Notes on Harmonics 1986

by bob jantzen [May June 1986]

University of Rome "La Sapienza", Dipartimento di Fisica, Corso di Fisica Teorico (Ruffini)

An attempt to make the previous notes on the Lifshitz perturbation analysis of Friedmann-Robertson-Walker cosmological models more digestible by first discussing representations of groups and the role they play in this process.

- harms1986.pdf 15 pages

Lecture Notes on HARMONICS
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Robert T Jantzen (bob)
Department of Mathematical Sciences
Villanova University
Villanova, PA 19085 USA

Corso di Fisica Teorica (Relativitá)
Dipartimento di Fisica
Universitá di Roma "La Sapienza"

p 2–4 why harmonics

p 5–14 key points of harmonics, taken from Lifshitz perturbation notes.

students give copy of Lifshitz perturbation notes (5/84)
and cosmological models notes ('84)

HARMONICS : representations of transformation groups on field spaces over a manifold

In physics the most familiar and useful groups are symmetry groups — transformation groups which leave the metric properties of a given space invariant. For a flat space, these are the translations (abelian) and the rotations (Euclidean) and pseudorotations (Lorentz).

Harmonics are useful in separating differential equations which share the symmetry group of a given space. The arise as eigenvectors of certain differential operators related to the firstorder differential operators (vector fields ~ "infinitesimal" transformations) which generate the transformation group. These operators are the "momentum operators" — linear momentum associated with translations and angular momentum with rotations in ordinary Euclidean space, for example. These operators generalize when acting on fields with "spin" (i.e., not scalar fields but tensor or spinor fields of any rank). For tensor fields they are Lie derivatives. We use them in nonrelativistic quantum mechanics when we consider the action of translations and rotations on wave functions. We use them in treating electromagnetic radiation (vector harmonic expansions).

[The word "harmonic" actually refers to eigenvectors of a Laplacian operator (D'Alembertian) ~~of~~ of a metric of a space or subspace.]

The most familiar harmonics are the spherical harmonics $Y_{\ell m}$ on the 2-sphere S^2 used in quantum mechanics and electrostatics (magnetostatics). Multiplication by the appropriate "radial functions" produces Euclidean harmonics. Less familiar are the vector harmonics constructed from the spherical harmonics to resolve vector field equations necessary to describe nonstatic electromagnetic fields in radiation problems.

Although we don't ordinarily consider differential equations for 2^{nd} rank tensor fields on Euclidean space, one can generalize the spherical harmonics to this case exactly as done for the vector case. For dimension 3, antisymmetric rank 2 tensor fields can be represented by vector fields (using the dual), leaving only the symmetric rank 2 tensor fields.

All of these harmonics can be generalized to nonflat 3-manifolds with spherical symmetry (the symmetry associated with the rotation group $SO(3,R)$). These include nonrotating black holes and FRW spacetimes, the space sections of which are Riemannian 3-manifolds with spherical symmetry.

Non rotating black holes (Schwarzschild, Riessner-Nordstrom) differ from flat Minkowski spacetime only in radial properties (in coordinates adapted to the spherical symmetry) so essentially only the radial functions change in the harmonics associated with the nonflat 3-Laplacian. The new complication is the necessity to consider also the symmetric 2nd rank tensor harmonics required to describe perturbations of the metric itself (linearized Einstein equations).

The same is true of FRW spacetimes if one considers solving linear differential equations by an expansion adapted to the spherical symmetry. The spacesections of these spacetimes are Riemannian 3manifolds of constant curvature, having a 6-dimensional symmetry group of rotations (associated with the isotropy) and translations (associated with the homogeneity). One may also consider harmonic expansions adapted to a translation subgroup. In the flat case one uses Fourier analysis to introduce "plane wave" expansions, in contrast with the "spherical wave" expansions associated with spherical symmetry. In the positive curvature case the space sections have the geometry of the 3-sphere S^3, a compact manifold, and the two kinds of expansions are very closely related. In the flat and negative curvature case this is not the case.

Why do we need harmonics in the black hole case? For the usual reasons as in Euclidean space — to solve "nonrelativistic" quantum mechanical equations and Maxwell's equations — and to consider gravitational waves, the new feature requiring the 2nd rank symmetric tensor harmonics (usually called simply the tensor harmonics). One may also introduce spinor harmonics to consider relativistic quantum mechanics (the Dirac equation), etc.

Why do we need harmonics in the FRW case? Here the motivation is different since we are not considering an isolated body but a uniform distribution of matter approximating the large scale structure of the universe. One is instead interested in the formation of inhomogeneities on the homogeneous background — gravitational clustering and galaxy formation — which may be studied by considering the linearized Einstein equations. Global anisotropies may be studied using the full Einstein equations leading to the spatially homogeneous cosmological models ("Bianchi cosmology") and using the linearized Einstein equations — to contribute to possible anisotropies in the cosmic background radiation which also arise from inhomogeneities (Sachs-Wolf effect).

$\boxed{\text{Harmonic Analysis on } S^{D-1} \subset E^D}$ $(\delta_{ab}X^aX^b = 1)$ $(a,b = 1,..,D)$

$$f(x) = \sum_{M=0}^{\infty} \frac{1}{M!} \underbrace{f_{,a_1\cdots a_M}}_{■\text{symmetric}}(0)\, x^{a_1}\cdots x^{a_M}$$

(power series at $x^a=0$)

(missing $M!$ in other notes)

Harmonic: $0 = \Delta f = -f_{,a}{}^{,a} \rightarrow f_{,a_1\cdots a_{M-2}a}{}^{a} = 0$ ■tracefree

Restrict to unit sphere $\delta_{ab}X^aX^b = 1$:

$$\hat{f} = \sum_{M=0}^{\infty} f_{,a_1\cdots a_M}(0)\, \hat{X}^{a_1}\cdots\hat{X}^{a_M}/M! \qquad \hat{X}^a \equiv r^{-1}x^a,\ r \equiv (\delta_{ab}X^aX^b)^{1/2}$$

$$= \sum_{M=0}^{\infty} f_{,a_1\cdots a_M}(0)\, \underbrace{x^{a_1}\cdots x^{a_M}}_{\mathbb{X}^{a_1\cdots a_M}} r^{-M}(M!)^{-1}$$

$$= \sum_{M=0}^{\infty} f_{,a_1\cdots a_M}(0) \underbrace{\left(X^{a_1}\cdots X^{a_M} - \frac{1}{c}\delta^{(a_1a_2}X^{a_3}\cdots X^{a_{M-2})}r^2\right)}_{\left(\substack{\text{tracefree part} \\ \mathbb{X}^{TF\, a_1\cdots a_M}}\right)} r^{-M}(M!)^{-1}$$

$$c = D + \frac{M(M-1)}{2} - 1$$

Note $\Delta\left(r^M \mathbb{X}^{TF\, a_1\cdots a_M}\right) = 0$ except at $r=0$, harmonic functions on E^D.

" cartesian harmonics "

<u>SPHERICAL COORDS</u>: $X^a = r\hat{X}^a \quad \hat{X}^a = \hat{X}^a(\underbrace{\ldots, \chi, \theta, \varphi}_{\theta^i})$

θ^i angular coords on S^{D-1}

$i,j = 1,\ldots,D-1$

on E^4:

$$\begin{aligned} x^1 &= r\ \sin\chi\ \sin\theta\ \cos\varphi \\ x^2 &= r\ \sin\chi\ \sin\theta\ \underbrace{\sin\varphi}_{\text{coords on } S^1} \\ x^3 &= r\ \sin\chi\ \underbrace{\cos\theta}_{\text{coords on } S^2} \\ x^4 &= r\ \underbrace{\cos\chi}_{\text{coords on } S^3} \end{aligned}$$

$$ds^2 = dr^2 + r^2\underbrace{[d\chi^2 + \sin^2\chi\ \underbrace{[d\theta^2 + \sin^2\theta\ \underbrace{d\varphi^2}_{\text{metric on } S^1}]}_{\text{metric on } S^2}]}_{\text{metric on } S^3}$$

$$g^{1/2} = \underbrace{(r\sin^2\chi)\underbrace{(r\sin\theta)\underbrace{(r)}_{E^2}}_{E^3}}_{E^4} = r^3\sin^2\chi\sin\theta$$

Let $g_{ij} = r^2 \underbrace{{}^{(D-1)}g_{ij}}_{\text{metric on } S^{D-1}}$ in spherical coords (metric on sphere of radius r)

$$g^{1/2} = r^{D-1}\ {}^{(D-1)}g^{1/2}$$

Recall $X^a{}_{;a} = g^{-1/2}(g^{1/2}X^a)_{,a}$

$$-{}^{D}\Delta\,\phi = \phi_{;a}{}^{;a} = g^{-1/2}(g^{1/2}g^{ab}\phi_{,b})_{,a}$$

$$= r^{-(D-1)}(r^{D-1}\phi_{,r})_{,r} + \frac{1}{r^2}\underbrace{\phi_{|i}{}^{|i}}$$

($-{}^{D-1}\Delta$ Laplacian on S^{D-1}
index raised with ${}^{(D-1)}g_{ij}$
$\phi_{|i}$ = cov der with respect to)

$\phi_{|i}{}^{|i} =$

$$\underbrace{\frac{1}{\sin^2\chi}\Big[(\sin^2\chi\,\phi_{,\chi})_{,\chi} + \underbrace{\frac{1}{\sin\theta}\Big\{\sin\theta\,\phi_{,\theta})_{,\theta} + \frac{1}{\sin\theta}\underbrace{(\phi_{,\varphi\varphi})}_{-{}^{1}\Delta\ \text{on}\ S^1}\Big\}\Big]}_{-{}^{2}\Delta\ \text{on}\ S^2}}_{-{}^{3}\Delta\ \text{on}\ S^3}$$

If $\phi = \phi(\theta^i)$ only, then $\phi_{;a}{}^{;a} \overset{\text{(cartesian coords)}}{=} \phi_{;a}{}^{;a} = r^{-2}\phi_{|i}{}^{|i}$

so can calculate the S^3 Laplacian of purely angular functions $X^{TF\,a_1\cdots a_m}$ (see other notes)

Let $A_{a_1\cdots a_M}$ be a constant tracefree tensor

so $A_{a_1\cdots a_M}\,\mathbb{X}^{TF\,a_1\cdots a_M} = A_{a_1\cdots a_M}\,x^{a_1}\cdots x^{a_M}\,r^{-M} \equiv Q^{(n)}$, $n \equiv M+1$

Calculate $\Delta Q^{(n)} = -Q^{(n)}{}_{|i}{}^{|i} = \lambda^{(n)} Q^{(n)}$ eigenfunctions of Laplacian

using cartesian coords. (see other notes).

D	sphere	$\lambda^{(n)}$	dim eigenspace $\lambda^{(n)}$	harmonics
2	S^1	M^2	1	ordinary harms: $e^{imφ}$
3	S^2	$M(M+1)$	$2M+1 = 2n-1$	spherical harms: $(\ell \equiv M)$ $Y_{\ell m}(\theta,\varphi) = \sum_m C_{\ell m} P_{\ell m}(\theta) e^{im}$
4	S^3	$M(M+2) = n^2-1$	$(M-1)^2 = n^2$	S^3 harms: $Q_{n\ell m}(\chi\theta\varphi) = \sum C_{n\ell m}\,\Pi_{n\ell}(\chi) Y_{\ell m}(\theta$

$$
\begin{cases}
\left(\frac{d}{d\varphi}\right)^2 e^{in\phi} + m^2 e^{im\phi} = 0 \\
\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP_{\ell m}}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right] P_{\ell m} = 0 \\
\frac{1}{\sin^2\chi}\left(\sin^2\chi\,\Pi_{n\ell,\chi}\right)_{,\chi} + \left[(n^2-1) - \frac{\ell(\ell+1)}{\sin^2\chi}\right]\Pi_{n\ell} = 0
\end{cases}
$$

separation of variables in spherical coords provides special functions

So the scalar harmonics on S^3 have the basis

$$\{\mathbb{X}^{TF\,a_1\cdots a_M}\}$$

eigenfunctions of Laplacian on S^3 with nonzero eigenvalues.

IRREDUCIBLE representations of $SO(4,\mathbb{R})$ = symmetry group of S^3 labeled by index n.

IRREDUCIBLE reps of subgroup $SO(3,\mathbb{R})$ acting on S^2 (θ,φ) labeled by index ℓ.

$X^a \rightarrow R^a{}_b X^b$ $SO(4,\ell)$ acting on E^4

$\hat{X}^a \rightarrow R^a{}_b \hat{X}^b$ $SO(4,\ell)$ acting on S^3

$\underline{R} = (R^a{}_b) = e^{\frac{1}{2}\theta^{ab} S_{ab}}$ $S_{ab} = (\bullet\, \delta^{cd}_{ab})$ $\theta^{ab} = \theta^{[ab]}$

matrix generators

$$L_{ab}(x) \equiv \frac{d}{d\theta^{ab}}\Big|_{\theta^{ab}=0} \underbrace{(R^c{}_d X^d)}_{\bar{X}^a} \frac{\partial}{\partial X^c} = S_{ab}{}^c{}_d X^d \frac{\partial}{\partial X^c}$$

vector field generators

"L_3" $= L_{12} = x^1\partial_2 - x^2\partial_1$, etc.

Let $F(S^3)$ be the "nice" functions on S^3.

Then $F(S^3) = \bigoplus_{n=1}^{\infty} \underbrace{F^{(n)}}$ direct sum of irreducible reps.

eigenspace of ${}^3\Delta$ labeled by n.

$\phi \in F^{(n)}$

$\Phi(x) = \phi \circ (R^{-1}x) =$ new function in $F^{(n)}$

A rotation leads to a linear trans of basis functions

$\mathfrak{X}^{TF} \rightarrow \mathcal{R}^{(n)} \mathfrak{X}^{TF}$ under R

$\mathfrak{X}^{TF} \rightarrow \ddot{\mathcal{L}}^{(n)}_{ab} \mathfrak{X}^{TF}$ under L_{ab}

} since a representation

G group acting on M as a transformation group

[$SO(4,R)$ on S^3, $SO(3,R)$ on S^2 or on S^3 (fourth axis fixed).]

$x \mapsto f_a(x)$ G acting on M

$\phi \mapsto \phi \circ f_a^{-1}$ G acting on $F(M)$ functions (scalars)

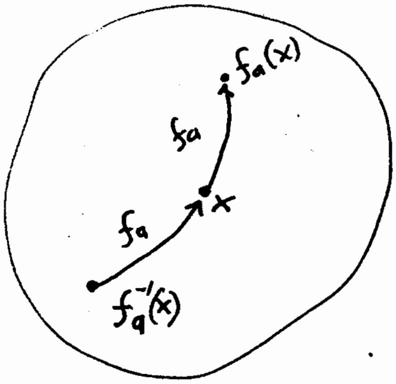
$X^i \mapsto \left(\frac{\partial f_a^i}{\partial x^j} X^j\right) \circ f_a^{-1}$ G acting on $\mathfrak{X}(M)$ vector fields

$\omega^i \mapsto \left(\frac{\partial f_a^{-1j}}{\partial x^i} \omega_j\right) \circ f_a^{-1}$ G acting on $\mathfrak{X}^*(M)$ 1-forms

$\vdots$

$T^{i\dots}{}_{j\dots} \mapsto$ tensor product representation

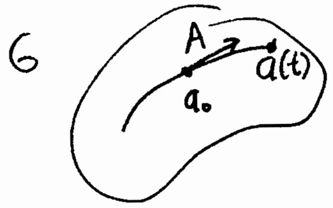
$\rightarrow$ ∞-dim representations of G (actions on linear spaces)



all fields pushed from $f_a^{-1}(x)$ to x.

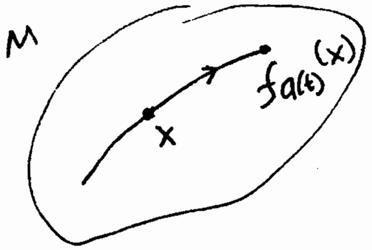
"dragged along by transformation"

"infinitesimal transformations" (tangent to curve of transformations at identity)



$a(0) = a_0$ curve of transformations

$$\left.\frac{d a^b(t)}{dt}\right|_{t=0} = A^b \qquad A = A^b \left.\frac{\partial}{\partial a^b}\right|_{a_0} = \text{tangent vector at identity.}$$



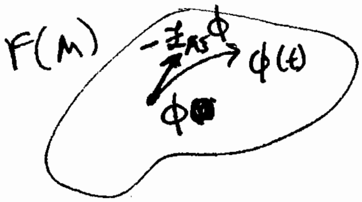
$f_{a(0)}(x) = x$ ~~curve of~~ path taken by x.

$$\left.\frac{dx^i(t)}{dt}\right|_{t=0} = \left.\left(\frac{\partial f^i_a}{\partial a^b}\frac{da^b}{dt}\right)\right|_{t=0} = \underbrace{\left.\frac{\partial f^i_a(x)}{\partial a^b}\right|_{a_0}}_{\xi^i_b} A^b$$

tangent $A^b \xi^i_b \partial_i = A^b \xi_b$

vector field on M giving direction of motion of each point.

$\bar\phi(t) = \phi \circ f^{-1}_{a(t)}$ curve of scalar fields in $F(M)$



$$\left.\frac{d}{dt}\bar\phi(t)\right|_{t=0} = \left.\frac{\partial\phi}{\partial x^i}\underbrace{\frac{\partial f^{-1\,i}}{\partial a^b}}_{-\xi^i_b}\underbrace{\frac{da^b}{dt}}_{A^b}\right|_{t=0}$$

$$= -A^b \xi_b \phi \equiv -\mathcal{L}_{A^b\xi_b}\phi$$

In vector spaces can identify tangent vectors with vectors

$x^i(t)$ $\frac{dx^i(t)}{dt}$ both vector

for a scalar only the evaluation at a new pt is involved.

for tensor fields, index trans enters picture.

$$\bar{X}^i(t) = \left(\frac{\partial f^i_{a(t)}}{\partial x^j} X^j \right) \circ f_a^{-1}$$

$$\left.\frac{d\bar{X}^i}{dt}\right|_{t=0} = -A^b \xi_b X^i + A^b \xi_b{}^i{}_{,j} X^j$$

$$= -A^b \underbrace{\left(\xi_b^j \partial_j X^i - X^j \partial_j A_b^i \right)}_{[\xi_b, X]}$$

Lie bracket = commutator of vectorfields

$$= -\left[A^b \xi_b , X \right]^i$$

$$\equiv -\left(\mathcal{L}_{A^b \xi_b} X \right)^i$$

etc for higher order tensors.

Representation $v \in V \to \bar{v} = \rho(a)\, v$ invertible linear trans.

$$\bar{v}(t) = \rho(a(t))\, v$$

$$\left.\frac{d\bar{v}(t)}{dt}\right|_{t=0} = \underbrace{d\rho(a_0)}_{\text{linear transformation}}\, v$$

$$\left.\frac{\partial \rho}{\partial a^b}\right|_{a_0} A^b = \sigma(A)$$

for each tangent vector A at identity of group

get linear tran

Lie algebra

Lie algebra representation

For all the field representations of G over M,

$-\underbrace{\mathcal{L}_{A^b \xi_b}}$ is the linear operator which represents the Lie algebra of G ~ "TM_{a_0}"

just the directional derivative of scalars, additional terms for tensors.

Ex $x^i \to R^i{}_j x^j$ $SO(3R)$ on E^3. identity representation.

$$\left.\frac{dx^i}{dt}\right|_{t=0} = \theta^a \underbrace{S_a{}^i{}_j}_{\epsilon_{aij}} x^j$$

$[S_a, S_b] = \epsilon_{abc} S_c$ Lie algebra of $SO(3,R)$ <u>matrix rep.</u>

$\to \xi_a = \epsilon_{aij} x^i \partial_j = L_a$ ang mom. (orbital)

$\mathcal{L}_{\xi_a} = J_a$ total angular mom = orb + spin.

$$J_a X^i = \mathcal{L}_{\xi_a} X^i = \xi_a^j \partial_j X^i - X^j \partial_j \underbrace{\xi_a^i}_{\epsilon_{aji} x^j}$$

$$= L_a X^i + \underbrace{S^i_{aj} x^j}_{} = (L_a + S_a) X$$

$$\sum_a S_a^2 = 2\,\underline{1} = 1(1+1)\,\underline{1} \qquad s = 1 \quad (\text{spin})$$

irreducible rep.

h_{ij} sym tensor, not irreducible rep

$$h_{ij} = \underbrace{h^{TF}_{ij}}_{} + \frac{1}{3}(\delta^{mn} h_{mn})\,\delta_{ij}$$

$$J_a h_{ij} = \mathcal{L}_{\xi_a} h_{ij} = L_a h_{ij} + (S_a h)_{ij}$$

$\downarrow$ spin operator.

$\sum_{ab} S_a^2$ acts as: $2(2+1)\,1$ on TF subsp $(s=2)$

0 on T subspace $(s=1)$

on E^3

$SO(3R)$ on S^3 these change but spin remain same.

~~SO(4,R)~~

scalar harms: $\hookrightarrow Q_{n\ell m}$ eigenvectors of $\overset{\text{scalar lapl}}{\Delta}, L^2, L_3$

$L_{12}^2 + L_{23}^2 + L_{31}^2$

$\quad\quad\quad\;\; \|$

$L_{41}^2 + L_{42}^2 + L_{43}^2$

L_{12}

$L_{43} = \tilde{L}_3$

$SO(4) \simeq SO(3) \times SO(3)$

$\downarrow$ locally.

$$\vec{X}_{n\ell m} \xrightarrow[\text{ingr}]{} \vec{T}_{sn\ell m}$$

$\uparrow$ spin

$\underset{\text{vector}}{\Delta}, J^2, J_3,$

one can find explicit expressions in spherical coords involving $Q_{\ell mn}$ & derivatives etc.

but not necessary. Can derive all properties from Cartesian coord form. & action of ang mm operators.

$dQ_{\ell mn}$ is a one fn fm an → exact $\underline{1}$-forms

~~$*d*Q_{\ell m}$ $*d*Q_{\ell m}$~~

$(Q_{\ell m})\,\delta_{ij}$ Trace $s=0$ tensor.

eh

~~eh~~

2006 Note: Looks like things got out of control at the end, eh?