future}) = f_{SO(3,1)}((1000))$$

$$SO(3,1)_{(1000)} = \text{Lorentz transformations which leave } x^0 \text{ axis fixed}$$
$$= \text{rotations of } (x^1x^2x^3) = SO(3,R)$$

$$H^3_-(\text{future}) \sim SO(3,1)/SO(3,R)$$

$$H^3_+ = f_{SO(3,1)}(0001), \ldots, \sim SO(3,1)/SO(2,1)$$

M^5



$$\text{DeSitter spacetime} = H^4_+ = f_{SO(4,1)}(0000R)$$

$$SO(4,1)_{(0000R)} = \text{pseudorotations which leave } x^4 \text{ fixed}$$
$$= \text{Lorentz transformations of } (x^0x^1x^2x^3) = SO(3,1)$$

$$H^4_+ \sim SO(4,1)/SO(3,1).$$

Similarly $E^3 \sim \underbrace{ISO(3,R)}_{\text{translations and rotations}} / \underbrace{SO(3,R)}_{\text{isotropy group at origin}}$

Suppose $\{y^i\}$ are coordinates on the orbit $f_G(x)$ of a point x of M and suppose $ds^2 = g_{ij}\,dy^i dy^j =$ "g" is a metric on the orbit.
Let $\varphi^* g$ be the metric on G/G_x which corresponds to g under the diffeomorphism $\ddot{\varphi}$.

$$\left(\begin{array}{l} \text{If } \{b^A\} \text{ are coords on } G/G_x \text{ then} \\ \varphi^* g = g_{ij}(\varphi(b))\, \dfrac{\partial\varphi^i}{\partial b^A}\dfrac{\partial\varphi^j}{\partial b^B}\, db^A db^B \\ \text{where } x^i = \varphi^i(b) \end{array}\right)$$

If g is invariant under the action of G on M, then $\varphi^* g$ must be invariant under the action of G on G/G_x by left multiplication, i.e. $\varphi^* g$ is a left invariant metric on G/G_x.

Thus all of the above examples yield left invariant metrics on the various coset spaces involved.

EXAMPLE. Suppose $G_x = \{a_0\}$ and $f_G(x) = M$, i.e. every transformation of G moves the point x and every point in M can be reached from x by a unique transformation. Since $G/G_x = \{a a_0 \mid a \in G\} = G$, G is diffeomorphic to M and one may identify the 2 spaces.
If g is a metric on M invariant under G, then it corresponds to a left invariant metric on G, which is specified entirely by the inner products of the basis left invariant vector fields e_a.

This generalizes the familiar case of the abelian group of translations of E^3 to arbitrary nonabelian groups in any dimension (or abelian groups); these all represent geometrically homogeneous spaces

$T_a(x) = x + a \quad , \quad x \in E^3$

Set $\varphi(a) = T_a(0) = a$ (identifies the point $x \in E^3$ with the translation by x)

Then $\varphi^*(\delta_{ij}\,dx^i dx^j) = \delta_{ij}\,da^i da^j$.

For 3-dimensions one obtains the family of spatially homogeneous space sections of the spatially homogeneous spacetimes.
Given a basis $\{e_a\}$ of the Lie algebra of a 3-dimensional Liegroup G, with structure constants $C^a{}_{bc}$, the spatial metric at each moment of time is specified by the spatially constant inner products

$$e_a \cdot e_b = g_{ab}(t). \qquad = 6 \text{ functions of time}$$

The Einstein equations become ordinary differential equations for these 6 functions, plus whatever functions of time are necessary to describe the matter of the universe.

For 4-dimensions one obtains the spacetime homogeneous spacetimes. Given a fixed set of structure constants $C^\alpha{}_{\beta\gamma}$ in a basis $\{e_\alpha\}$ of the 4-dimensional Lie algebra, the 10 constants

$$g_{\alpha\beta} = e_\alpha \cdot e_\beta$$

determine completely the Lorentz signature metric. Einstein's equations then become a set of algebraic equations for $g_{\alpha\beta}$ and $C^\alpha{}_{\beta\gamma}$ which may not necessarily have solutions for every group. (For nonvacuum spacetimes, other constants describing the matter enter the equations.)

For both the spacetime homogeneous and spatially homogeneous spacetimes, one need only consider a representative group from each equivalence class of isomorphic Lie groups of dimension 4 and 3 respectively. In 3 dimensions the classification of inequivalent 3-dimensional Lie groups is called the Bianchi classification and determines the various symmetry types possible for homogeneous 3-spaces just as $k = 1, 0, -1$ classifies the symmetry types possible for homogeneous and isotropic 3-spaces. In 4-dimensions the situation is more complicated: usually an orthonormal basis is assumed and the Einstein equations place constraints on the structure constants.

THE CARTAN-KILLING METRIC

Let us now return to the spaces and spacetimes (for simplicity: spaces) of constant curvature, realized as coset spaces of the full group of motions by an isotropy group, with left invariant metrics of the appropriate signature.

The Cartan-Killing form provides a natural 1-parameter family of nondegenerate inner products: $\{ g_{ab} = \sigma \gamma_{ab} \mid \sigma \in R - \{0\} \}$

$$\gamma_{ab} = C^d{}_{ac} C^c{}_{bd}$$

on the Lie algebras of all the semisimple groups, including all of the pseudo-orthogonal groups:

$$e_a \cdot e_b = g_{ab} = \sigma \gamma_{ab}, \qquad \{e_a\} \text{ a basis of the Lie algebra } \mathfrak{g} \text{ of Left invariant vector fields on } G$$

This specifies the inner products of a basis of left invariant vector fields on G at every point of G and hence determines a left invariant metric G, the <u>Cartan-Killing metric</u>, up to a scale factor.
This metric turns out to be <u>bi-invariant</u>, i.e. both left and right invariant.

Since it is right invariant, it is invariant under right translation by any subgroup H of G. Whenever a metric is invariant under a group of transformations, it projects to a metric on the quotient space of the original space by the group, namely on the space of orbits.
In this case the metric g projects to a metric on the space of left cosets of H in G.
Because g is also left invariant it remains invariant under left translation by G on G/H.
For a certain choice of σ, this is exactly the metric on all of the coset spaces corresponding to the pseudospheres discussed above.

THUS THE FULL GROUP OF MOTIONS OF THE SPACES OF CONSTANT NONZERO CURVATURE ITSELF CONTAINS ALL THE GEOMETRY OF THESE SPACES.

The Laplacian of this metric $\quad g^{ab} e_a e_b = \sigma^{-1} \gamma^{ab} e_a e_b$
(where $(g^{ab}) = (g_{ab})^{-1}$) is a second order differential operator which commutes with all of the left and right invariant vector fields (CASIMIR OPERATOR)

EXAMPLE. $\quad S^2 \sim SO(3,R)/SO(2,R)$
The Laplacian is $\quad \mathcal{J}^2 = \delta^{ij} \mathcal{J}_i \mathcal{J}_j$

The Lorentz group $SO(3,1)$ has a matrix Lie algebra which is a direct sum vector space of the 3-dimensional Lie subalgebra of rotation generators $\mathcal{r}$ and the 3-dimensional subspace of boost generators $\mathcal{b}$.

$$so(3,1) = \underbrace{\mathcal{r}}_{\text{Lie subalgebra}} \oplus \mathcal{b} \quad \nwarrow \text{orthogonal subspace}$$

This is an orthogonal direct sum with respect to the Cartan-Killing inner product since the natural basis of rotation and boost generators is orthogonal.
The commutation relations of these subspaces may be summarized as follows

$[\mathcal{r},\mathcal{r}] \subset \mathcal{r}$ (rotation generators rotate like a 3-vector) (rotations form a subgroup)
$[\mathcal{r},\mathcal{b}] \subset \mathcal{b}$ (boost generators rotate like a 3-vector)
$[\mathcal{b},\mathcal{b}] \subset \mathcal{r}$ (Wigner rotation – boosts don't form a group)

such a decomposition is called a <u>CARTAN decomposition</u> of the Lie algebra

At least near the identity one can write any Lorentz transformation as

$$L = \underbrace{e^{\theta^{i0}J_{i0}}}_{\text{pure boosts}} \underbrace{e^{\frac{1}{2}\theta^{ij}J_{ij}}}_{SO(3,R)} \qquad \begin{array}{l} \theta^{i0}J_{i0} \in \mathcal{b} \\ \frac{1}{2}\theta^{ij}J_{ij} \in \mathcal{r} \end{array}$$

For fixed θ^{i0}, varying θ^{ij} sweeps out the left coset $e^{\theta^{i0}J_{i0}}SO(3,R)$ of $SO(3,R)$ in $SO(3,1)$, so the variables θ^{i0} parametrize the left coset space $SO(3,1)/SO(3,R) \sim H^3_-$ (future).
Equivalently the submanifold of $SO(3,1)$ consisting of pure boosts, namely $\{e^{\theta^{i0}J_{i0}} \in SO(3,1) \mid \theta^{i0} \in R^4\}$
is in a 1-1 correspondence with this coset space.

In fact one can evaluate the exponentiation to find

$$e^{\theta^{i0}J_{i0}} = \left(\begin{array}{c|c} \cosh\theta & n^i \sinh\theta \\ \hline n^i\sinh\theta & \text{uninteresting JUNK} \end{array}\right) \qquad \text{where} \begin{cases} \theta^{i0} = \theta n^i \\ \delta_{ij}n^i n^j = 1 \\ \theta = (\delta_{ij}\theta^{i0}\theta^{j0})^{1/2} \end{cases}$$

Now define $X^0 = \cosh\theta$, $X^i = n^i \sinh\theta$
and find $\eta_{\mu\nu}X^\mu X^\nu = -1$, ie. this gives us an explicit map from $SO(3,1)/SO(3,R)$ to H^3_- (future).

This may be repeated for each of the constant curvature spaces discussed above, with similar results.

Unfortunately I have run out of time and have not been able to derive all of the facts presented in the second two lectures. However, I hope the sketch I have given makes you aware of the utility of Lie group theory in this subject, exploited <u>beyond</u> the minimum level required to work with the various spaces we have considered.

There is also no time to go over the physical side of this subject but I'm sure that will be adequately covered in due course. These mathematical ideas will not be so I wanted to expose them to you so you might go on to study them yourself.

The next step is of course <u>harmonic analysis</u>, well known in the simple examples of Fourier analysis on E^3 (the group of translations) or spherical harmonic expansion on $S^2 \sim SO(3,R)/SO(2)$, and used in cosmology to solve the perturbation equations (creation of inhomogeneous structure) or consider quantum fields or particle production on background spacetimes.

A useful book in this context is

A.O. Barut, R. Raczka *Theory of Group Representations and Applications*
Polish Scientific Publishers.

Good luck.

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SUPPLEMENT TO NOTES ON COSMOLOGICAL MODELS (by bob jantzen)

CURVATURE OF PSEUDOSPHERES

Given an arbitrary metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, or simply "$g_{\mu\nu}$", with inverse $g^{\mu\nu}$, one can raise and lower all indices in the standard fashion:

$$V_\mu = g_{\mu\nu} V^\nu \,, \quad V^\nu = g^{\nu\mu} V_\mu \,.$$

The Riemann curvature tensor of such a metric may be computed by the following algorithm: (1) first compute

$$\Gamma^\rho{}_{\mu\nu} = \tfrac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \equiv g^{\rho\sigma} \Gamma_{\sigma\mu\nu} = \Gamma^\rho{}_{\nu\mu}$$

[where $\Gamma_{\sigma\mu\nu} = \tfrac{1}{2} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$]

(2) Next compute $\quad R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\epsilon} \Gamma^\epsilon{}_{\nu\sigma}$.

This is the curvature tensor.

(3) Next raise the 2^{nd} index : $\quad R^{\rho\sigma}{}_{\mu\nu} = g^{\sigma\epsilon} R^\rho{}_{\epsilon\mu\nu}$.

That's a lot of work. For a space of constant curvature (a maximally symmetric space), one finds the simple result:

$$R^{\rho\sigma}{}_{\mu\nu} = \lambda \, \delta^{\rho\sigma}_{\mu\nu} = \lambda (\delta^\rho_\mu \delta^\sigma_\nu - \delta^\rho_\nu \delta^\sigma_\mu) ,$$

where λ is a constant, the "constant curvature." If we accept this formula, a trick can be used to evaluate λ for pseudospheres.

Generalizing the work of page 5 on M^4, consider pseudospherical coordinates on $M^{p,q}$ with flat metric ${}^{(p,q)}\eta_{\mu\nu}$. $\left(({}^{(p,q)}\eta_{\mu\nu}) = \mathrm{diag}(\underbrace{1,\ldots 1}_{p}, \underbrace{-1,\ldots,-1}_{q}) \right)$

The metric becomes:

$$ds^2 = {}^{(p,q)}\eta_{\mu\nu} dx^\mu dx^\nu = \underbrace{\epsilon}_{\epsilon = \pm 1} (dy^0)^2 + (y^0)^2 \underbrace{\gamma_{ij}(y^k) dy^i dy^k}_{\substack{\text{metric on} \\ \text{unit pseudosphere} \\ {}^{(p,q)}\eta_{\mu\nu} X^\mu X^\nu = \epsilon}} = \epsilon (dy^0)^2 + \underbrace{g_{ij} dy^i dy^k}_{\substack{\text{metric on} \\ \text{pseudosphere} \\ \text{of radius } y^0 \\ {}^{(p,q)}\eta_{\mu\nu} X^\mu X^\nu = \epsilon (y^0)^2}}$$

$$= g_{\mu\nu} dy^\mu dy^\nu \qquad (i,j,k = 1, \ldots, N-1 \ , \text{where } N = p+q.)$$

Thus in pseudospherical coordinates $\{y^\mu\} = \{y^0, y^i\}$, the metric satisfies:

$$g_{00} = \epsilon \qquad g_{0i} = 0 \qquad g_{ij} = (y^0)^2 \gamma_{ij}.$$

$\uparrow$ depends only on $\{y^k\}$ = coords on unit pseudosphere.

So now we compute $R^{\mu\nu}{}_{\rho\sigma}$ for $g_{\mu\nu}$; the nonzero Γ's are given:

$$\left.\begin{array}{l}\Gamma_{0ij} = -y^0\gamma_{ij}\\ \Gamma_{i0j} = y^0\gamma_{ij}\end{array}\right\} \to \left\{\begin{array}{l}\Gamma^0{}_{ij} = -\epsilon y^0\gamma_{ij} = -\epsilon(y^0)^{-1}g_{ij}\\ \Gamma^i{}_{0j} = (y^0)^{-1}\delta^i{}_j\\ \Gamma^i{}_{jk} = \Gamma(\gamma)^i{}_{jk} = \Gamma(g)^i{}_{jk}\end{array}\right.$$

(namely the Γ symbols for either γ_{ij} or g_{ij})

Then compute the following components of $R^{\mu\nu}{}_{\rho\sigma}$:

$$R^i{}_{jkl} = \partial_k\Gamma^i{}_{lj} - \partial_l\Gamma^i{}_{kj} + \underbrace{\Gamma^i{}_{k\mu}\Gamma^\mu{}_{lj} - \Gamma^i{}_{l\mu}\Gamma^\mu{}_{kj}}$$

(sum over all coordinates)

$$= \underbrace{\qquad\Gamma^i{}_{km}\Gamma^m{}_{lj} - \Gamma^i{}_{lm}\Gamma^m{}_{kj}}_{\substack{R(\gamma)^i{}_{jkl} = R(g)^i{}_{jkl}\\ \text{namely the curvature of either } \gamma_{ij} \text{ or } g_{ij}}} + \underbrace{\Gamma^i{}_{k0}\Gamma^0{}_{lj} - \Gamma^i{}_{l0}\Gamma^0{}_{kj}}_{\substack{((y^0)^{-1}\delta^i{}_k)(-\epsilon(y^0)^{-1}g_{lj})\\ -(y^0)^{-1}\delta^i{}_l(-\epsilon(y^0)^{-1}g_{kj})}}$$

$$R^i{}_{jkl} = R(g)^i{}_{jkl} - \epsilon(y^0)^{-2}(\delta^i{}_k g_{lj} - \delta^i{}_l g_{kj})$$

(THIS IS IN FACT CALLED THE GAUSS EQUATION)

$$R^{ij}{}_{kl} = g^{j\mu}R^i{}_{\mu kl} = g^{jm}R^i{}_{mkl} + \underbrace{g^{j0}}_{0}R^i{}_{0kl}$$

$$= R(g)^{ij}{}_{kl} - \epsilon(y^0)^{-2}(\delta^i{}_k\delta^j{}_l - \delta^i{}_l\delta^j{}_k) = R(g)^{ij}{}_{kl} - \epsilon(y^0)^{-2}\delta^{ij}_{kl}$$

$\| \; 0$

But $M^{p,q}$ is flat so $R^{\mu\nu}{}_{\rho\sigma} = 0$, so $\underbrace{R(g)^{ij}{}_{kl}}_{\substack{\text{curvature of}\\ \text{pseudosphere of radius } y^0}} = \underbrace{\epsilon(y^0)^{-2}}_{\lambda\, =\, \text{constant of curvature}}\delta^{ij}{}_{kl}$

In short a pseudosphere ${}^{(p,q)}\eta_{\mu\nu}X^\mu X^\nu = \epsilon\mathcal{R}^2$ has constant curvature $\lambda = \epsilon\mathcal{R}^{-2}$.

For metrics which are not positive definite, we are free to change the sign of the metric depending on our choice of convention; this has the effect of interchanging p and q. (the signature $S = p-q \to -S$ when $g_{\mu\nu} \to -g_{\mu\nu}$)

For such metrics, $g_{\mu\nu} \to -g_{\mu\nu}$ has the effect $\epsilon \to -\epsilon$, $g_{ij} \to -g_{ij}$ and hence $\lambda \to -\lambda$, so the sign is convention dependent.

So now we compute $R^{\mu\nu}{}_{\rho\sigma}$ for $g_{\mu\nu}$; the nonzero Γ's are given:

$$\left.\begin{aligned}\Gamma_{0ij} &= -y^0\gamma_{ij}\\ \Gamma_{i0j} &= y^0\gamma_{ij}\end{aligned}\right\} \rightarrow \begin{cases}\Gamma^0{}_{ij} = -\epsilon y^0 \gamma_{ij} = -\epsilon (y^0)^{-1} g_{ij}\\ \Gamma^i{}_{0j} = (y^0)^{-1}\delta^i{}_j\\ \Gamma^i{}_{jk} = \Gamma(\gamma)^i{}_{jk} = \Gamma(g)^i{}_{jk}\end{cases}$$

(namely the Γ symbols for either γ_{ij} or g_{ij})

Then compute the following components of $R^{\mu\nu}{}_{\rho\sigma}$:

$$R^i{}_{jkl} = \partial_k\Gamma^i{}_{lj} - \partial_l\Gamma^i{}_{kj} + \underbrace{\Gamma^i{}_{k\mu}\Gamma^\mu{}_{lj} - \Gamma^i{}_{l\mu}\Gamma^\mu{}_{kj}}$$

(sum over all coordinates)

$$= \underbrace{\qquad \Gamma^i{}_{km}\Gamma^m{}_{lj} - \Gamma^i{}_{lm}\Gamma^m{}_{kj}}_{R(\gamma)^i{}_{jkl} = R(g)^i{}_{jkl}} + \underbrace{\Gamma^i{}_{k0}\Gamma^0{}_{lj} - \Gamma^i{}_{l0}\Gamma^0{}_{kj}}_{((y^0)^{-1}\delta^i{}_k)(-\epsilon(y^0)^{-1}g_{lj}) - (y^0)^{-1}\delta^i{}_l(-\epsilon(y^0)^{-1}g_{kj})}$$

namely the curvature of either γ_{ij} or g_{ij}

$$R^i{}_{jkl} = R(g)^i{}_{jkl} - \epsilon(y^0)^{-2}(\delta^i{}_k g_{lj} - \delta^i{}_l g_{kj})$$

(THIS IS IN FACT CALLED THE GAUSS EQUATION)

$$R^{ij}{}_{kl} = g^{j\mu}R^i{}_{\mu kl} = g^{jm}R^i{}_{mkl} + \underbrace{g^{j0}}_{0}R^i{}_{0kl}$$

$$= R(g)^{ij}{}_{kl} - \epsilon(y^0)^{-2}(\delta^i{}_k\delta^j{}_l - \delta^i{}_l\delta^j{}_k) = R(g)^{ij}{}_{kl} - \epsilon(y^0)^{-2}\delta^{ij}{}_{kl}$$

$\| \; 0$ But $M^{p,q}$ is flat so $R^{\mu\nu}{}_{\rho\sigma} = 0$, so $\underbrace{R(g)^{ij}{}_{kl}}_{\text{curvature of pseudosphere of radius } y^0} = \underbrace{\epsilon(y^0)^{-2}}_{\lambda \,=\, \text{constant of curvature}}\delta^{ij}{}_{kl}$

In short a pseudosphere ${}^{(p,q)}\eta_{\mu\nu}X^\mu X^\nu = \epsilon\mathcal{R}^2$ has constant curvature $\lambda = \epsilon\mathcal{R}^{-2}$.

For metrics which are not positive definite, we are free to change the sign of the metric depending on our choice of convention; this has the effect of interchanging p and q. (the signature $s = p-q \rightarrow -s$ when $g_{\mu\nu} \rightarrow -g_{\mu\nu}$)

For such metrics, $g_{\mu\nu} \rightarrow -g_{\mu\nu}$ has the effect $\epsilon \rightarrow -\epsilon$, $g_{ij} \rightarrow -g_{ij}$ and hence $\lambda \rightarrow -\lambda$, so the sign is convention dependent.

ANTIDESITTER SPACETIME

A spacetime of signature $(-+++)$ and constant curvature $\lambda = -\frac{1}{\mathcal{R}^2}$

This may be realized as a pseudosphere in $M^{3,2}$

$$^{(3,2)}\eta_{\mu\nu} X^\mu X^\nu = -(X^0)^2 - (X^4)^2 + \delta_{ij} X^i X^j = \underset{\uparrow\ \epsilon = -1}{-}\mathcal{R}^2$$

This sign is wrong in Hawking and Ellis.

The point $(0000\overset{X^4}{\mathcal{R}})$ lies on this pseudosphere.

At this point
$$d(\eta_{\mu\nu}X^\mu X^\nu) = 2\eta_{\mu\nu}X^\mu dX^\nu$$
$$= 2\eta_{44}X^4 dX^4 = -2\mathcal{R}\, dX^4 \rightarrow dX^4 = 0$$

so $\underbrace{(dX^0, dX^i)}$ arbitrary

this gives us the right signature.

SLICING BY CONSTANT X^4 (TIMELIKE HYPERPLANES)

timelike slicing anti DeSitter $\quad |X^4| > \mathcal{R}$

$$\left.\begin{array}{l} X^4 = \mathcal{R}\cosh\lambda \\ \sigma = \mathcal{R}\sinh\lambda \end{array}\right\} \qquad -(X^4)^2 + \underbrace{\sigma^2}_{\eta_{\alpha\beta}X^\alpha X^\beta} = -\mathcal{R}^2$$

$\alpha, \beta = 0\,123$

$$ds^2 = -(dX^4)^2 + d\sigma^2 + \sigma^2 d\Gamma_+^2 = \ldots$$
$$= \mathcal{R}^2 d\lambda^2 + \mathcal{R}^2 \sinh^2\lambda \, d\Gamma_+^2$$

spacelike slicing anti DeSitter $\quad |X^4| < \mathcal{R}$

$$\left.\begin{array}{l} X^4 = \mathcal{R}\cos\lambda \\ \tau = \mathcal{R}\sin\lambda \end{array}\right\} \qquad -X^{4\,2} - \underbrace{\tau^2}_{-\eta_{\alpha\beta}X^\alpha X^\beta} = -\mathcal{R}^2$$

$\alpha, \beta = 0\,123$

$$ds^2 = -(dX^4)^2 - d\tau^2 + \tau^2 d\Gamma^2$$
$$= \ldots = -\mathcal{R}^2 d\lambda^2 + \mathcal{R}^2 \sin^2\lambda \, d\Gamma^2$$
$$= -dt^2 + \mathcal{R}^2 \sin^2(\mathcal{R}^{-1}t)\, d\Gamma^2$$

So this yields at $k=-1$ FRW spatially homogeneous slicing which becomes null and then timelike at the beginning and end ($t=0$ and $\mathcal{R}^{-1}t = \pi$).

CARTAN-KILLING METRIC ON $gl(N,R)$

Let $gl(N,R) \ni A = (A^a{}_b) \equiv A^a{}_b e^b{}_a$ define the natural basis of $gl(N,R)$ [$a \leftrightarrow$ row index , $b \leftrightarrow$ column index]. Then

$$e^a{}_b e^c{}_d = \delta^a{}_d e^c{}_b = \delta^a{}_d \delta^c{}_g \delta^f{}_b e^g{}_f$$

$$\left[\text{since} \quad (A^b{}_a e^a{}_b)(B^d{}_c e^c{}_d) = (A^b{}_a B^a{}_c) e^c{}_b\right] \qquad \text{so}$$

$$[e^a{}_b, e^c{}_d] = C^{f_g}{}_{a_b c_d}\, e^g{}_f = (\delta^a{}_d \delta^c{}_g \delta^f{}_b - \delta^c{}_b \delta^a{}_g \delta^f{}_d)\, e^g{}_f$$

$$\gamma(e^a{}_b, e^c{}_d) = \gamma_{a_b c_d} = C^{f_g}{}_{a_b c_d}\, C^{d_c}{}_{m_n g_f} = (\delta^a{}_d \delta^c{}_g \delta^f{}_b - \delta^c{}_b \delta^a{}_g \delta^f{}_d)(\delta^m{}_f \delta^g{}_c \delta^d{}_n - \delta^g{}_n \delta^m{}_c \delta^d{}_f)$$

$$= N \delta^a{}_n \delta^m{}_b - \delta^a{}_b \delta^m{}_n - \delta^a{}_b \delta^m{}_n + \delta^a{}_b \delta^m{}_b N = 2N \delta^a{}_n \delta^m{}_b - 2 \delta^a{}_b \delta^m{}_n$$

$$\gamma(A,B) = \gamma_{a_b c_d} A^b{}_a B^d{}_c = 2\,(N\,\mathrm{Tr}AB - \mathrm{Tr}A\,\mathrm{Tr}B) \quad = \text{Cartan-Killing form on } gl(N,R)$$

But there is a 2-parameter family (1-parameter modulo scale) of Ad-invariant inner products on $gl(N,R)$:

$$\gamma_{\lambda,\sigma} = (2N\sigma)\left[\mathrm{Tr}\,AB - \tfrac{\lambda}{N}\mathrm{Tr}A\,\mathrm{Tr}B\right] \qquad \gamma_{1,1} = \gamma \qquad \begin{pmatrix} \sigma \in R^+ \\ \lambda \in R \end{pmatrix}$$

Note $gl(N,R) = \mathrm{span}\{\mathbb{1}\} \oplus sl(N,R)$ is an orthogonal direct sum with respect to all of these inner products, and

$$\gamma_{\lambda,\sigma}(\mathbb{1},\mathbb{1}) = 2N^2\sigma(1-\lambda)$$

$\gamma(\mathbb{1},\mathbb{1}) = 0$, γ is the only degenerate member of this family.

Note $gl(N,R) = diag(N,R) \oplus offdiag(N,R) = so(N,R) \oplus sym(N,R)$ are *also orthogonal direct sums.*

$(\sigma,\lambda) = (\frac{1}{2N}, N)$ gives the <u>DeWitt</u> inner product.

Let $\langle A,B\rangle_{(\lambda,\sigma)} = \gamma_{\lambda,\sigma}(A,B)$.

Let $\mathcal{M}^{p,q}_N \equiv \{ (g_{ab}) = (g_{ba}) \in GL(N,R)_0 \mid \text{signature}(g_{ab}) = s = p-q \ , N \equiv p+q \}$

= space of component matrices of real inner products of signature s

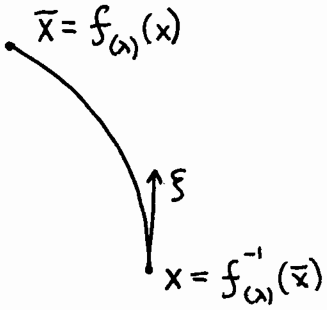
Let $T\mathcal{M}^{p,q}_N = \{ (h_{ab}) = (h_{ab}) \in gl(N,R) \}$ identified as tangent space to $\mathcal{M}^{p,q}_N$.

Given $(h_{ab}) \in T\mathcal{M}^{p,q}_N$, let $h = (h^a{}_b) = (g^{ac} h_{cb})$ = matrix of mixed components

Set $\mathcal{G}_{\lambda,\sigma}(h_1,h_2) = (g^{\frac{1}{2}})^{\rho} \langle h_1, h_2 \rangle_{\lambda,\sigma}$. This defines a natural pseudoRiemannian metric on $\mathcal{M}^{p,q}_N$ $(\rho \in R)$. $\rho = 0$ $\cancel{(\lambda,\sigma,\rho) = (\tfrac{1}{8}, 4, 0)}$ gives the metric associated with the harmonic mapping from Minkowski spacetime which occurs in the bi-metric Rosen theory for $N=4$, $|p-q| = 2$.

LIE DERIVATIVE

$x^\mu \to \bar{x}^\mu = f^\mu_{(\lambda)}(x)$ 1-parameter family of point transformations ("diffeomorphisms")



$f^\mu_{(0)}(x) = x^\mu$ identity transformation

GENERATING VECTOR FIELD :

$$\xi^\mu(x) \equiv \frac{df^\mu_{(0)}(x)}{d\lambda}\,, \qquad \xi = \xi^\mu \frac{\partial}{\partial x^\mu}$$

(vector fields identified with differential operators)

power series expansion :

$$\bar{x}^\mu = f^\mu_{(0)}(x) + \lambda \frac{df^\mu_{(0)}(x)}{d\lambda} + \frac{1}{2}\lambda^2 \frac{d^2 f^\mu_{(0)}(x)}{d\lambda^2} + \dots \quad = x^\mu + \lambda \xi^\mu(x) + \dots$$

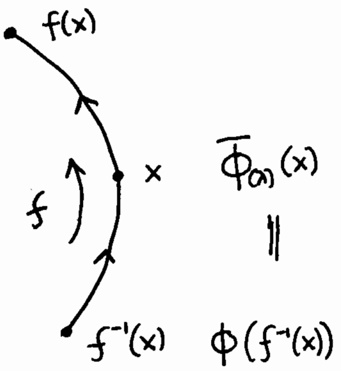
$$\approx x^\mu + \lambda \xi^\mu(x) \quad \text{for } \lambda \ll 1$$

inverse transformation satisfies $f^{-1\,\mu}_{(\lambda)}(f_{(\lambda)}(x)) = x^\mu$, $f^{-1\,\mu}_{(0)}(x) = x^\mu$,
so by chain rule, $\frac{d}{d\lambda}\Big|_{\lambda=0}$ of the first equation gives :

$$\frac{df^{-1\,\mu}_{(0)}}{d\lambda}(\underbrace{f_{(0)}(x)}_{x}) + \frac{d}{d\lambda}\Big|_{\lambda=0} f^{-1}_{(0)}(\underbrace{f_{(\lambda)}(x)}_{f^\mu_{(\lambda)}(x)}) = 0 \to \frac{df^{-1\,\mu}_{(0)}(x)}{d\lambda} = -\frac{df^\mu_{(0)}(x)}{d\lambda} = -\xi^\mu(x)$$

so a similar power series expansion yields

$$f^{-1\,\mu}_{(\lambda)}(x) = x^\mu - \lambda \xi^\mu(x) + \dots \quad \approx x^\mu - \lambda \xi^\mu(x) \qquad \lambda \ll 1.$$



If $\phi(x)$ is a (scalar) function, let $\bar{\Phi}_{(\lambda)}(x)$ be the function transformed by the point transformation

$$\bar{\Phi}_{(\lambda)}(\bar{x}) = \phi(x) \qquad \text{or} \qquad \bar{\Phi}_{(\lambda)}(x) = \phi(f^{-1}_{(\lambda)}(x))$$

new value at new point — old value at old point

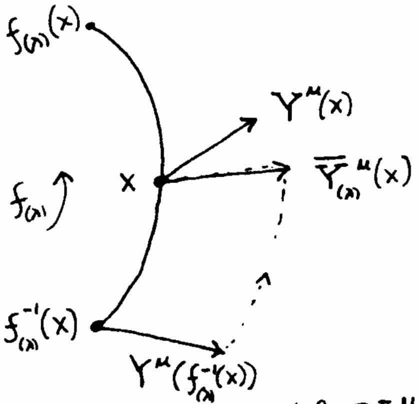
This definition moves the function in the direction of the point transformation.

The rate of change of $\bar{\Phi}_{(\lambda)}$ with respect to λ at $\lambda = 0$ tells how ϕ begins to change under the point transformation and defines the negative of the Lie derivative of ϕ with respect to the generating vector field

$$\frac{d}{d\lambda}\Big|_{\lambda=0} \bar{\Phi}_{(\lambda)}(x) = \frac{\partial\phi}{\partial x^\mu}(f^{-1}_{(0)}(x)) \frac{df^{-1\,\mu}_{(0)}(x)}{d\lambda} = -\xi^\mu(x)\frac{\partial\phi(x)}{\partial x^\mu} = -\xi(x)\,\phi$$

$$\pounds_\xi \phi \equiv -\frac{d}{d\lambda}\Big|_{\lambda=0} \bar{\Phi}_{(\lambda)} = \xi\phi = \xi^\mu \frac{\partial}{\partial x^\mu}\phi = \phi_{,\mu}\xi^\mu$$

The Lie derivative of a scalar by a vector field ξ is just the directional derivative of the scalar along that vector field



A vector field is transformed by the point transformation as follows:

$$\underbrace{\overline{Y}^{\mu}_{(\lambda)}(x)}_{\text{value at } x} = \frac{\partial f^{\mu}_{(\lambda)}}{\partial x^{\nu}}(f^{-1}_{(\lambda)}(x))\; Y^{\nu}(\underbrace{f^{-1}_{(\lambda)}(x)}_{\text{value at point mapped onto } x \text{ by } f_{(\lambda)}})$$

Now calculate its Lie derivative exactly as for the scalar:

$$(\mathcal{L}_{\xi} Y^{\mu})(x) = -\frac{d}{d\lambda}\Big|_{\lambda=0} \overline{Y}^{\mu}_{(\lambda)}(x)$$

$$= -\Big[\underbrace{\frac{\partial f^{\mu}_{(0)}}{\partial x^{\nu}}(\underbrace{f^{-1}_{(0)}(x)}_{x})}_{\frac{\partial x^{\mu}}{\partial x^{\nu}} = \delta^{\mu}{}_{\nu}}\Big] \underbrace{\underbrace{\frac{\partial Y^{\nu}}{\partial x^{\rho}}(f^{-1}_{(0)}(x))\, \frac{d f^{-1\,\rho}_{(0)}}{d\lambda}(x)}_{-\xi^{\rho}(x)\frac{\partial Y^{\nu}}{\partial x^{\rho}}(x)}}_{\text{just like in scalar case}} - \frac{\partial}{\partial x^{\nu}}\Big[\underbrace{\frac{d f^{\mu}_{(0)}}{d\lambda}}_{\xi^{\mu}}\Big](f^{-1}_{(0)}(x))\; Y^{\nu}(\underbrace{f^{-1}_{(0)}(x)}_{x})$$

$$= \Big[\underbrace{\xi^{\rho}\frac{\partial Y^{\nu}}{\partial x^{\rho}}}_{\text{directional derivative of components}} - \frac{\partial \xi^{\mu}}{\partial x^{\nu}} Y^{\nu}\Big](x)$$

$$\boxed{\mathcal{L}_{\xi}\phi = \phi_{,\rho}\,\xi^{\rho}}$$

$$\boxed{\mathcal{L}_{\xi} Y^{\mu} = Y^{\mu}{}_{,\rho}\,\xi^{\rho} - \xi^{\mu}{}_{,\rho}\,Y^{\rho}}$$

$$\boxed{\mathcal{L}_{\xi} Z_{\mu} = Z_{\mu,\rho}\,\xi^{\rho} + Z_{\rho}\,\xi^{\rho}{}_{,\mu}}$$

$$\boxed{\mathcal{L}_{\xi} g_{\mu\nu} = \underbrace{g_{\mu\nu,\rho}}_{\Gamma_{\nu\mu\rho}+\Gamma_{\mu\nu\rho}}\,\xi^{\rho} + g_{\rho\nu}\,\xi^{\rho}{}_{,\mu} + g_{\mu\rho}\,\xi^{\rho}{}_{,\nu}}$$

If we do the same thing for a covariant vector field

$$\overline{Z}^{(\lambda)}_{\mu} = \underbrace{\frac{\partial f^{-1\,\nu}_{(\lambda)}}{\partial x^{\mu}}}(x)\, Z_{\nu}(f^{-1}_{(\lambda)}(x))$$

the λ-derivative of this term leads instead to $-\xi^{\nu}{}_{\mu} Z_{\nu}$ so we get

For the metric we get one of these terms for each covariant index, but always the first term is just the directional derivative of the components

The expression for the $\mathcal{L}_{\xi} g_{\mu\nu}$ can be rewritten using

to yield:

$$\mathcal{L}_{\xi} g_{\mu\nu} = g_{\rho\nu}\,\xi^{\rho}{}_{;\mu} + g_{\mu\rho}\,\xi^{\rho}{}_{;\nu} = (g_{\rho\nu}\xi^{\rho})_{;\mu} + (g_{\mu\rho}\xi^{\rho})_{;\nu}$$

$$= \xi_{\nu;\mu} + \xi_{\mu;\nu} \qquad (\text{since } g_{\rho\nu;\alpha} = 0)$$

* Note that $\mathcal{L}_{\xi} Y = [\xi, Y] = -\mathcal{L}_{Y}\xi$, where [,] is the commutator of the vector fields as differential operators.

page 11 insert

In the March notes, I neglected to talk explicitly about the invariance of a field under a transformation.

In the case of a 1-parameter family of point transformations, invariance of a tensor field $T^{\mu\ldots}{}_{\nu\ldots}$ means that

$$\overline{T}_{(\lambda)}{}^{\mu\ldots}{}_{\nu\ldots} = T^{\mu\ldots}{}_{\nu\ldots}$$

the transformed field equals the original field for all λ, hence taking the λ-derivative at $\lambda=0$, one has vanishing Lie derivative:

$$\mathcal{L}_\xi T^{\mu\ldots}{}_{\nu\ldots} = 0.$$

For a metric, invariance means:

$$\overline{g}_{(\lambda)\,\mu\nu} = g_{\mu\nu}$$

$$\mathcal{L}_\xi g_{\mu\nu} = 0. \qquad \text{or} \qquad 0 = \xi_{(\mu;\nu)}.$$

ξ is called a Killing vector field, and the equation Killing's equation. Its solutions are the generators of the full group of motions of the metric. We evaluated explicity the Killing vectors for the flat spaces of arbitrary signature, and consequently, for the imbedded pseudospheres, homogeneous and isotropic spaces with maximum symmetry.

On a group G, the generators of right translations $\{e_a\}$ and the generators of left translations $\{\tilde{e}_a\}$ commute since the left translations commute with the right translations:

$$[e_a,\tilde{e}_b]=0: \begin{cases} \rightarrow \mathcal{L}_{\tilde{e}_b} e_a = 0 & \text{means } \{e_a\} \text{ are left invariant} \\ \rightarrow \mathcal{L}_{e_a} \tilde{e}_b = 0 & \text{means } \{\tilde{e}_a\} \text{ are right invariant} \end{cases}$$

CONSTANTS OF THE MOTION FOR GEODESICS

A very useful property of Killing vector fields is that each independent KVF yields a conserved momentum for a geodesic.

Suppose $x^\mu = x^\mu(\tau)$ is a timelike geodesic parametrized by the proper time τ. The unit four-velocity $u^\mu = dx^\mu(\tau)/d\tau$ satisfies $\frac{Du^\mu}{d\tau} \equiv u^\mu{}_{;\nu} u^\nu = 0$, where $\frac{D}{dt} =$ " $_{;\nu} u^\nu$ " is the covariant derivative along the tangent.

If ξ^μ is a KVF, then the momentum like quantity $p = \xi_\mu u^\mu$, is sort of component of the velocity along the symmetry direction, is conserved:

only symmetric part contributes

$$\frac{D}{d\tau}\left(\xi_\mu u^\mu\right) = \left(\xi_\mu u^\mu\right)_{;\nu} u^\nu = \xi_\mu \underbrace{\left(u^\mu{}_{;\nu} u^\nu\right)}_{\substack{=0 \\ \text{(geodesic)}}} + \underbrace{\xi_{(\mu;\nu)}}_{\substack{=0 \\ \text{(Killing eq)}}} u^\mu u^\nu = 0$$

If ξ^μ is timelike, then $-p$ can be interpreted as an energy, and if instead spacelike, as some kind of momentum (linear or angular).