

INTRODUCTION TO COSMOLOGICAL MODELS

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Dipartimento di Fisica
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INTRODUCTION TO COSMOLOGICAL MODELS: PART 1

What is Cosmology? Overview.

Flat Euclidean/Lorentz geometry for $N=2$

$N > 2$: (pseudo-)spherical, (pseudo-)cylindrical coordinates

Orthogonal coordinates on M^4 and constant curvature subspaces

Maximally symmetric spaces

Friedmann-Robertson-Walker spacetimes

Higher Dimensions?

Special coordinates on DeSitter spacetime

Special coordinates on AntiDeSitter spacetime

Gaussian normal coordinates and constant curvature, Einstein curvature

Interpretation of ~~an~~ intrinsic curvature

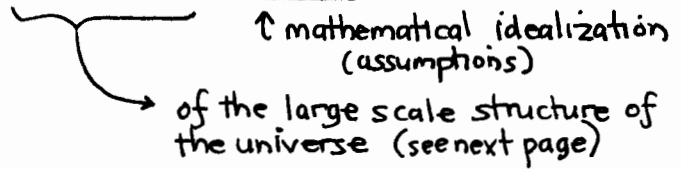
Arbitrary signature flat spaces and pseudo-orthogonal groups

Part 1 is designed to generalize the student's knowledge of the geometry of special relativity to the case of cosmological spacetimes, from the point of view of metric and coordinates, with some mention of groups of motion. Only geometries of constant curvature are considered.

The geometry of symmetry groups and its application to less symmetric spacetime geometries relevant in cosmological models in 4 or more dimensions will be developed in part 2.

WHAT IS COSMOLOGY?

Cosmology is the study of the large scale structure of the universe, most often accomplished using cosmological models.



Classical Cosmological models involve kinematics and dynamics:

1) KINEMATICS: spacetime metric + symmetry conditions + adapted coordinates

2) DYNAMICS: field equations

EINSTEIN EQUATIONS:

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta}$$

↑ curvature (Einstein tensor) ↑ "cosmological constant" term ↑ energy-momentum
 geometry matter

MATTER EQUATIONS: $T^{\alpha\beta}_{;\beta} = 0$, etc.

Complications: underlying gravitational theory, dimension of spacetime, "early cosmology" modifications of old "standard picture"

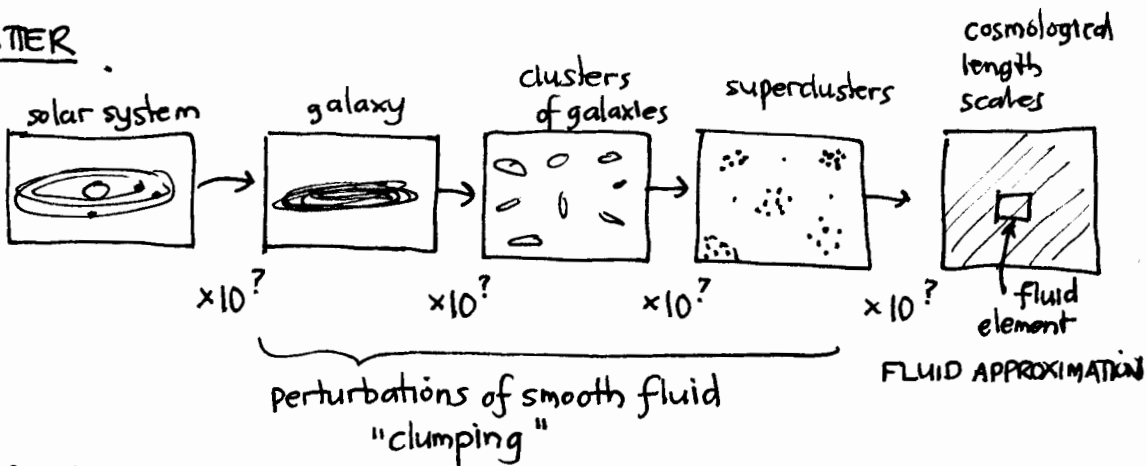
GUTS, Kaluza-Klein, supergravity, string theories,

topological effects: monopoles, ~~etc~~ cosmic strings, domain walls (associated with phase transitions, semiclassical quantum calculations, ...)

MODEL FOR MATTER

now:

ASTRONOMICAL
DISTANCE
SCALES:



magnification:

$\times 10^?$ $\times 10^?$ $\times 10^?$ $\times 10^?$

now dust fluid model:

pressure $p=0$, $\rho =$ average energy density at cos. length. scales
 $U^\alpha =$ average 4-velocity of "dust" ($U^\alpha U_\alpha = -1$ timelike)
 zero temperature , actual ^{constituent} ~~total~~ motions relative to average motion are on scales much less than cos. length scales at speeds much less than 1 (lightspeed) (nonrelativistic matter)
 (existence is an assumption — selfsimilarity & fractal structure can lead to problems)

"Big Bang" — plasma gas before creation of nuclei, stars, galaxies, etc.
 — high temperature relativistic matter, pressure p nonzero

COSMOLOGICAL EQUATION OF STATE :

$p = (\gamma - 1) \rho$	$(\gamma - 1) = 0$ $\gamma = 1$ $p = 0$ — dust (NR matter)
\uparrow	$(\gamma - 1) = \frac{1}{3}$ $\gamma = \frac{4}{3}$ $p = \frac{1}{3} \rho$ — relativistic matter
const.	$(\gamma - 1) = 1$ $\gamma = 2$ $p = \rho$ — "stiff matter"

very near "Big Bang" — quantum vacuum energy dominates particle physics is key [QUANTUM MATTER]

veryvery near "Big Bang" — quantum gravitational effects become important [QUANTUM GEOMETRY]
 ↳ "quantum cosmology"

MODEL FOR GEOMETRY (\rightarrow MATTER): SYMMETRIES

When equations too difficult to solve — impose symmetry to simplify.
Highest symmetry \rightarrow easiest to solve. (PRACTICAL APPROACH)

OR "cosmological principle": why should our position and time in the universe be special? (PHILOSOPHICAL APPROACH)

1) FIRST TRY: universe same at all places, all times, all directions
HOMOGENEOUS AND ISOTROPIC SPACETIME

like ^{static} Einstein Universe allowed by $\Lambda \neq 0$ [$R \times S^3$]

BUT NOT SAME AT ALL TIMES: EXPANSION

2) SECOND TRY: universe same at all places and in all directions at a given time but evolves with time [Break homogeneity and isotropy in time direction]

SPATIALLY HOMOGENEOUS AND ISOTROPIC SPACETIME

like Friedmann-Robertson-Walker models

3) CURIOSITY: maybe universe not same in all (spatial) directions.

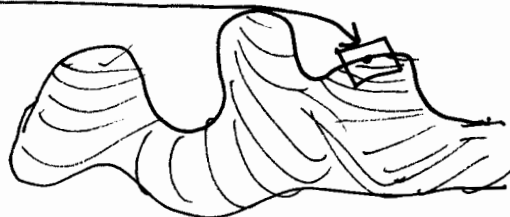
SPATIALLY HOMOGENEOUS (anisotropic) UNIVERSE [Break isotropy in space]
like Bianchi models

4) MORE CURIOSITY: maybe universe not same at every place (at cosmological distance scales)

INHOMOGENEOUS UNIVERSE [Break homogeneity in space]

SYMMETRY BREAKING \downarrow

POSSIBLE PERSPECTIVE: perhaps observable universe is only a small part of an inhomogeneous anisotropic spacetime which seems very symmetric at large scales [MICROWAVE BACKGROUND]



maybe good enough to study highly symmetric models.

UNIVERSE MODEL

So one assumes a highly symmetric universe model for the "largest scale" structure of the universe, the dynamics of which can be handled exactly, qualitatively or numerically (ORDINARY DIFFERENTIAL EQUATIONS).

The structure at smaller distance scales which breaks this symmetry evolves by much more complicated dynamics (PARTIAL DIFFERENTIAL EQUATIONS). Only by perturbation techniques (coupled with harmonic analysis) can they be handled, but maybe that is good enough to study certain questions like GALAXY FORMATION and CLUSTERING, since these are relatively small deviations from the large scale behavior of the universe.

Classical relativity together with the observed expansion tells us there should be an initial cosmological singularity of some kind (Big Bang?) but the conditions of high energy & density near such an event invalidate the classical theory there. One needs quantum gravity which still does not exist, so semiclassical calculations or possibly unrealistic model quantum calculations are done.

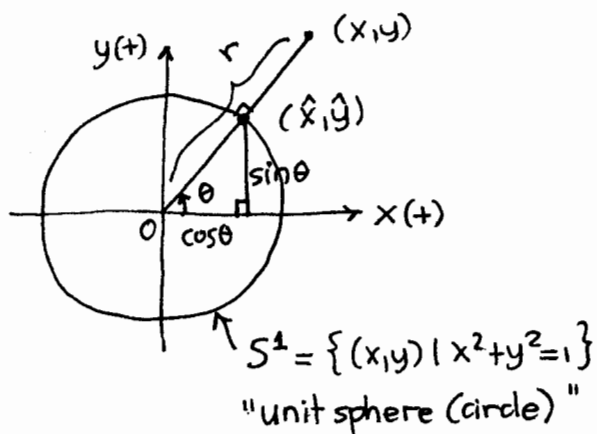
At very early times the theory of matter becomes extremely important GUTS (Grand Unified Theories) and their consequences, string theories, higher-dimensional spacetimes, etc all complicate the picture, offering the possibility of explaining why the universe is the way it is.

WE WILL BEGIN BY STUDYING THE
Kinematics of highly symmetric spacetimes.

FLAT PSEUDO-RIEMANNIAN GEOMETRY ON \mathbb{R}^N

The geometry of cosmological models is best approached by first understanding well the geometry of flat pseudo-Riemannian spaces. The dimension of a space must be at least $N=2$ to have an indefinite signature metric. This case illustrates the key differences between Riemannian and pseudo-Riemannian spaces. Consider the case of flat metrics on \mathbb{R}^2 . There are two distinct, inequivalent signatures: $(+,+)$ and $(-,+)$.

$N=2 (+,+)$: [spherical coordinates on $E^2 = \mathbb{R}^2$ with Euclidean metric (all directions equivalent)]



distance² from 0: $r^2 = x^2 + y^2$

metric: $ds^2 = dx^2 + dy^2$

project onto S^1 , parametrize S^1 :

$$x = r \hat{x} = r \cos \theta$$

$$y = r \hat{y} = r \sin \theta$$

(\hat{x}, \hat{y}) are constrained coordinates on S^1 (called "projective coordinates"):

$$\hat{x}^2 + \hat{y}^2 = 1 \xrightarrow{d} 0 = \hat{x} d\hat{x} + \hat{y} d\hat{y} = "(\hat{x}, \hat{y}) \cdot (d\hat{x}, d\hat{y}) = 0"$$

(orthogonality of radial and tangential directions)

$\begin{matrix} \cos^2 \theta & \sin^2 \theta \\ \text{(identification with)} \\ \text{trig identity} \end{matrix}$

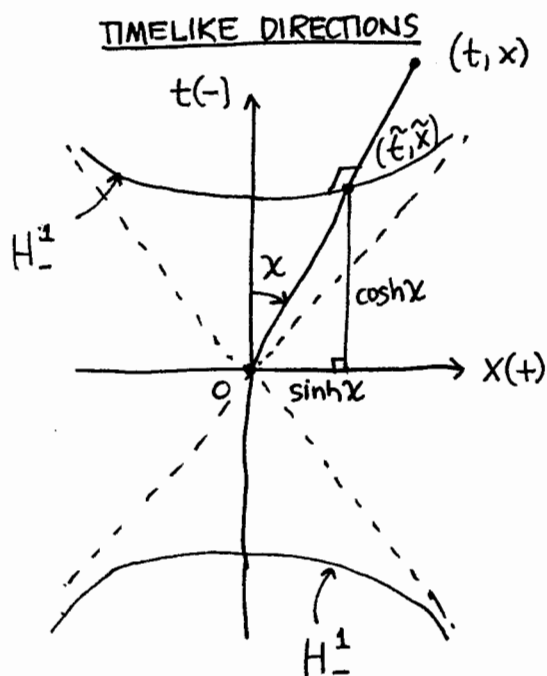
radial / tangential decomposition of metric:

$$\begin{aligned} \left. \begin{aligned} dx &= dr \hat{x} + r d\hat{x} \\ dy &= dr \hat{y} + r d\hat{y} \end{aligned} \right\} \rightarrow ds^2 &= (dr \hat{x} + r d\hat{x})^2 + (dr \hat{y} + r d\hat{y})^2 \\ &= \underbrace{(\hat{x}^2 + \hat{y}^2)}_1 dr^2 + \underbrace{(\hat{x} d\hat{x} + \hat{y} d\hat{y})}_0 \cdot 2r dr + r^2 (d\hat{x}^2 + d\hat{y}^2) \\ &= dr^2 + r^2 \underbrace{(d\hat{x}^2 + d\hat{y}^2)}_{d\theta^2 \text{ metric on } S^1} \end{aligned}$$

These coordinates provide locally an orthogonal decomposition $S^1 \times \mathbb{R} \leftrightarrow \mathbb{R}^2$ which breaks down at the coordinate singularity $r=0$.

$N=2 (-, +)$: [pseudospherical coordinates on $M^2 = R^2$ with Lorentz metric (distinct equivalence classes of directions)]

metric: $ds^2 = -dt^2 + dx^2$, $-t^2 + x^2 \begin{cases} < 0 & \text{timelike } (-), \tau^2 = t^2 - x^2 \\ = 0 & \text{null } (0) \quad \tau = 0 = \sigma \\ > 0 & \text{spacelike } (+), \sigma^2 = x^2 - t^2 \end{cases}$



unit spacelike pseudosphere of unit timelike directions:

$$H_-^1 = \{(t, x) \mid -t^2 + x^2 = -1\}$$

or $|\tau| = 1$.

Let $\text{sgn } \tau = \text{sgn } t$, so $\tau > 0 \Leftrightarrow$ future and $\tau < 0 \Leftrightarrow$ past.

project onto H_-^1 ; parametrize H_-^1

$$\begin{aligned} t &= \tau \tilde{t} = \tau \cosh \chi \\ x &= \tau \tilde{x} = \tau \sinh \chi \end{aligned}$$

(\tilde{t}, \tilde{x}) are constrained coordinates on H_-^1 (called "projective coordinates"):

$$-\tilde{t}^2 + \tilde{x}^2 = -1 \xrightarrow{d} 0 = -\tilde{t} d\tilde{t} + \tilde{x} d\tilde{x} = "(\tilde{t}, \tilde{x}) \cdot (d\tilde{t}, d\tilde{x})"$$

(orthogonality of radial and tangential directions)

(identification with hyperbolic identity)

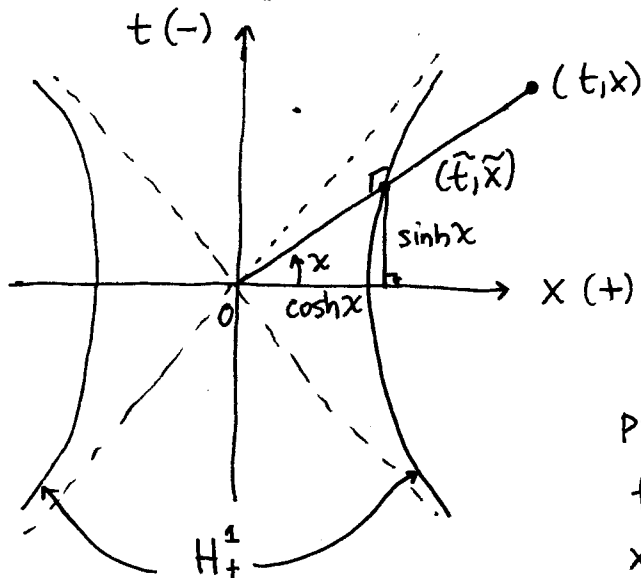
radial/tangential decomposition of metric:

$$\begin{aligned} \left. \begin{aligned} dt &= d\tau \tilde{t} + \tau d\tilde{t} \\ dx &= d\tau \tilde{x} + \tau d\tilde{x} \end{aligned} \right\} ds^2 &= -(d\tau \tilde{t} + \tau d\tilde{t})^2 + (d\tau \tilde{x} + \tau d\tilde{x})^2 \\ &= \underbrace{(-d\tilde{t}^2 + d\tilde{x}^2)}_{-1} d\tau^2 + \underbrace{(-\tilde{t} d\tilde{t} + \tilde{x} d\tilde{x})}_0 \cdot 2\tau d\tau + \tau^2 (-d\tilde{t}^2 + d\tilde{x}^2) \\ &= -d\tau^2 + \tau^2 \underbrace{(-d\tilde{t}^2 + d\tilde{x}^2)}_{dx^2 \text{ metric on } H_-^1} = \end{aligned}$$

These coordinates provide locally an orthogonal decomposition

$H_-^1 \times R \Leftrightarrow R^2$ which breaks down at the lightcone coordinate singularity $\tau = 0$. They cover only half of M^2 .

SPACELIKE DIRECTIONS



unit timelike pseudosphere of unit spacelike directions

$$H_+^1 = \{ (t, x) \mid -t^2 + x^2 = 1 \}$$

or $|\sigma| = 1$.

Let $\text{sgn } \sigma = \text{sgn } x$.

Project onto H_+^1 , parametrize H_+^1 :

$$t = \sigma \tilde{t} = \sigma \sinh \chi$$

$$x = \sigma \tilde{x} = \sigma \cosh \chi$$

(\tilde{t}, \tilde{x}) are constrained coordinates on H_+^1 ("projective coordinates"):

$$-\tilde{t}^2 + \tilde{x}^2 = 1 \xrightarrow{d} 0 = -\tilde{t} d\tilde{t} + \tilde{x} d\tilde{x} = "(\tilde{t}, \tilde{x}) \cdot (d\tilde{t}, d\tilde{x})"$$

(orthogonality of radial and tangential directions)

(identification with hyperbolic identity)

radial/tangential decomposition of metric:

$$\left. \begin{aligned} dt &= \dots \\ dx &= \dots \end{aligned} \right\} ds^2 = \dots$$

$$= \underbrace{(-d\tilde{t}^2 + d\tilde{x}^2)}_1 d\sigma^2 + \underbrace{(-\tilde{t}d\tilde{t} + \tilde{x}d\tilde{x})}_{0} \cdot 2\sigma d\sigma + \sigma^2 (-d\tilde{t}^2 + d\tilde{x}^2)$$

$$= d\sigma^2 + \sigma^2 \underbrace{(-d\tilde{t}^2 + d\tilde{x}^2)}_{-d\chi^2 \text{ metric on } H_+^1}$$

$-d\chi^2$ metric on H_+^1

These coordinates provide locally an orthogonal decomposition

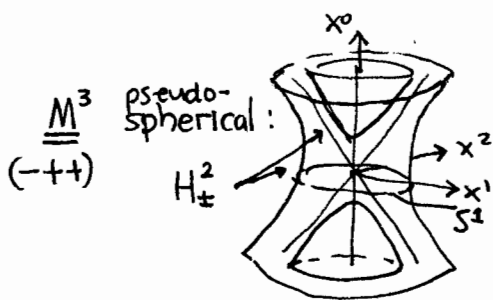
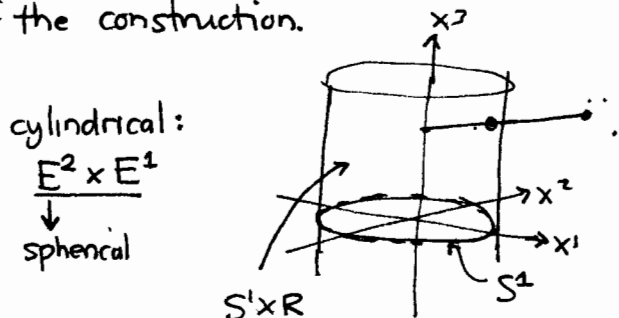
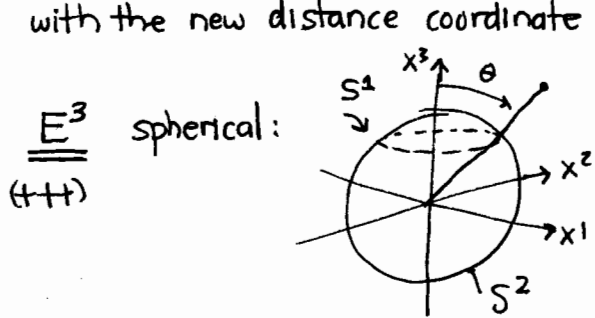
$H_+^1 \times \mathbb{R} \leftrightarrow \mathbb{R}^2$ which breaks down at the lightcone coordinate singularity $\sigma = 0$. These cover the second half of M^2 . However, the lightcone is still left uncovered.

$N > 2$ FLAT PSEUDO-RIEMANNIAN METRICS ON \mathbb{R}^N

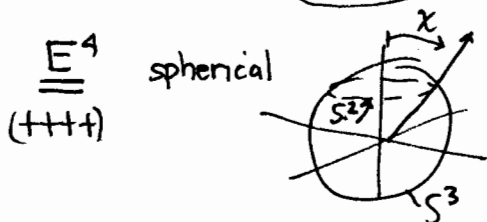
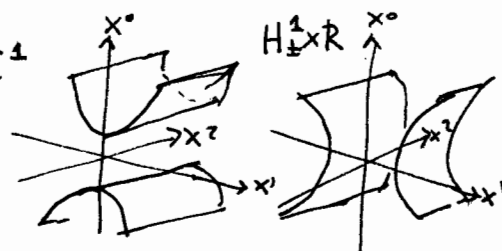
The (pseudo-)spherical coordinates may be generalized in an iterative way to yield either (pseudo-)spherical or (pseudo-)cylindrical coordinates on higher dimensional flat spaces. The previous calculations generalize simply introducing indices.

(Pseudo-)spherical coordinates result from decomposing \mathbb{R}^N into a family of $(N-1)$ -dimensional (pseudo-)spheres, which are isotropic and homogeneous subspaces of constant curvature.

(Pseudo-)cylindrical coordinates result from first decomposing $\mathbb{R}^N = \mathbb{R}^d \times \mathbb{R}^{N-d}$ and introducing (pseudo-)spherical coordinates on \mathbb{R}^d , keeping the cartesian coordinates on \mathbb{R}^{N-d} . One therefore decomposes \mathbb{R}^N a family of $(N-1)$ -dimensional anisotropic but homogeneous hypersurfaces (cylinders) with flat directions associated with the remaining cartesian coordinates. These hypersurfaces are associated with the new distance coordinate of the construction.



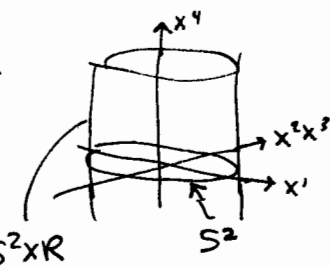
cylindrical: $\underline{M^2 \times E^1}$
or $\underline{M^1 \times E^2}$
(same as for E^3)



cylindrical: $\underline{E^3 \times E^1}$
↓
spherical

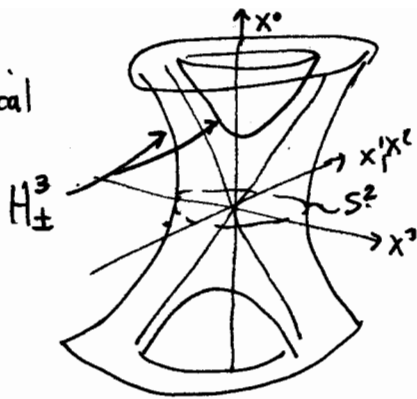
$\underline{E^2 \times E^2}$
↓
spherical

$S^1 \times \mathbb{R}^2$



M⁴ pseudo-spherical

(-+++)

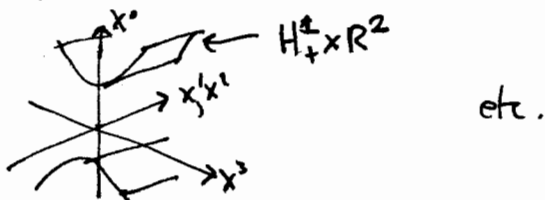


cylindrical :

$M^1 \times E^3$ as in E^3 spherical
 ↓
 spherical

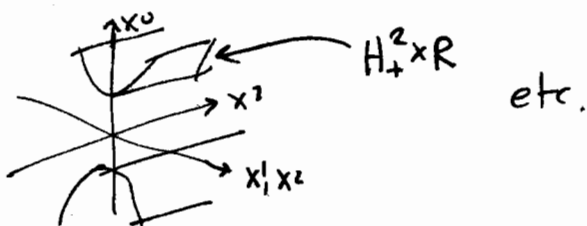
$M^2 \times E^2$ as in E^3 cylindrical
 ↓
 spherical

$M^2 \times E^2$
 ↓
 pseudospherical



etc.

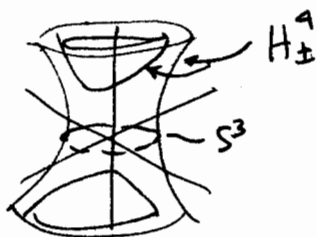
$M^3 \times E^1$
 ↓
 pseudospherical



etc.

M⁵ pseudospherical

(-++++)



[+ DeSitter universe]

cylindrical



[Einstein universe]

$M^2 \times E^3$
 ↓

$M^2 \times E^3$
 ↓

$M^3 \times E^2$
 ↓

$M^3 \times E^2$
 ↓

$M^4 \times E^1$
 ↓

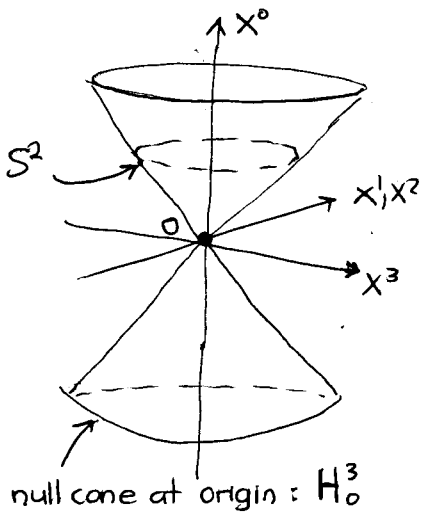
etc.

(more details later)

Minkowski spacetime M^4

= R^4 with standard cartesian coordinates $\{X^\mu\}$ $\mu=0,1,2,3$
and flat Lorentz $(-+++)$ metric

$\eta = \eta_{\mu\nu} dx^\mu \otimes dx^\nu$ or more simply " $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ "
where $(\eta_{\mu\nu}) = \text{diag}(-1, 1, 1, 1)$.



spacetime intervals

From origin to event with coordinates $\{X^\mu\}$:

timelike directions: $\eta_{\mu\nu} X^\mu X^\nu < 0 \rightarrow \tau^2 \equiv -\eta_{\mu\nu} X^\mu X^\nu$

τ is the proper time along path of inertial observer connecting origin to $\{X^\mu\}$

null (lightlike) directions: $\eta_{\mu\nu} X^\mu X^\nu = 0$

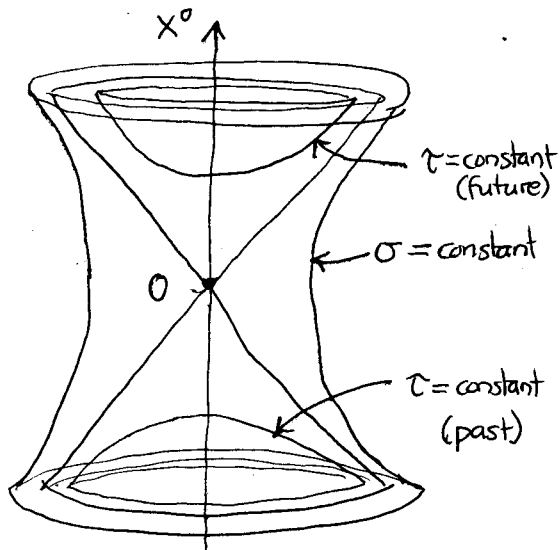
paths of light rays lie on null cone

spacelike directions: $\eta_{\mu\nu} X^\mu X^\nu > 0 \rightarrow \sigma^2 = \eta_{\mu\nu} X^\mu X^\nu$

σ is the proper distance between origin and $\{X^\mu\}$ in an inertial frame in which they are simultaneous events.

pseudospheres (at origin):

hypersurfaces of constant spacetime interval from the origin



spacelike pseudospheres (future, past) ($\sim R^3$)
 $\tau = \text{constant}$

unit spacelike pseudospheres: H_{-}^3 (future/past)

$\eta_{\mu\nu} X^\mu X^\nu = -1$ $\begin{cases} X^0 > 0 & \text{future} \\ X^0 < 0 & \text{past} \end{cases}$

timelike pseudospheres ($\sim S^2 \times R$)

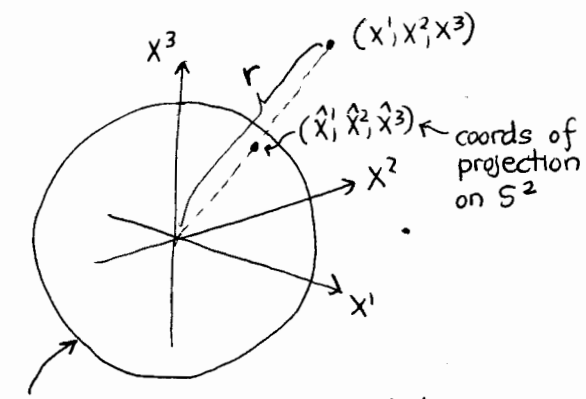
$\sigma = \text{constant}$

unit timelike pseudosphere: H_{+}^3

$\eta_{\mu\nu} X^\mu X^\nu = 1$.

null pseudosphere = null cone = light cone (H^3)

$\eta_{\mu\nu} X^\mu X^\nu = 0$ ($\sim S^2 \times R \cup \{0\} \cup S^2 \times R$)
↑
vertex of cone



spherical coordinates on R^3 ($i, j = 1, 2, 3$)

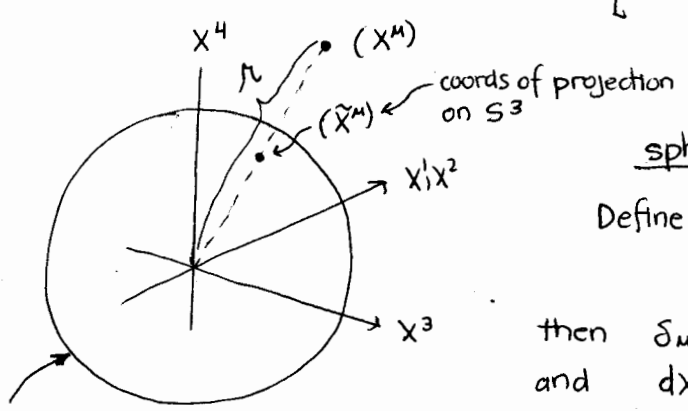
Define $\begin{cases} r^2 = \delta_{ij} x^i x^j \\ \hat{x}^i = x^i / r \text{ or } x^i = r \hat{x}^i \end{cases}$

then $\delta_{ij} \hat{x}^i \hat{x}^j = 1 \rightarrow 2\delta_{ij} \hat{x}^i d\hat{x}^j = 0$
and $dx^i = \hat{x}^i dr + r d\hat{x}^i$

$S^2 = \{ (x^1, x^2, x^3) \in R^3 \mid \delta_{ij} x^i x^j = 1 \}$

$ds^2 = \delta_{ij} dx^i dx^j = \delta_{ij} (\hat{x}^i dr + r d\hat{x}^i) (\hat{x}^j dr + r d\hat{x}^j)$
 $= (\delta_{ij} \hat{x}^i \hat{x}^j) dr^2 + (2\delta_{ij} \hat{x}^i d\hat{x}^j) r dr + r^2 \underbrace{\delta_{ij} d\hat{x}^i d\hat{x}^j}_{d\Omega^2}$
 $= dr^2 + r^2 d\Omega^2$ (metric on unit sphere S^2)

$\left. \begin{aligned} & \underbrace{(\hat{x}^3)^2}_{\cos^2\theta} + \underbrace{[(\hat{x}^1)^2 + (\hat{x}^2)^2]}_{\sin^2\theta} = 1 \\ & \underbrace{S^1 \text{ of radius } \sin\theta} \end{aligned} \right\} \begin{aligned} \hat{x}^1 &= \sin\theta \cos\varphi \\ \hat{x}^2 &= \sin\theta \sin\varphi \\ \hat{x}^3 &= \cos\theta \end{aligned}$
 $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$
 metric on S^1 of radius $\sin\theta$



spherical coordinates on R^4 ($\mu, \nu = 1, 2, 3, 4$)

Define $\begin{cases} r^2 = \delta_{\mu\nu} x^\mu x^\nu \\ \tilde{x}^\mu = x^\mu / r \text{ or } x^\mu = r \tilde{x}^\mu \end{cases}$

then $\delta_{\mu\nu} \tilde{x}^\mu \tilde{x}^\nu = 1 \rightarrow 2\delta_{\mu\nu} \tilde{x}^\mu d\tilde{x}^\nu = 0$
and $dx^\mu = \tilde{x}^\mu dr + r d\tilde{x}^\mu$

$S^3 = \{ (x^\mu) \in R^4 \mid \delta_{\mu\nu} x^\mu x^\nu = 1 \}$

Exactly as above:
 $ds^2 = \delta_{\mu\nu} dx^\mu dx^\nu = \dots = dr^2 + r^2 d\Sigma^2$ $d\Sigma^2 = \delta_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu$ {metric on S^3 }

$\left. \begin{aligned} & \underbrace{(\tilde{x}^4)^2}_{\cos^2\chi} + \underbrace{\delta_{ij} \tilde{x}^i \tilde{x}^j}_{\sin^2\chi} = 1 \\ & \underbrace{S^2 \text{ of radius } \sin\chi} \end{aligned} \right\}$
 set $\tilde{x}^i = \sin\chi \hat{x}^i$
 $\tilde{x}^4 = \cos\chi$ (same as above)

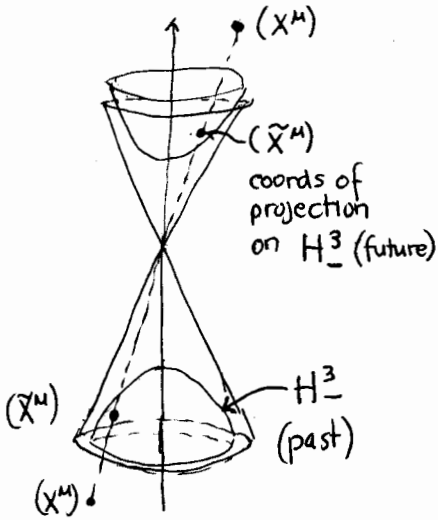
$d\Sigma^2 = \delta_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = \dots = d\chi^2 + \sin^2\chi \underbrace{\delta_{ij} d\hat{x}^i d\hat{x}^j}_{d\Omega^2}$

NOTE: R^3 and R^4 with the Euclidean metric are usually referred to as E^3 and E^4 .

pseudospherical coordinates on $M^4 = \text{Minkowski spacetime}$

$\mu, \nu = 0, 1, 2, 3$

INSIDE LIGHT CONE

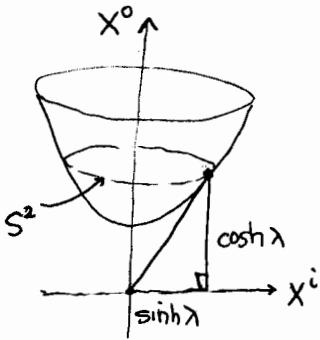


Define $\tau^2 = -\eta_{\mu\nu} X^\mu X^\nu$ $\begin{cases} \tau > 0 \text{ future} \\ \tau < 0 \text{ past} \end{cases}$
 and $\tilde{X}^\mu = X^\mu / \tau$ or $X^\mu = \tau \tilde{X}^\mu$,

then $\eta_{\mu\nu} \tilde{X}^\mu \tilde{X}^\nu = -1 \rightarrow 2\eta_{\mu\nu} \tilde{X}^\mu d\tilde{X}^\nu = 0$
 and $dx^\mu = \tilde{X}^\mu d\tau + \tau d\tilde{X}^\mu$,

$H^3_{\text{future/past}} = \{(X^\mu) \in R^4 \mid \eta_{\mu\nu} X^\mu X^\nu = -1, X^0 > 0\}$

$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \dots = -d\tau^2 + \tau^2 \underbrace{\eta_{\mu\nu} d\tilde{X}^\mu d\tilde{X}^\nu}_{d\Omega^2}$ metric on unit hyperboloid H^3



$-(\tilde{X}^0)^2 + (\delta_{ij} \tilde{X}^i \tilde{X}^j) = -1$
 $\underbrace{\quad}_{\cosh^2 \lambda} \quad \underbrace{\quad}_{\sinh^2 \lambda}$

S^2 of radius $\sinh \lambda$

set $\tilde{X}^i = \sinh \lambda \hat{X}^i$
 $\tilde{X}^0 = \cosh \lambda$

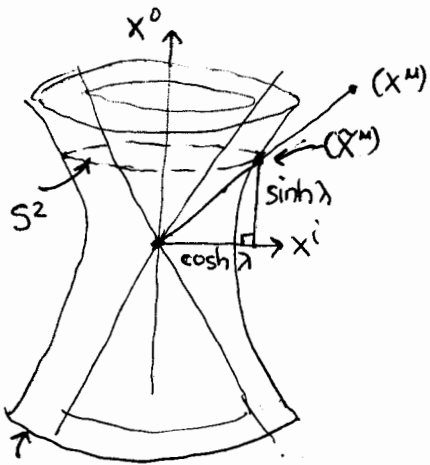
(same as above)

$d\tilde{X}^i = \cosh \lambda d\lambda \hat{X}^i + \sinh \lambda d\hat{X}^i$
 $d\tilde{X}^0 = \sinh \lambda d\lambda$

$d\Omega^2 = -d(\tilde{X}^0)^2 + \delta_{ij} d\tilde{X}^i d\tilde{X}^j = -\sinh^2 \lambda d\lambda^2 + \delta_{ij} (\cosh \lambda d\lambda \hat{X}^i + \sinh \lambda d\hat{X}^i) (\cosh \lambda d\lambda \hat{X}^j + \sinh \lambda d\hat{X}^j)$
 $= +d\lambda^2 (\underbrace{-\sinh^2 \lambda + \cosh^2 \lambda}_{1} \underbrace{\delta_{ij} \hat{X}^i \hat{X}^j}_{1}) + 2\sinh \lambda d\lambda \underbrace{\delta_{ij} \hat{X}^i d\hat{X}^j}_0 + \sinh^2 \lambda \underbrace{\delta_{ij} d\hat{X}^i d\hat{X}^j}_{d\Omega^2}$
 $= d\lambda^2 + \sinh^2 \lambda d\Omega^2$

$ds^2 = -d\tau^2 + \tau^2 [d\lambda^2 + \sinh^2 \lambda d\Omega^2]$

OUTSIDE LIGHT CONE



Define $\sigma^2 = \eta_{\mu\nu} X^\mu X^\nu$, $\tilde{X}^\mu = X^\mu / \sigma$ or $X^\mu = \sigma \tilde{X}^\mu$
 then $\eta_{\mu\nu} \tilde{X}^\mu \tilde{X}^\nu = 1$, $2\eta_{\mu\nu} \tilde{X}^\mu d\tilde{X}^\nu = 0$, $dx^\mu = \tilde{X}^\mu d\sigma + \sigma d\tilde{X}^\mu$

$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \dots = d\sigma^2 + \sigma^2 \underbrace{\eta_{\mu\nu} d\tilde{X}^\mu d\tilde{X}^\nu}_{d\Omega^2_+}$ metric on unit hyperboloid H^3_+

$-(\tilde{X}^0)^2 + (\delta_{ij} \tilde{X}^i \tilde{X}^j) = 1$
 $\underbrace{\quad}_{\sinh^2 \lambda} \quad \underbrace{\quad}_{\cosh^2 \lambda}$

S^2 of radius $\cosh \lambda \rightarrow \tilde{X}^i = \cosh \lambda \hat{X}^i$
 $\tilde{X}^0 = \sinh \lambda \rightarrow$

$H^3_{\pm} = \{(X^\mu) \in R^4 \mid \eta_{\mu\nu} X^\mu X^\nu = \pm 1\}$

$d\Omega^2_{\pm} = -d(\tilde{X}^0)^2 + \delta_{ij} d\tilde{X}^i d\tilde{X}^j = \dots = -d\lambda^2 + \cosh^2 \lambda d\Omega^2$

In each of the above cases we introduced a radial coordinate measuring distance from the origin and a family of pseudospheres (spheres in the positive definite case) orthogonal to the radial direction. A given pseudosphere is characterized by a fixed value of the radial coordinate, say the constant \mathcal{R} ,

i.e.

$$r = \mathcal{R} \quad (E^3) \quad \tau = \mathcal{R} \quad (M^4)$$

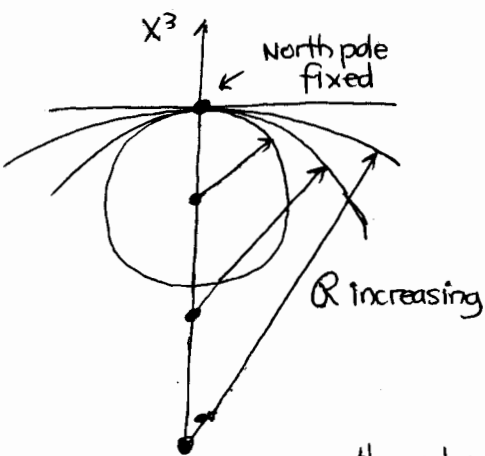
$$\lambda = \mathcal{R} \quad (E^4) \quad \sigma = \mathcal{R}$$

The metric on these pseudospheres is obtained from the metric on the full flat space by setting the differential of the radial coordinate to zero:

$$dr=0, \quad d\lambda=0, \quad \underbrace{d\tau=0, \quad d\sigma=0}_{\mathcal{R}^2 d\Omega_{\pm}^2}$$

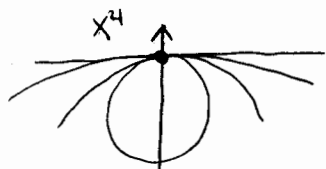
$$ds^2 = \mathcal{R}^2 d\Omega^2, \quad \mathcal{R}^2 d\Sigma^2, \quad \mathcal{R}^2 d\Omega_{\pm}^2.$$

All of these pseudospheres are spaces of constant curvature $k = \pm 1/\mathcal{R}^2$; the unit pseudospheres S^2, S^3, H_{\pm}^3 have curvature $k = \pm 1$.



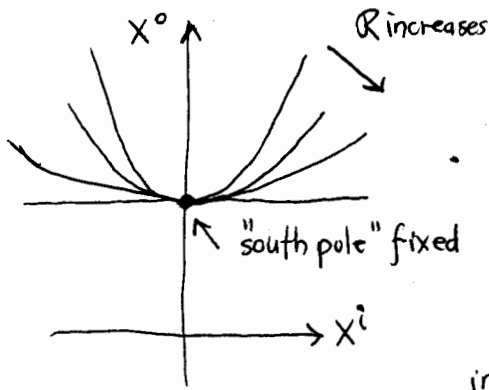
The 2-sphere of radius \mathcal{R} is a Riemannian (positive definite metric) space of constant positive curvature $k = 1/\mathcal{R}^2$. If one considers a family of spheres with the same north pole, but increasing \mathcal{R} , those with small \mathcal{R} have large curvature, those with large \mathcal{R} have small curvature, and in the limit $\mathcal{R} \rightarrow \infty$

the sphere flattens out into the plane $x^3 = \text{constant}$ which has zero curvature.

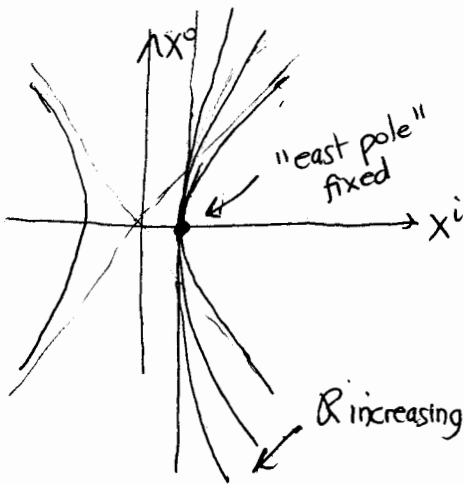


The 3-sphere of radius \mathcal{R} is a Riemannian manifold of constant positive curvature $k = 1/\mathcal{R}^2$. As above, in the limit $\mathcal{R} \rightarrow \infty$ with north pole fixed, the sphere flattens out into the hyperplane $x^4 = \text{constant}$ which is just E^3 of zero curvature.

("spacelike")



The future and past pseudospheres of constant $\tau = \mathcal{R}$ in M^4 are Riemannian spaces of constant negative curvature $k = -\frac{1}{\mathcal{R}^2}$ ($d\Gamma^2$ is positive definite, H^3 is a spacelike hypersurface in M^4). Again increasing \mathcal{R} with a fixed "south" (future) or "north" (past) pole, in the limit $\mathcal{R} \rightarrow \infty$, the spacelike pseudosphere flattens out into the flat hypersurface $x^0 = \text{constant}$, i.e. E^3 just as in the case of the 3-sphere in E^4 . (All of these are simply connected)



The timelike pseudospheres (timelike since they contain 1 timelike and 2 spacelike directions) have a Lorentz metric

$$\mathcal{R}^2 d\Gamma_+^2 = \mathcal{R}^2 (-dx^2 + \cosh^2 \lambda d\ell^2)$$

\uparrow 1 timelike \uparrow 2 spacelike

They are pseudo-Riemannian (Lorentz) spaces of constant positive curvature $k = 1/\mathcal{R}^2$, i.e.

curved 3-dimensional spacetimes (neglecting one spacelike coordinate). (3-dim DeSitter spacetime)

In fact if we generalize to pseudospheres in 5-dimensions, we can obtain the 4-dimensional spacetimes of constant curvature $k = \pm 1/\mathcal{R}^2$. The positive case is called the DeSitter universe and the negative case the Anti-DeSitter universe, both of which flatten out to Minkowski spacetime in the limit $\mathcal{R} \rightarrow \infty$, which has of course $k=0$.

On the other hand the spacelike pseudospheres in 4-dimensions, together with the flat limit E^3 , provide the geometry of the space sections of the Friedmann-Robertson-Walker spacetimes.

Riemannian spaces of constant curvature are maximally symmetric.

They are both homogeneous and isotropic:

motions
or
isometries
of the
space
(transformations
which leave the
metric invariant)

homogeneous: there exists a translation which can move a point to any other point of the space, without changing any relative distances or angles, i.e. so that the metric (and therefore the geometry) of the space is invariant

isotropic: at any given point there exists a rotation about the point which maps any given direction to any other direction; when expressed as a linear transformation (of the directions) in an orthonormal basis, the matrix of such a rotation lies in the special orthogonal group of the same dimension as the space

An n -dimensional space must have an n -dimensional group of translations for homogeneity. The dimension of the group of rotations about each point may be calculated in two ways:

$$(1) = \dim(SO(n, \mathbb{R})) = \text{number of linearly independent antisymmetric } n \times n \text{ matrices (which generate the rotations)} = \frac{n(n-1)}{2}$$

$$(2) = \text{number of linearly independent } 2\text{-planes in } n\text{-dimensions} \\ (\text{each rotation occurs in a } 2\text{-plane}) \\ = \binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

The dimension of the full group of motions is therefore

$$r = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

For E^3 there are 3 translations and 3 rotations. ($r=6$)

For E^4 there are 4 translations and 6 rotations. ($r=10$)

For S^2 there are 2 translations and 1 rotation ($r=3$), but notice there is no 2-dimensional subgroup of the rotations of S^2 that can be called a group of translations — each translation is in fact a rotation about some point.

For S^3 there are 3 translations and 3 rotations but it turns out that there are 2 independent translation subgroups which commute with each other

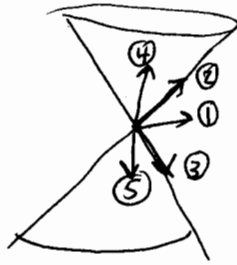
For H^3 there are 3 rotations but a several parameter family of different translation subgroups. Thus in curved spaces the notion of "the group of translations" is ambiguous, if such a group exists at all.

For a pseudo-Riemannian space (indefinite metric), the only difference is that some of the rotations are replaced by boosts (pure Lorentz transformations); the rotations and boosts will be referred to in general as pseudorotations. Of course as far as isotropy is concerned, the causality structure of the space must be respected, i.e. timelike directions remain timelike under pseudorotation and spacelike directions remain spacelike, etc.

Example: M^4

The space of directions at the origin decomposes into 5 distinct subspaces of

- 1) spacelike directions
- 2) future directed null directions
- 3) past directed null directions
- 4) future directed timelike directions
- 5) past directed timelike directions.



origin decomposes into equivalent directions:

- 2) future directed null
- 4) future directed timelike

FRIEDMANN-ROBERTSON-WALKER UNIVERSES (FRW)

The FRW Universes are spacetimes whose space sections are Riemannian 3-spaces of constant curvature, which as hypersurfaces in the spacetime are "geodesically parallel" exactly as in the case of the pseudospheres of E^3 , E^4 and M^4 . The curvature can be positive, zero or negative but will now be a function of time determined by the Einstein equations. The only difference one must make in the Minkowski metric is to replace the Euclidean metric $\delta_{ij} dx^i dx^j$ by the metric of a space of constant curvature with time dependent scale factor $\mathcal{R}(t)$:

$$ds^2 = -dt^2 + \mathcal{R}^2(t) \begin{pmatrix} d\Sigma^2 \\ \delta_{ij} dx^i dx^j \\ d\Gamma^2 \end{pmatrix} \quad \begin{array}{l} k=1 \leftrightarrow S^3 \\ k=0 \leftrightarrow E^3 \\ k=-1 \leftrightarrow H^3 \end{array}$$

The scale factor $\mathcal{R}(t)$ scales the constant spatial curvature from the values $k=1, 0, -1$ to $k = 1/\mathcal{R}^2, 0, -1/\mathcal{R}^2$.

CONFORMAL TIME

By defining $dt = \mathcal{R}(t) d\tau$ or $\tau = \int_0^t \frac{dt}{\mathcal{R}(t)}$ one obtains:

$$ds^2 = \bar{\mathcal{R}}^2(\tau) \left(-d\tau^2 + \begin{pmatrix} d\Sigma^2 \\ \delta_{ij} dx^i dx^j \\ d\Gamma^2 \end{pmatrix} \right)$$

Note that for $k=0$, the metric is proportional to the metric of Minkowski space, i.e. it is conformally flat.

For $k=1$, the metric is conformal to the metric $-d\tau^2 + d\Sigma^2$, namely $R \times S^3$ with the natural metric (the Einstein cylindrical Universe)

This is also conformal to Minkowski spacetime, as is the case $k=-1$ (HAWKING & ELLIS, for example).

In particular any conformally invariant equations can be solved on Minkowski spacetime (easy) and conformally transformed to the FRW spacetimes. Maxwell's equations and the zero mass Dirac equation are such equations. (In general, zero mass particles of arbitrary spin have conformally invariant field equations).

EXAMPLE: The flat spaces with metric of arbitrary signature

Let $M^{p,q} = \mathbb{R}^N$, where $p, q \geq 0$ are positive integers and $p+q=N$, with flat metric: $ds^2 = {}^{(p,q)}\eta_{\mu\nu} dx^\mu dx^\nu$ ($\mu, \nu = 1, \dots, N$)

where: ${}^{(p,q)}\eta = ({}^{(p,q)}\eta_{\mu\nu}) = \text{diag} \left(\underbrace{1, \dots, 1}_p, \underbrace{-1, \dots, -1}_q \right)$
spacelike directions timelike directions

These contain the special cases:

$E^N = M^{N,0}$ = Euclidean space (or $M^{0,N}$ if we change the signature)

$M^N = M^{N-1,1}$ (or $M^{1,N-1}$ allowing the opposite signature)
 = N-dimensional Minkowski space

The remaining flat pseudoriemannian spaces have more than 1 equivalent (independent) timelike directions and hence have no distinction between future and past timelike directions as in Minkowski space. [In other words the spacelike pseudospheres are connected hypersurfaces rather than having 2 disjoint components as in Minkowski space]

The metric can be written:

$$ds^2 = \left(\sum_{\mu, \nu=1}^p \delta_{\mu\nu} dx^\mu dx^\nu \right) - \left(\sum_{\mu, \nu=p+1}^{p+q} \delta_{\mu\nu} dx^\mu dx^\nu \right).$$

This is clearly invariant under rotations of the timelike directions among themselves (except for E^N and M^N which are degenerate cases) and of the spacelike directions among themselves. The remaining independent pseudorotations are boosts involving a timelike and spacelike direction.

The pseudo-orthogonal group $O(p,q)$ consists of all linear transformations $x^\mu \rightarrow L^\mu{}_\nu x^\nu$ such that the metric is invariant, i.e.

$$ds^2 \rightarrow \eta_{\mu\nu} L^\mu{}_\alpha L^\nu{}_\beta dx^\alpha dx^\beta = \eta_{\alpha\beta} dx^\alpha dx^\beta$$

i.e. $\eta_{\mu\nu} L^\mu{}_\alpha L^\nu{}_\beta = \eta_{\alpha\beta}$

more precisely: $O(p,q) = \{ (L^\mu{}_\nu) \in GL(N, \mathbb{R}) \mid \eta_{\mu\nu} L^\mu{}_\alpha L^\nu{}_\beta = \eta_{\alpha\beta} \}$

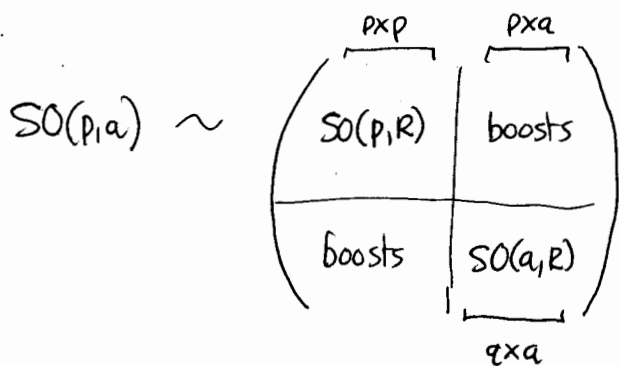
↑
nonsingular matrices in N dimensions

Note: $\det(\Lambda_{\mu\nu}) [\det(L^{\mu\nu})]^2 = \det(\Lambda_{\mu\nu}) \rightarrow \det(L^{\mu\nu}) = \pm 1$.

Set $SO(p,q) = \{L^{\mu\nu} \in O(p,q) \mid \det(L^{\mu\nu}) = 1\}$ = special pseudo-orthogonal group

[$SO(p,q)$ consists of 2 disconnected pieces when both p and q are odd]
see p. T99 of GILMORE.

special cases: $\begin{cases} q=0 & SO(N,0) \equiv SO(N,R) \text{ special orthogonal group} \\ q=1 & SO(N-1,1) = \text{Lorentz or DeSitter group} \end{cases}$



contains $SO(p,R)$ and $SO(q,R)$ which rotate the spacelike and timelike directions among themselves plus boosts in each pair of timelike and spacelike directions

$$\dim SO(p,q) = \underbrace{\dim SO(p,R)}_{= \frac{p(p-1)}{2}} + \underbrace{\dim SO(q,R)}_{= \frac{q(q-1)}{2}} + \underbrace{p \cdot q}_{= \text{number of independent pairs of 1 spacelike and 1 timelike direction.}}$$

The rotations $SO(p,R) \times SO(q,R)$ form a subgroup of $SO(p,q)$ but the boosts do not; boosts along different directions combine to give a boost plus a rotation (Wigner rotation)

Adding the N translations $x^\mu \rightarrow x^\mu + a^\mu$ to the pseudo-orthogonal group $O(p,q)$ yields the full group of motions of $M^{p,q}$ called the inhomogeneous pseudo-orthogonal group $IO(p,q)$ (and the inhomogeneous special pseudo-orthogonal group $ISO(p,q)$) of dimension $\frac{N(N-1)}{2} + N = \frac{N(N+1)}{2} = r$.

More precisely these may be realized as matrix groups in $N+1$ dimensions by adding a trivial row and a nontrivial column to $O(p,q)$

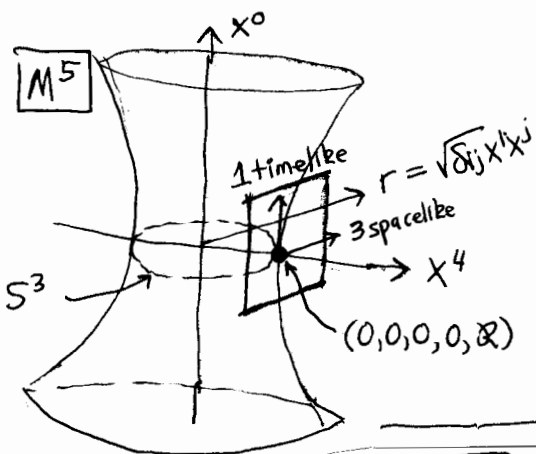
$$IO(p,q) \sim \left(\begin{array}{c|c} O(p,q) & \begin{matrix} a^1 \\ \vdots \\ a^N \end{matrix} \\ \hline 0 \dots 0 & 1 \end{array} \right)$$

special cases: $q=0$ $IO(N,0) = IO(N,R) =$ Euclidean group in N dimensions
 $q=1$ $IO(N-1,1) =$ Poincare group in N dimensions
 (or inhomogeneous Lorentz group)

The pseudospheres at the origin of $M^{p,q}$ satisfy
 $\eta_{\mu\nu} X^\mu X^\nu = \text{constant}$.

Each connected component of a pseudosphere is an $(N-1)$ -dimensional hypersurface on which an $\frac{N(N-1)}{2} = \frac{(N-1)(N-1+1)}{2}$ dimensional group acts under which all points are equivalent, so they are all spaces of constant curvature and of various signatures (which can be easily determined by considering coordinate directions at their intersection with one of the cartesian coordinate axes)

When $N=5$ we can obtain pseudospheres of Lorentz signature and thus 4-dimensional spacetimes of constant curvature. There are only 2 inequivalent indefinite signatures for $N=5$: $(p,q) = (4,1) \sim (1,4)$ and $(3,2) \sim (2,3)$, so it suffices to consider $M^5 = M^{4,1}$ and $M^{3,2}$.

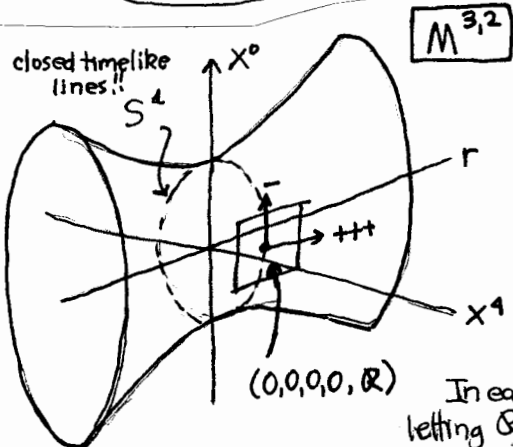


Let $\mu\nu = 0,1,2,3,4$ with 0 timelike. The pseudosphere of unit spacelike directions
 $\eta_{\mu\nu} X^\mu X^\nu = -(x^0)^2 + \underbrace{\delta_{ij} x^i x^j}_{\equiv r^2} + (x^4)^2 = R^2$

has Lorentz signature (see diagram) and turns out to have constant positive curvature $k = 1/R^2$.

DE SITTER SPACETIME

ANTI DE SITTER SPACETIME



Again let $\mu,\nu = 0,1,2,3,4$ but 0,4 timelike. The pseudosphere of unit timelike directions

$$\eta_{\mu\nu} X^\mu X^\nu = \underbrace{-(x^0)^2 + (x^4)^2}_{\equiv r^2} + \delta_{ij} x^i x^j = R^2$$

has Lorentz signature and turns out to have constant negative curvature $k = -1/R^2$.

error in Hawking & Ellis

In each case keeping the reference point of the diagram fixed but letting $R \rightarrow \infty$ flattens these pseudospheres out into a hyperplane i.e. $M^4 =$ Minkowski spacetime.

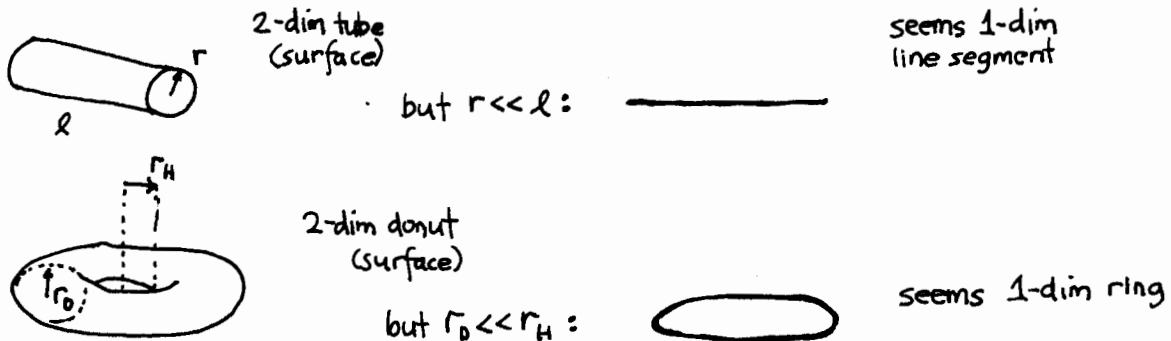
Tangent space at $(0,0,0,0,R)$ in either diagram is: $x^4 = R$ which is Minkowski spacetime

Why consider $N > 5$?

Many people are considering unified field theories (^{generalized} Kaluza-Klein) on spacetimes with extra dimensions which spontaneously compactify to very small dimensions so that the result is essentially 4-dimensional. The majority of these people are elementary particle physicists.

In quantum field theory dimensional regularization lets the dimension of spacetime be a real number. Other techniques involve substituting the nonsemisimple Poincare group by the semisimple DeSitter group and then contracting the latter group to the Poincare group at the end of the calculation. (What happens to the DeSitter group when $R \rightarrow \infty$ flattens out DeSitter space.) Higher dimensional Riemannian spaces are relevant to Euclidean path integrals in quantum gravity.

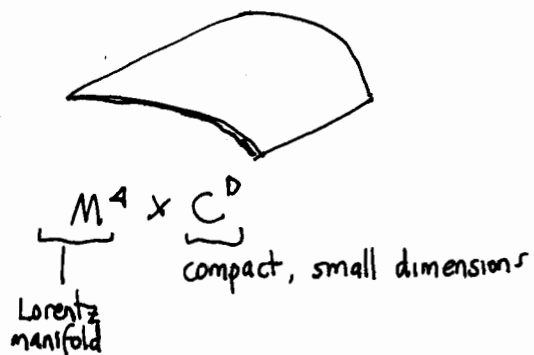
EFFECTIVE DIMENSION OF SPACETIME (?)



4+D dim spacetime

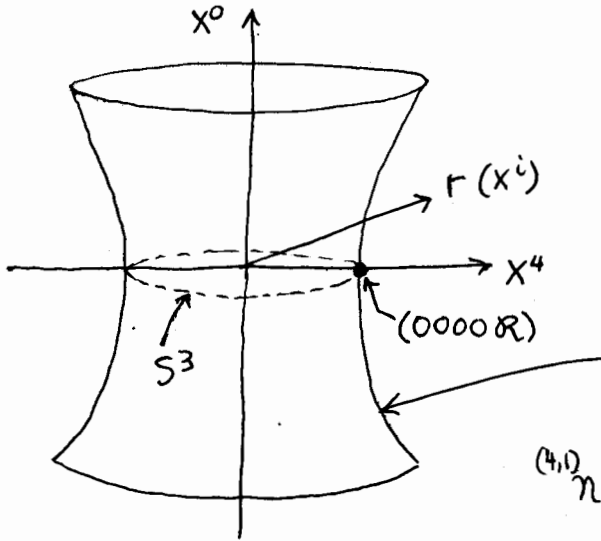
?

but now seems:



SPECIAL COORDINATES ON DESITTER SPACETIME

Because of GUTS and inflation, as will be described later, DeSitter spacetime is of special importance. Using the various pseudospherical coordinates on E^3, E^4 , and M^4 , all of which are contained in M^5 , we can introduce 3 kinds of special coordinates on the DeSitter hyperboloid in M^5 .



Let $\alpha, \beta = 0, 1, 2, 3, 4$ with 0 timelike

$$A, B = 1, 2, 3, 4$$

$$M, N = 0, 1, 2, 3$$

$$i, j = 1, 2, 3$$

$$H_+^4 = \{ (x^\alpha) \in \mathbb{R}^5 \mid {}^{(4,1)}\eta_{\alpha\beta} x^\alpha x^\beta = \mathcal{R}^2 \}$$

$$\delta_{AB} X^A X^B = \mathcal{R}^2$$

$${}^{(4,1)}\eta_{\alpha\beta} x^\alpha x^\beta = -(x^0)^2 + \underbrace{\delta_{ij} x^i x^j}_{r^2} + (x^4)^2$$

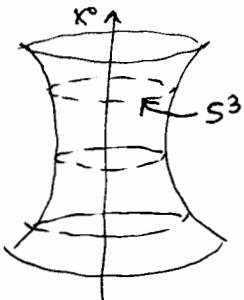
$$\eta_{\mu\nu} x^\mu x^\nu = -\mathcal{R}^2 = \sigma^2$$

$$\delta_{AB} dx^A dx^B = dr^2 + r^2 d\Sigma^2$$

$$ds^2 = {}^{(4,1)}\eta_{\alpha\beta} dx^\alpha dx^\beta = -(dx^0)^2 + \underbrace{\delta_{ij} dx^i dx^j}_{dr^2 + r^2 d\Sigma^2} + (dx^4)^2$$

$$\eta_{\mu\nu} dx^\mu dx^\nu = -d\mathcal{R}^2 + \mathcal{R}^2 d\Sigma^2$$

1) SLICING OF HYPERBOLOID BY FAMILY OF PARALLEL SPACELIKE HYPERPLANES $x^0 = \text{constant}$ (Cauchy slicing by compact space sections $\sim S^3$).



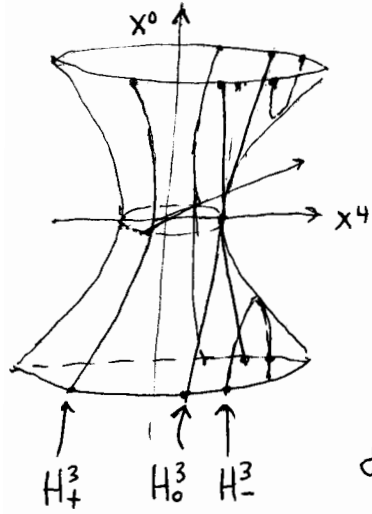
$$-(x^0)^2 + \mathcal{R}^2 = \mathcal{R}^2 \rightarrow \begin{cases} x^0 = \mathcal{R} \sinh \lambda \\ \mathcal{R} = \mathcal{R} \cosh \lambda \end{cases} \text{ where } x^A = \mathcal{R} X^A$$

$$(\delta_{AB} X^A X^B = 1 \leftrightarrow S^3)$$

$$ds^2 = -(dx^0)^2 + (dr^2 + r^2 d\Sigma^2) = -\mathcal{R}^2 \cosh^2 \lambda d\lambda^2 + (\mathcal{R}^2 \sinh^2 \lambda d\lambda^2 + \mathcal{R}^2 \cosh^2 \lambda d\Sigma^2)$$

$$= -\mathcal{R}^2 d\lambda^2 + \mathcal{R}^2 \cosh^2 \lambda d\Sigma^2 = -dt_+^2 + \mathcal{R}^2 \cosh^2(\mathcal{R}^{-1} t_+) \underbrace{d\Sigma^2}_{\text{metric on } S^3}$$

2) SLICING OF HYPERBOLOID BY FAMILY OF PARALLEL TIMELIKE HYPERPLANES $x^4 = \text{constant}$



$$\cdot |x^4| \begin{cases} > \mathcal{R} & \text{(i) } \tau^2 = -\eta_{\mu\nu} X^\mu X^\nu > 0 \sim H_-^3 & \text{(2 disjoint spacelike) pseudospheres} \\ = \mathcal{R} & \eta_{\mu\nu} X^\mu X^\nu = 0 \sim H_0^3 & \text{(null cone)} \\ < \mathcal{R} & \text{(ii) } \sigma^2 = \eta_{\mu\nu} X^\mu X^\nu > 0 \sim H_+^3 & \text{(timelike pseudosphere)} \end{cases}$$

$$\text{(i) } -\tau^2 + (x^4)^2 = \mathcal{R}^2 \rightarrow \begin{cases} x^4 = \pm \mathcal{R} \cosh \lambda \\ \tau = \mathcal{R} \sinh \lambda \end{cases}, \text{ where } x^A = \tau \tilde{x}^A \quad (\eta_{\mu\nu} \tilde{x}^\mu \tilde{x}^\nu = -1 \leftrightarrow H_-^3)$$

$$ds^2 = (-d\tau^2 + \tau^2 d\Gamma_-^2) + (dx^4)^2 = (-\mathcal{R}^2 \cosh^2 \lambda d\lambda^2 + \mathcal{R}^2 \sinh^2 \lambda d\Gamma_-^2) + \mathcal{R}^2 \sinh^2 \lambda d\lambda^2 \\ = -\mathcal{R}^2 d\lambda^2 + \mathcal{R}^2 \sinh^2 \lambda d\Gamma_-^2 = -dt_-^2 + \mathcal{R}^2 \sinh^2(\mathcal{R}^{-1} t_-) d\Gamma_-^2 \quad \textcircled{C}$$

metric on H_-^3

This represents 4 disjoint coordinate patches ($\tau \lesssim 0, x^4 > \mathcal{R}$ or $x^4 < -\mathcal{R}$) which cover the region $|x^4| > \mathcal{R}$.

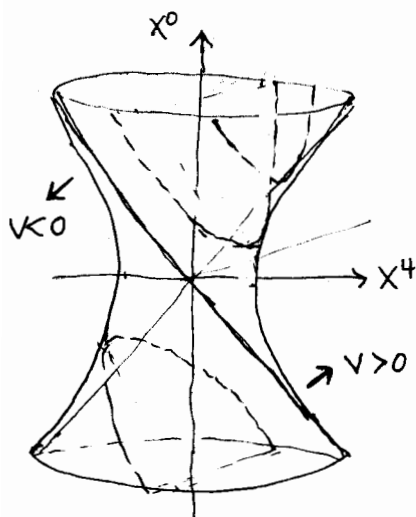
There remains a single patch for $|x^4| < \mathcal{R}$

$$\text{(ii) } \sigma^2 + (x^4)^2 = \mathcal{R}^2 \rightarrow \begin{cases} x^4 = \mathcal{R} \cos \lambda \\ \sigma = \mathcal{R} \sin \lambda \end{cases}$$

$$ds^2 = (d\sigma^2 + \sigma^2 d\Gamma_+^2) + (dx^4)^2 = \dots = \mathcal{R}^2 d\lambda^2 + \mathcal{R}^2 \sin^2 \lambda d\Gamma_+^2 \quad \textcircled{D}$$

The hypersurfaces $\lambda = \text{constant}$ are timelike hypersurfaces in De Sitter spacetime which continue the spacelike hypersurfaces $t_- = \text{constant}$ through the null hypersurface $t_- = 0$.

3) SLICING OF HYPERBOLOID BY FAMILY OF PARALLEL NULL HYPERPLANES $x^0 + x^4 = \text{constant}$



Introduce null coordinates $\begin{cases} v = x^0 + x^4 \\ u = x^0 - x^4 \end{cases}$

$$(4.1) \eta_{\alpha\beta} dx^\alpha dx^\beta = -uv + r^2 = \mathcal{Q}^2$$

$$ds^2 = -dudv + \underbrace{\delta_{ij} dx^i dx^j}_{dr^2 + r^2 d\Omega^2}$$

set $v = \pm \mathcal{Q} e^\lambda$, $r = |\mathcal{Q}| \tilde{r}$, $x^i = \mathcal{Q} \tilde{x}^i$

then

$$-uv + v^2 \tilde{r}^2 = \mathcal{Q}^2 \rightarrow u = v \tilde{r}^2 - v^{-1} \mathcal{Q}^2 = \pm \mathcal{Q} (e^\lambda \tilde{r}^2 - e^{-\lambda})$$

$$du = \pm \mathcal{Q} (e^\lambda \tilde{r}^2 + e^{-\lambda}) d\lambda \pm 2\mathcal{Q} e^\lambda \tilde{r} d\tilde{r}$$

$$dv = \pm \mathcal{Q} e^\lambda d\lambda$$

$$dr = \mathcal{Q} e^\lambda (d\tilde{r} + \tilde{r} d\lambda)$$

$$ds^2 = - [\mathcal{Q} (e^\lambda \tilde{r}^2 + e^{-\lambda}) d\lambda + 2\mathcal{Q} e^\lambda \tilde{r} d\tilde{r}] [\mathcal{Q} e^\lambda d\lambda] + [\mathcal{Q} e^\lambda (d\tilde{r} + \tilde{r} d\lambda)]^2 + \mathcal{Q}^2 e^{2\lambda} \tilde{r}^2 d\Omega^2$$

$$= \dots = -\mathcal{Q}^2 d\lambda^2 + \mathcal{Q}^2 e^{2\lambda} \underbrace{(d\tilde{r}^2 + \tilde{r}^2 d\Omega^2)}_{\text{metric on } E^3}$$

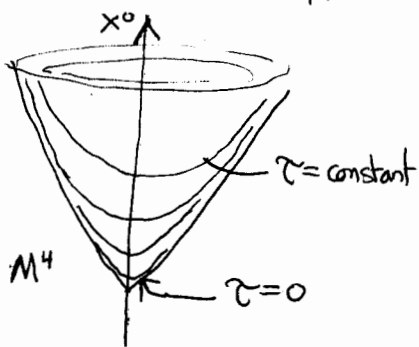
$$= -dt_0^{\pm 2} + \mathcal{Q}^2 e^{\pm 2\mathcal{Q}^{-1} t_0^{\pm}} \underbrace{(\delta_{ij} d\tilde{x}^i d\tilde{x}^j)}_{\text{metric on } E^3} \quad \textcircled{E}$$

$t_0^- \in (-\infty, \infty]$ covers $v \in (-\infty, 0)$ } two disjoint
 $t_0^+ \in [-\infty, \infty)$ covers $v \in (0, \infty)$ } coordinate patches

Thus we have 2 families of spacelike hypersurfaces of DeSitter spacetime separated by a null hypersurface. ($v=0$)

Notice that all of these special coordinates represent all or part of DeSitter spacetime as FRW spacetimes, i.e. represent spatially homogeneous and isotropic slicings of DeSitter spacetime. Except for case 1) (metric (A)), all of these FRW representations reveal fictitious cosmological singularities where the spacelike hypersurfaces of the slicing become null and then either timelike or again spacelike. However, from the form of the metric alone, it is not obvious that these are not true physical singularities (big bangs) as occur in the standard models.

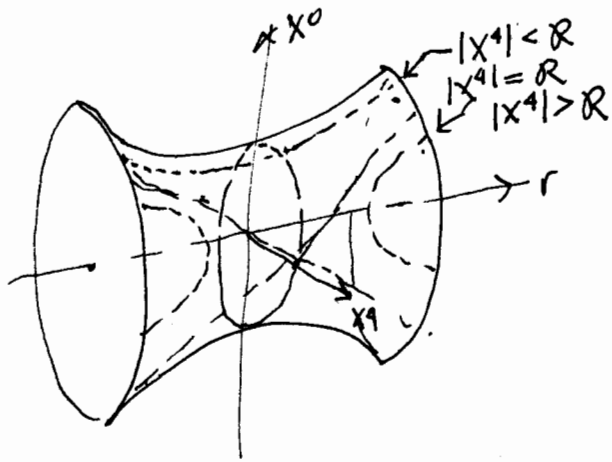
Examination of curvature invariants of these metrics show them to be well behaved at these singularities but the conformal techniques of Penrose (Hawking & Ellis) actually show how to continue the spacetime past these coordinate singularities. In fact pseudospherical coordinates represent the interior of the future null cone of Minkowski spacetime as a $k=-1$ FRW model with an apparent singularity at $\tau=0$:



$$ds^2 = -d\tau^2 + \tau^2 d\Omega^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

However, the classical singularity theorems of Penrose and Hawking predict that the universe must have had a true singularity in the past at which the curvature of spacetime becomes infinite and the laws of physics break down. In other words classical general relativity must break down under such conditions and one would expect that quantum gravitational effects would become significant.

SPECIAL COORDINATES ON ANTI-DESITTER SPACETIME: $-(x^0)^2 - (x^4)^2 + \delta_{ij} x^i x^j = -R^2$



slicing by constant x^4

SPACELIKE SLICING $|x^4| < R$

$$-(x^4)^2 + r^2 = -R^2$$

$$\begin{cases} x^4 = R \cos \lambda \\ r = R \sin \lambda \end{cases}$$

$$ds^2 = -(dx^4)^2 + dr^2 + r^2 d\Gamma^2$$

$$= \dots = -R^2 d\lambda^2 + R^2 \sin^2 \lambda d\Gamma^2$$

$$= -dt^2 + R^2 \sin^2(R^{-1}t) d\Gamma^2$$

TIMELIKE SLICING $|x^4| > R$

$$-(x^4)^2 + \sigma^2 = -R^2$$

$$\begin{cases} x^4 = R \cosh \lambda \\ \sigma = R \sinh \lambda \end{cases}$$

$$ds^2 = -(dx^4)^2 + d\sigma^2 + \sigma^2 d\Gamma_+^2$$

$$= R^2 d\lambda^2 + R^2 \sinh^2 \lambda d\Gamma_+^2$$

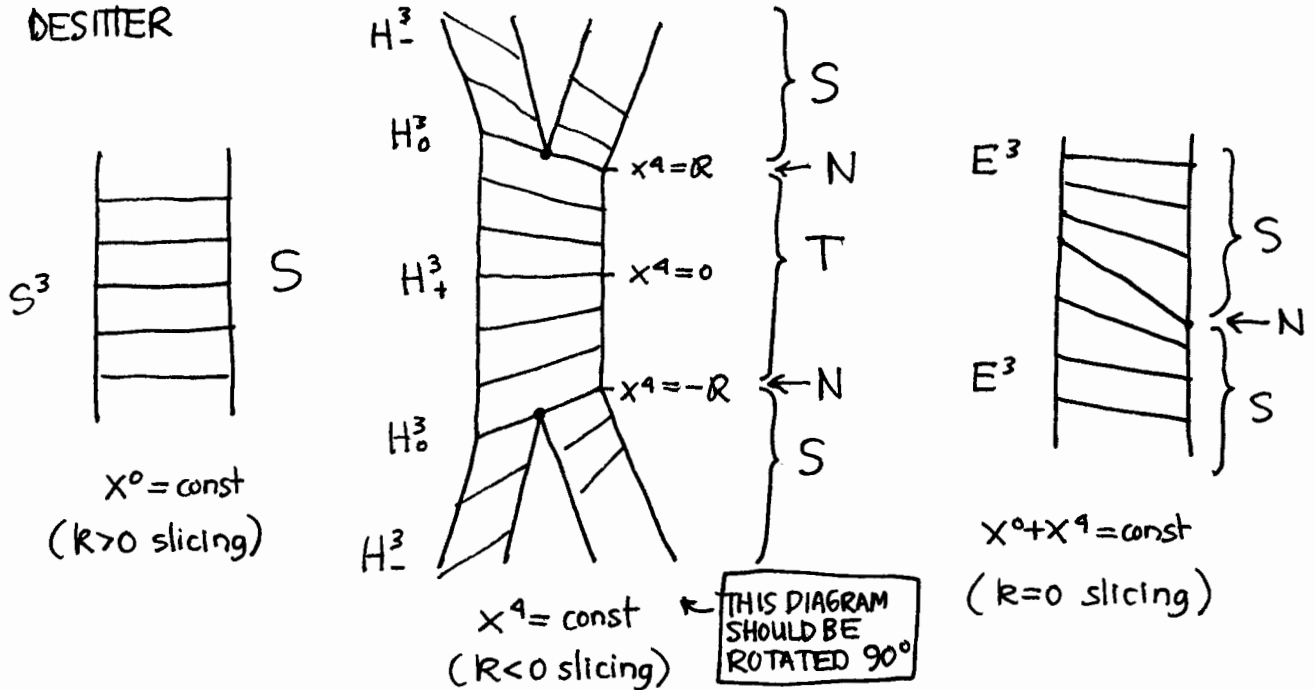
So this yields a $k=-1$ FRW spatially homogeneous slicing which becomes null and then timelike at the beginning $t=0$ and the end $R^{-1}t = \pi$.

As an exercise you might try other constant curvature slicings of anti-deSitter spacetime. You will not find the result in any text or article that I know of.

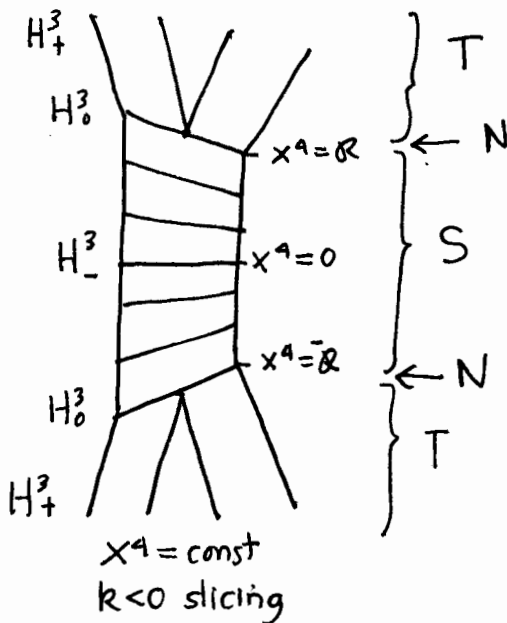
SUGGESTIVE SPACETIME DIAGRAMS

One can use suggestive 2-dimensional diagrams to describe the various slicings of deSitter and anti-deSitter spacetimes, using one time and one spatial coordinate, indicating null directions by a 45° angle (Penrose diagrams). (time vertical, space horizontal)

DESITTER

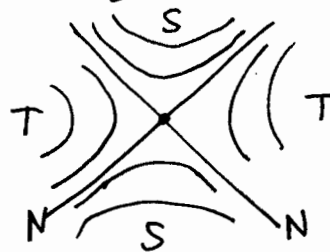


ANTIDESITTER



Here S, N, T on the right means spacelike, null and timelike sections or slices and the (conformal) geometry is indicated by the explicit manifolds on the left.

Of these, only the first is really right topologically. For example, the $R = 0$ deSitter slicing is best described by



and we have only given half above. Only the $R > 0$ deSitter slicing is singularity free.

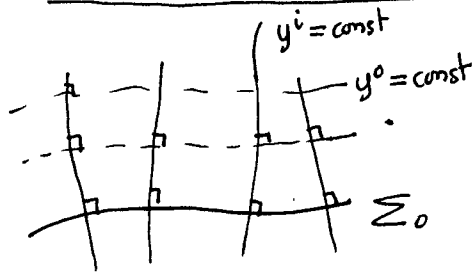
Notice how difficult it is to reconstruct the global structure from local slicings.

PSEUDO-SPHERICAL COORDINATES

In each of the examples of pseudo-spherical coordinates, we started with a flat space and decomposed it locally into two orthogonal families of subspaces — the pseudospheres (of constant curvature) about a given point and the radial geodesics orthogonal to them. This enables us to project all points to the unit pseudo-sphere(s) which we in turn decomposed locally into two orthogonal families of subspaces — the pseudospheres (of constant curvature) about a given point and the radial geodesics orthogonal to them relative to the intrinsic geometry of the unit pseudo-sphere of the original flat space, and so on iteratively until the process stops at 1-dimensional pseudo-spheres. The DeSitter/AntiDeSitter spacetimes start at the second step since they are themselves pseudospheres in the next higher dimension.

This iterative slicing of these spaces seems to be characterized by the intrinsic geometry of the slices, but they may also be characterized by the extrinsic geometry of the slices, namely, how the slices bend in the enveloping space in which they sit. To discuss this one needs to consider how to decompose metric and curvature into "radial and tangential parts", or into "extrinsic and intrinsic parts". This is accomplished with Gaussian normal coordinates.

GAUSSIAN NORMAL COORDINATES IN (N+1)-DIMENSIONAL SPACE



Take initial hypersurface Σ_0 with a nondegenerate metric (spacelike or timelike in Lorentz case) and let y^0 measure the signed length along the normal geodesics which will be $y^0 = \text{const}$ for any local coordinates on Σ_0 .

Then $ds^2 = \epsilon (dy^0)^2 + g_{ij} dy^i dy^j = g_{\alpha\beta} dx^\alpha dx^\beta$
 $\begin{cases} \alpha, \beta = 0, 1, \dots, N \\ i, j = 1, \dots, N \end{cases}$

i.e. $g_{00} = \epsilon = \begin{cases} 1, & y^0 \text{ lines spacelike } (\Sigma_0 \text{ timelike in Lorentz case}) \\ -1, & y^0 \text{ lines timelike } (\Sigma_0 \text{ spacelike in Lorentz case}) \end{cases}$

and $g_{0i} = 0$.

Define the extrinsic curvature (a spatial tensor like the intrinsic metric g_{ij})

$K_{ij} = -\frac{1}{2} \frac{\partial}{\partial y^0} g_{ij}$ (rate of change of intrinsic metric in the direction perpendicular to the slices)

This tells us how a given slice is bent in the enveloping space as will be described below.

Now evaluate Christoffel symbols:

$\Gamma^{\alpha\beta\gamma} = \frac{1}{2} (\partial_\beta g_{\gamma\alpha} + \partial_\gamma g_{\alpha\beta} - \partial_\alpha g_{\beta\gamma})$

$\Gamma^0_{ij} = K_{ij}$
 $\Gamma^i_{j0} = \Gamma^i_{0j} = -K^i_j$
 $\Gamma^i_{jk} = \text{Christoffel symbols of intrinsic metric}$

} only nonzero components

Raise first index:

$\Gamma^0_{ij} = \epsilon K_{ij}$
 $\Gamma^i_{j0} = \Gamma^i_{0j} = -K^i_j$
 $\Gamma^i_{jk} = \text{(ditto)}$

Now evaluate Riemann curvature; only nonzero components are $\underbrace{R^0_{ijk}, R^i_{0jk}, R^i_{j0k}, R^i_{jok}}_{1 \text{ zero}}, \underbrace{R^0_{ipk}, R^i_{0ok}, \dots}_{2 \text{ zero's}}$

and no zeros, namely the intrinsic components:

$R^i_{jke} = \partial_k \Gamma^i_{ej} - \partial_e \Gamma^i_{kj} + \Gamma^i_{k\alpha} \Gamma^\alpha_{ej} - \Gamma^i_{e\alpha} \Gamma^\alpha_{kj}$
 $\quad \quad \quad = \Gamma^i_{km} \Gamma^m_{ej} - \Gamma^i_{em} \Gamma^m_{kj} + \Gamma^i_{k0} \Gamma^0_{ej} - \Gamma^i_{e0} \Gamma^0_{kj}$
 $\quad \quad \quad = \Gamma^i_{km} \Gamma^m_{ej} - \Gamma^i_{em} \Gamma^m_{kj} - \epsilon K^i_k K_{ej} + \epsilon K^i_e K_{kj}$

* R^i_{jke} (Riemann tensor of g_{ij})

leading to the formula

$$R^i{}_{jke} = {}^*R^i{}_{jke} - \epsilon (K^i{}_k K_{je} - K^i{}_e K_{jk})$$

or $R^i{}_{jke} = {}^*R^i{}_{jke} - \epsilon (K^i{}_k K^j{}_e - K^i{}_e K^j{}_k)$

↑
intrinsic components of

$$= {}^*R^i{}_{jke} - 2\epsilon K^i{}_{[k} K^j{}_{e]}$$

↑
intrinsic curvature + quadratic expression in extrinsic curvature.

[] = antisymmetrization of indices.

This is called the GAUSS equation.

For a flat enveloping space $R^i{}_{jke} = 0$, so we get a relation between the intrinsic and extrinsic curvatures of a hypersurface:

$${}^*R^i{}_{jke} = 2\epsilon K^i{}_{[k} K^j{}_{e]}$$

APPLICATION TO PSEUDO-SPHERICAL COORDINATES, SPACES OF CONSTANT CURVATURE

For pseudo-spherical coordinates, $g_{ij} = (y^0)^2 \gamma_{ij}$ where γ_{ij} is the metric on the unit pseudosphere, depending only on the coordinates $\{y^i\}$. Therefore

$$K_{ij} = -\frac{1}{2} \frac{\partial}{\partial y^0} (y^0)^2 \gamma_{ij} = -y^0 \gamma_{ij}$$

$$K^i{}_j = g^{ik} K_{kj} = (y^0)^{-2} \gamma^{ik} K_{kj} = -(y^0)^{-1} \delta^i{}_j$$

so ${}^*R^i{}_{jke} = \frac{2\epsilon}{(y^0)^2} \delta^i{}_{[k} \delta^j{}_{e]} = \frac{\epsilon}{(y^0)^2} \delta^i{}_{ke}$

$K^i{}_j = -(y^0)^{-1} \delta^i{}_j \iff {}^*R^i{}_{jke} = \epsilon (y^0)^{-2} \delta^i{}_{ke}$

isotropic constant extrinsic curvature \iff isotropic constant intrinsic curvature

Each pseudosphere has $|y^0| = R = \text{constant}$, and in a flat space this means constant curvature $k = \epsilon R^{-2}$ implies that the extrinsic curvature is also isotropic and constant, with trace

$$K = K^i{}_i = \pm n R^{-1}. \quad K \text{ is called the "mean extrinsic curvature",}$$

and slicings for which K is a constant on any given slice are called "constant mean curvature slicings". Thus all of the pseudospherical coordinates slice a space in this way in an iterative fashion.

EINSTEIN TENSOR FOR SPACE OF CONSTANT CURVATURE of dimension N

From the Riemann tensor $R^\alpha{}_{\beta\gamma\delta}$, one contracts indices to obtain

$$\text{the Ricci tensor } R_{\beta\delta} = R^\alpha{}_{\beta\alpha\delta}$$

$$\text{the curvature scalar } R = g^{\beta\delta} R_{\beta\delta} = R^{\alpha\beta}{}_{\alpha\beta}.$$

For a space of constant curvature, $R^{\alpha\beta}{}_{\gamma\delta} = K \delta^{\alpha\beta}{}_{\gamma\delta} = K(\delta^\alpha_\gamma \delta^\beta_\delta - \delta^\alpha_\delta \delta^\beta_\gamma)$

$$R^\beta{}_\delta = R^{\alpha\beta}{}_{\alpha\delta} = K(\delta^\alpha_\alpha \delta^\beta_\delta - \delta^\alpha_\delta \delta^\beta_\alpha) = K(\delta^\alpha_\alpha - 1)\delta^\beta_\delta = K(N-1)\delta^\beta_\delta$$

$$R = K(N-1)\delta^\beta{}_\beta = KN(N-1)$$

The Einstein tensor is

$$G^\alpha{}_\beta = R^\alpha{}_\beta - \frac{1}{2}R\delta^\alpha{}_\beta = K(N-1)\delta^\alpha{}_\beta - \frac{1}{2}KN(N-1)\delta^\alpha{}_\beta$$

$$= \underbrace{K(1 - \frac{N}{2})(N-1)}_{= -\Lambda} \delta^\alpha{}_\beta$$

$$\text{so } \boxed{G^\alpha{}_\beta + \Lambda\delta^\alpha{}_\beta = 0}$$

This is Einstein's vacuum equation with cosmological constant Λ when the space is Lorentz, i.e. a spacetime, and of constant curvature.

For DeSitter ($\epsilon=1$) and AntiDeSitter ($\epsilon=-1$), we have

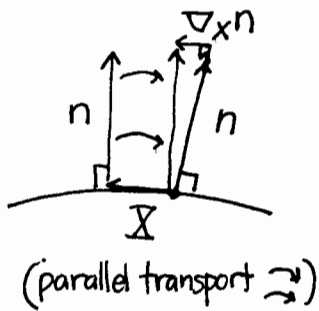
$$\Lambda = -\frac{\epsilon}{R^2} (1 - \frac{N}{2})(N-1) \stackrel{N=4}{=} +\frac{3\epsilon}{R^2}$$

Signature conventions:

Both $\Gamma^{\cdot\cdot} \sim g^{\cdot\cdot} \partial g_{\cdot\cdot}$ and $R^{\cdot\cdot\cdot\cdot} \sim \partial \Gamma^{\cdot\cdot} + \Gamma^{\cdot\cdot} \Gamma^{\cdot\cdot}$ are unchanged by a signature reversal $g_{\alpha\beta} \rightarrow -g_{\alpha\beta}$, but $R^{\alpha\beta}{}_{\gamma\delta}$ changes sign due to the single raised index. Since $\delta^{\alpha\beta}{}_{\gamma\delta}$ is unchanged, $K \rightarrow -K$, so the sign of K in the indefinite case is not significant by itself, only together with the signature.

Both DeSitter and AntiDeSitter spacetimes generalize to dimension N as the constant curvature $K > 0$, $K < 0$ (respectively) spacetimes of signature $(-+++ \dots)$. All of the geometry can be iterated to describe these cases.

INTERPRETATION OF THE EXTRINSIC CURVATURE



The rate of change of the unit normal to a hypersurface in the tangential direction must be tangential to the hypersurface (the tip of the normal can only (pseudo)rotate due to the unit length condition). The extrinsic curvature describes this change as a linear transformation of the tangential direction

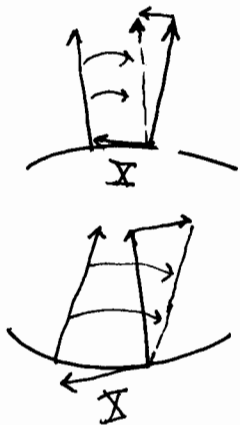
Assume the hypersurface is described by $y^0 = \text{const}$ in a Gaussian normal coordinate system

$$ds^2 = \epsilon (dy^0)^2 + g_{ij} dy^i dy^j$$

$$n^\alpha = \delta^\alpha_0 \quad \text{unit normal}, \quad g_{\alpha\beta} n^\alpha n^\beta = \epsilon$$

$$\text{tangential vector } X^\alpha = X^i \delta^\alpha_i$$

$$\nabla_X n^\alpha = n^\alpha_{;\beta} X^\beta + \Gamma^\alpha_{\beta\gamma} X^\beta n^\gamma = \Gamma^\alpha_{\beta 0} X^\beta = \delta^\alpha_i (\Gamma^i_{j0} X^j) = \delta^\alpha_i (-K^i_j X^j)$$



Thus positive eigenvalues of $(-K^i_j)$ correspond to bending away from the normal along the eigenvector direction

while negative eigenvalues of $(-K^i_j)$ correspond to bending toward the normal along the eigenvector direction.

When all the eigenvalues are equal, every direction is an eigenvector and the extrinsic curvature is "isotropic":

$$K^i_j = \left(\frac{1}{N} K^k_k \right) \delta^i_j$$

This is the case for pseudospheres.

For a family of hypersurfaces which are geodesically parallel as those in a Gaussian normal coordinate system one has the interpretation of expansion (positive eigenvalues) or contraction (negative eigenvalues) along the normal direction:

