Curvilinear Coordinates and Curvature

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A brief motivation for how connection and curvature arise by considering transforming from Cartesian coordinates and ordinary derivatives to spherical coordinates and the corresponding covariant derivatives. Written as a preference to existing notes to be reused.

- ccc1989.pdf: 5 pages, 125K

INTRODUCTORY LECTURE
Corso di Fisica Teorica (relativitá generale)

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There are many occasions in Euclidean 3-space where curvilinear coordinates prove more useful than cartesian coordinates. In order to use familiar derivative operations like grad, curl, div, ... ~~or~~ consider constant tensor fields, we must transform them to the new coordinates. This immediately leads to a mathematical machinery identical to that of curved spaces. One is led to a "covariant derivative", "apparent curvature", a "nonconstant" metric, parallel transport, all of which extend to nonflat spaces where the curvature tensor is nonzero.

The subsequent notes will consider a large family of orthogonal coordinates on spaces of arbitrary dimension and signature, built using flat geometry and pseudospheres on the full space and on linear subspaces, generalizing cylindrical and spherical coordinates in 3 Euclidean dimensions. This leads not only to coordinates for which flat Minkowski space looks like a cosmological spacetime, but also to constant curvature spacetimes which play an important theoretical role in present cosmology, and to constant curvature 3-spaces which are used to build the standard cosmologies.

The next 3 pages are rough notes used to lecture from and of course lack all the explanation and added geometrical pictures of the lecture. But perhaps they serve as a useful reminder of what we did.

JANUARY, 1989

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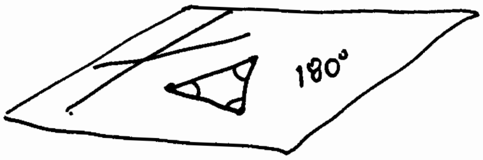
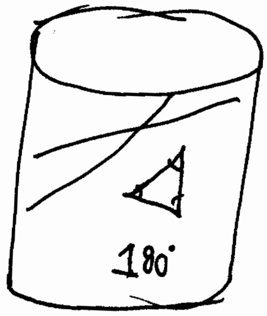
metrics and curvature

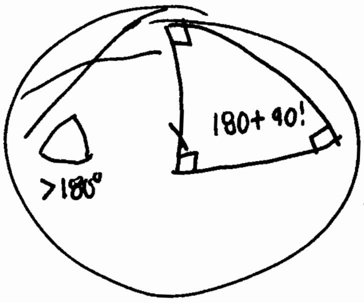
PLANE



geometry same locally



CYLINDER



geometry different.

SPHERE

intrinsic geometry of surface!! $\boxed{\text{intrinsic curvature}}$

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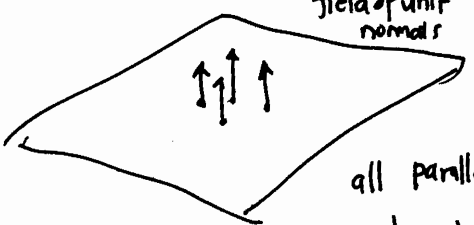
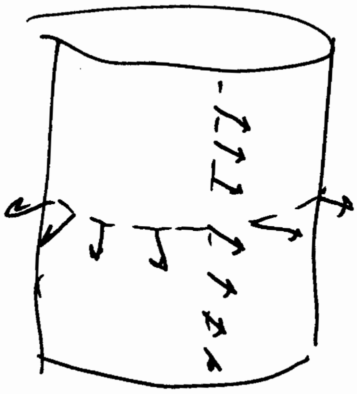
field of unit normals



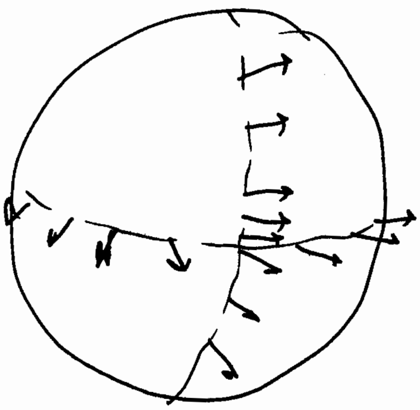
all parallel

no bending



parallel ~~move~~ axial direction

bending perpendicular to axis direction.



bending in both directions.

$\boxed{\text{extrinsic curvature}}$

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surfaces embedded in flat Euclidean space:

intrinsic + extrinsic curvature of ~~layer~~ surface connected by flat intrinsic ~~geometry~~ geometry of Euclidean space.

In flat Euclidean space or Minkowski space, often convenient to work in nonflat coordinate systems. These introduce "apparent curvature", need machinery of curved manifolds even though total space is flat.

$\{x^i\}$ cartesian coordinates

$\partial_i X^j$ components of grad $\vec{X}$.

$\vec{X}$ constant if grad $\vec{X} = 0$, constant cartesian components

But suppose use spherical coordinates $\{x^{i'}\}$: $X^{i'} = \frac{\partial x^{i'}}{\partial x^j} X^j$ now functions of position even for a constant vector field.

$$\underbrace{\frac{\partial x^\ell}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^m}}\left(\frac{\partial X^m}{\partial x^\ell}\right) = \nabla_{i'} X^{j'} \quad \text{(new components of grad } \vec{X}\text{)}$$

$$\frac{\partial x^{j'}}{\partial x^m}\frac{\partial X^m}{\partial x^{i'}} = \frac{\partial}{\partial x^{i'}}\left(\frac{\partial x^{j'}}{\partial x^m} X^m\right) - \underbrace{\frac{\partial^2 x^{j'}}{\partial x^{i'}\partial x^m}\frac{\partial x^m}{\partial x^{k'}}}_{= \Gamma^{j'}_{i'k'}} X^{k'}$$

$$= \frac{\partial}{\partial x^{i'}} X^{j'} + \underbrace{\Gamma^{j'}_{i'k'}}_{\text{"components of connection"}} X^{k'}$$

$\boxed{\text{"covariant derivative"}}$

parallel transport...

$$(\partial_i \partial_j - \partial_j \partial_i) X^k = 0$$

$\downarrow$ same calculation, more complicated.

$$(\nabla_{i'}\nabla_{j'} - \nabla_{j'}\nabla_{i'}) X^{k'} = \underbrace{R^{k'}{}_{\ell' i' j'}}_{=\,0} X^{\ell'} = 0$$

"curvature tensor"

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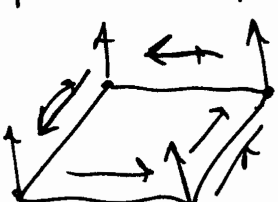
interpretation: "parallel transport"

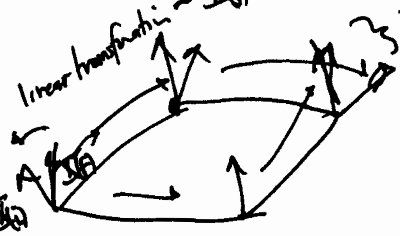


flat space

return to same initial vector

linear transformation $\sim X^k_{(f)} - X^k_{(i)} \sim R^k{}_{\ell ij}\,\Pi^{i}{}_{m}{}^{j}\,X^{\ell}$

projection into 2 planes, linear transf. of $\vec{X}$



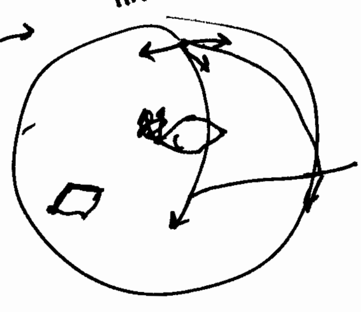
$X_{(i)}$

~~flat space~~

(i) ~ initial

(f) ~ final

curved space



δ_{ik}

metric $\quad ds^2 = \delta_{ij}\, dx^i dx^j = \delta_{ij}\left(\frac{\partial x^i}{\partial x^{m'}}\, dx^{m'}\right)\left(\frac{\partial x^j}{\partial x^{n'}}\, dx^{n'}\right) = \underbrace{\left(\delta_{ij}\frac{\partial x^i}{\partial x^{m'}}\frac{\partial x^j}{\partial x^{n'}}\right)}_{\equiv\, g_{i'j'}} dx^{m'} dx^{n'}$

new components of the flat metric $\longrightarrow \; g_{i'j'}$

$\partial_k \delta_{ij} = 0 \qquad$ metric is "constant", i.e. "covariant constant"

$\nabla_k \delta_{ij} = 0 \longrightarrow$ transform to new coordinates, find:

$$\nabla_{k'} g_{m'n'} = \partial_{k'} g_{m'n'} - g_{l'n'}\Gamma^{l'}{}_{k'm'} - g_{m'l'}\Gamma^{l'}{}_{k'n'} \qquad = 0$$

But $$\Gamma^{j'}{}_{i'k'} = \frac{\partial}{\partial x^{i'}}\left(\frac{\partial x^{j'}}{\partial x^m}\right)\frac{\partial x^m}{\partial x^{k'}} = \underbrace{\frac{\partial}{\partial x^{i'}}\Bigg(\underbrace{\frac{\partial x^{j'}}{\partial x^m}\frac{\partial x^m}{\partial x^{k'}}}_{\delta^{j'}{}_{k'}}\Bigg)}_{0} - \frac{\partial x^{j'}}{\partial x^m}\underbrace{\frac{\partial^2 x^m}{\partial x^{i'}\partial x^{k'}}}_{\text{symmetric in } i',k'}$$

called a "symmetric connection"

For a symmetric connection one can solve the above constraint between $g_{m'n'}$ and $\Gamma^{j'}{}_{i'k'}$ by a trick of classical geometry, finding:

$$\Gamma^{k'}{}_{i'j'} = \tfrac{1}{2} g^{k'l'}\left(\partial_{i'} g_{l'j'} - \partial_{l'} g_{j'i'} + \partial_{j'} g_{i'l'}\right)$$

$+\, i'l'j' - l'j'i' + j'i'l' \leftrightarrow$ anticyclic permutation of indices

The connection used for parallel transport is completely determined by the metric.

By iterating the formula $\nabla_{i'} X^{j'} = \partial_{i'} X^{j'} + \Gamma^{j'}{}_{i'k'} X^{k'}$ extended to higher rank tensors one can recalculate the curvature in terms of the connection from $(\nabla_{i'}\nabla_{j'} - \nabla_{j'}\nabla_{i'}) X^{k'} = R^{k'}{}_{l'i'j'} X^{l'}$ to find

$$R^{k'}{}_{l'i'j'} = \partial_{i'}\Gamma^{k'}{}_{j'l'} - \partial_{j'}\Gamma^{k'}{}_{i'l'} + \Gamma^{k'}{}_{i'm'}\Gamma^{m'}{}_{j'l'} - \Gamma^{k'}{}_{j'm'}\Gamma^{m'}{}_{i'l'}$$

so the curvature is a function of the connection which is itself a function of the metric.

For a nonflat manifold the mathematical machinery is the same as in curvilinear coordinates on flat space, except the curvature tensor is nonzero.

(The connection cannot be a tensor since it is identically zero in cartesian coordinates and nonzero in curvilinear coordinates.)