Let \( e = (e_1, e_2, e_3) \) be the frame with dual basis:
\[
\begin{align*}
\tilde{e}(e_1) &= e_2 + e_3 \\
\tilde{e}(e_2) &= e_1 + e_3 \\
\tilde{e}(e_3) &= e_1 + e_2
\end{align*}
\]

Let:
\[
\begin{align*}
\nabla_{e_1} e_2 &= e_3 \\
\nabla_{e_2} e_3 &= -e_1 \\
\nabla_{e_3} e_1 &= e_2
\end{align*}
\]

Define "\text{CONNECTION ONE-FORMS}" with respect to this frame:
\[
\begin{align*}
\omega^1 &= \tilde{e}^2 - \tilde{e}^3 \\
\omega^2 &= \tilde{e}^3 - \tilde{e}^1 \\
\omega^3 &= \tilde{e}^1 - \tilde{e}^2
\end{align*}
\]

so that:
\[
\nabla_{e_a} e_b = \omega^c_{ab} e_c
\]

**Torsion Tensor:**
\[
T(x,y) = \nabla_x Y - \nabla_y X + [X,Y] = \omega^c(x,y) e_c
\]

**Ricci Tensor:**
\[
T(e_a) = T^b_{ab} e_b = \tfrac{1}{2} T^{bc} (\nabla_c \omega^b)_a
\]

*Note: \( T^{ab} = R^{ab} - C_a^{ab} \)*

**Curvature Tensor**
\[
R(x,y) z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z = \omega^c(x,y) \omega^d(x,y) e_c
\]

**Ricci Curvature**
\[
R(x,y) e_a = \tfrac{1}{2} \tilde{R}^{cd} (\nabla_c \omega^d)_a
\]

**Christoffel Symbols:**
\[
\begin{align*}
\Gamma^a_{bc} &= \gamma^a_{bc} + \omega^a (x) \omega^c (x) + \omega^c (x) \omega^a (x) \\
&= \left\{ \gamma^a_{bc} + \omega^a (x) \omega^c (x) + \omega^c (x) \omega^a (x) \right\} e_a
\end{align*}
\]

**Result:**
\[
\nabla^2 = \omega^a \omega^b + \omega^a \omega^b 
\]

**Ricci-Christoffel Formula:**
\[
\tilde{R}^{cd} = 2 \delta^c (x) \omega^d (x) + 2 \tilde{R}^{cd} e_c = \tilde{R}^{cd} e_c
\]
Bianchi Identities

\[ \Theta^a = d \Theta^a + \omega^{ab} \wedge \Theta_b \]

COMP.

\[ d \wedge \omega = -\omega \wedge \omega \]

RESULT

\[ d \Theta^a + \omega^{ab} \Theta_b = \Omega \wedge \Theta^a \]

\[ \Theta = \omega \wedge \omega \]

\[ d \omega = d \Theta^a \wedge \Theta_a = 0 \]

Note. For \( \Theta = 0 \),\( 0 = 2 \Theta^a \wedge \Theta_a = 2 \wedge \epsilon_{abc} \Sigma \wedge \Theta^a = 0 \rightarrow \epsilon_{abc} \Sigma = 0. \]

The Cartan Structural Equations are equivalent to the definition of the torsion and curvature tensors, but stated in the language of forms. The Bianchi Identities are a trivial consequence following immediately from the Cartan Structural Equations.
TENSOR VALUED FORMS

In the third chapter of the book of Alvaro, it is said that all indices are not created equal. Different indices perform different functions. Examples:

(i) \( \nabla_a e_b = \Gamma^c_{ab} e_c \) defines the connection coefficients, but

(ii) \( \nabla_a e_b = \Gamma^c_{ab} X^c e_b = \Gamma^c_{ab} \delta_b^c(X) e_b \) shows that the index \( c \) has a natural function as a one-form index, so we define the connection one-forms: \( \omega^b_a = \Gamma^c_{ab} \theta^c \)

so that \( \nabla_a e_b = \omega^c_{ab}(X) e_b \).

This equation shows that \( \omega^c_{ab}(X) \) acts as a linear transformation of the frame vectors, so we call \( \omega^c_{ab} \) as a \( g(DN)-valued \) one-form, since evaluating it on a vector leads to a \( g(DN) \)-valued transformation of the tangent space.

Then the covariant differential of \( e_b \) is:

\( \nabla e_b = e_a \omega^a_b \),

So the covariant differential of a general vector field is:

\( \nabla X = e_a \omega^a_b \theta^b \equiv (e_a \omega^a_b \theta^b)(X) \)

where we need to define:

\( \omega = e_a \theta^a \theta_b \theta^b = -\partial_a e_b \).

Because \( T^a_b = T^a_b^\gamma \theta^\gamma \),

\( T = T^a_{\alpha \beta} \theta^a \theta^\alpha \theta^\beta \),

\( e_a = e_{a \theta} \theta^\alpha \theta^\beta \theta^\gamma \).

So the antisymmetric pair of indices leads us naturally to defining a set of \( n \)-shaped forms \( \Omega^a \) or a vector-valued two-form: \( \Theta = e_a \theta^a \theta^b \). (1)

Because \( R^a_{b \gamma} = R^a_{b \gamma} \theta^\gamma \),

\( R = R^a_{b \gamma} \theta^a \theta^b \theta^\gamma \theta^\delta \),

\( e_a = e_{a \theta} \theta^\alpha \theta^\beta \theta^\gamma \).

So the antisymmetric pair of indices leads us naturally to defining a set of \( \Omega^a \) or a \( g(DN) \)-valued two-form: (1) tensor-valued form \( \Omega = R \).

So the torsion and curvature tensors are most naturally interpreted as operators on 2-surfaces embedded in \( \mathbb{R}^n \) as follows: \( \text{tangential variation under parallel transport around the parallelogram} \).

Similarly \( T(\Theta) \) assigns a vector \( T(\Theta)X \) to the area spanned by \( X \), which represents the failure of the "parallel transport parallelogram" to be determined by \( X \) to close.

In general on (5) tensor-valued p-forms is simply on (5) tensors which is alternating in p of its covariant indices (in fact more indices may be alternating but not all). Therefore, \( S^a_{\alpha \beta} \rightarrow \delta^a_{\alpha \beta} \delta^c_{\alpha \beta} \).

\( S^a_{\alpha \beta} \rightarrow \delta^a_{\alpha \beta} \delta^c_{\alpha \beta} \).

That \( S^a_{\alpha \beta} \) is a set of p-forms or on (5) tensor valued p-forms.

Alternative interpretation. Let \( e^a, e^b \) be the standard basis and dual basis of \( \mathbb{R}^n \) in \( \mathbb{C}^n \) is a column vector \( [e^a] \) and \( e^b \) is a row vector \( (e^b) \), and \( e^a = e^a \otimes e^b \) is their inner product with \( \mathbb{R}^n \otimes \mathbb{R}^n \), duals, and elsewhere, i.e. \( (e^a) \) is the
canonial basis of $\mathbb{R}^n$. It turns out that we shall be primarily concerned with vector, covector - and (1) tensor-valued forms. In the expression for a tensor-valued $p$ form replace $e_\alpha, \theta^\alpha$ by $e_\alpha e^\beta, \theta^\alpha \theta_\beta$ in the tensor-valued factor:

$$
\begin{align*}
\epsilon &= (e_\alpha, \epsilon_\alpha) = e_\alpha e^\alpha^* \quad \Theta^\alpha = \theta^\alpha \theta_\beta \\
\epsilon^\alpha &= e_\alpha e^\alpha^* \\
\Omega &= \theta^\alpha \theta_\beta = \theta^\alpha e^\beta \theta_\beta \\
X &= \epsilon \epsilon^\alpha
\end{align*}
$$

Note that the tensor-valued 1-form, but a contravariant-valued 1-form, because it does transform under a change of basis. Similarly we could define other geometric object valued forms.

$X$ is a tensor-valued 1-form, $e_\alpha$ a covector-valued 0-form.

Using this notation and the conventions of matrix multiplication we can write matrix equations.

If we place certain equations involving such forms, Examples:

$$
\begin{align*}
\begin{bmatrix}
\theta_\alpha
\end{bmatrix} &= \begin{bmatrix}
\theta^\alpha
\end{bmatrix} \\
\omega_\alpha &= \theta^\beta \\
\Sigma_\alpha &= \theta^\beta \\
\epsilon &= \epsilon^\alpha e_\alpha
\end{align*}
$$

Note: occasional suppression of $\otimes$.

Multiplication (wrim) can occur only if adjacent indices are in the proper position. Thus: $\theta^\alpha \epsilon = \Sigma$ for $\theta^\alpha \epsilon = \epsilon^\alpha$.

$\Sigma$ is convenient to have both interpretations in mind when writing tensor-valued forms.

Note that if $T$ is on (1) tensor-valued zero form, $\nabla T = \theta^\beta S \theta_\beta$ is on (1) tensor-valued 1-form.

Also $\epsilon \Theta = \epsilon \epsilon \theta^\alpha$ is just what we called $\Theta$ in the former interpretation, and it is just the identity operator on the tangent space with frame components:

$$
\begin{align*}
\delta^\alpha &= \begin{bmatrix} e_\alpha \theta^\alpha \end{bmatrix} \\
\delta^\alpha &= \begin{bmatrix} \theta^\alpha \theta_\beta \end{bmatrix} \delta^\alpha \theta_\beta = \epsilon \theta^\alpha
\end{align*}
$$

Note that if $S$ is an (1) tensor-valued zero form, $\nabla S = \theta^\beta S \theta_\beta$ is on (1) tensor-valued 1-form.

Also $\epsilon \Theta = \epsilon \epsilon \theta^\alpha$ is just what we called $\Theta$ in the former interpretation, and it is just the identity operator on the tangent space with frame components:

$$
\begin{align*}
\delta^\alpha &= \begin{bmatrix} e_\alpha \theta^\alpha \end{bmatrix} \\
\delta^\alpha &= \begin{bmatrix} \theta^\alpha \theta_\beta \end{bmatrix} \delta^\alpha \theta_\beta
\end{align*}
$$

The torsion and curvature tensors "transform as tensors" under the frame change:

$$
\begin{align*}
\Omega^\alpha &= (\alpha^a)_{\alpha}^{\beta} \omega^\beta \\
\Omega^\alpha &= (\alpha^a)_{\alpha}^{\beta} \omega^\beta
\end{align*}
$$

The above called a $(1,2)$-form or $\Theta$ of type, $(1,2)$.
COVARIANT EXTERIOR DERIVATIVE

We would like to extend the exterior derivative $d$ defined for ordinary forms to a covariant exterior derivative $D$ defined for tensor-valued forms.

On the one extreme we have ordinary forms for which $D = d$.

On the other we have $n$-tensor-valued zero-forms; $D_{\alpha} = \partial_{\alpha}$.

\[ D_{\alpha} \equiv \partial_{\alpha} \quad (D_f = \partial_f = df \text{ for functions } f) \]

Remark. If $\Theta$ is a tensor-valued zero-form and $\omega$ a form, we can write

\[ \Theta \wedge \omega = \Theta \wedge \omega^* \text{ analogously to writing } f \wedge \omega = \omega^* f \text{ for functions } f \text{ for zero-forms}. \]

We extend $D$ to tensor-valued $p$-forms $S$ by:

\[ \dot{D} = (S \wedge \Theta) = D(S \wedge \Theta) = \Theta(S \wedge \Theta) \text{ in } (n-p) \text{ forms}. \]

where $\Theta$ is an ordinary form. Note that for $\Theta$ a scalar-valued $p$-form, i.e., ordinary form, this is the usual rule for forms.

Thus:

\[ S \equiv D(S \wedge \Theta), \quad \Theta \wedge \omega = \omega^* \Theta \]

Fact: \[ \nabla \omega = \partial \omega + \omega(\Theta) \]

Thus almost by inspection:

\[ D_{\Theta} = \{ S \wedge \Theta \} \{ \partial \omega + \omega(\Theta) \} \]

Thus the covariant exterior derivative of an $(n)$ tensor-valued $p$-form is an $(n)$ tensor-valued $(p+1)$-form, and we will loop into writing $D(S \wedge \Theta)$ for $D = (S \wedge \Theta)$ exactly as we write $\nabla X^\Theta$ for $(\nabla X)^{\Theta}_{\Theta}$.

\[ \Theta, \omega \text{ are vector-valued } 1 \text{-forms and } 2 \text{-forms respectively so:} \]

\[ D_{\Theta} = \Theta + \omega(\Theta) \leftrightarrow D\Theta = \Theta + \omega(\Theta \Theta) \]

Thus $D_{\Theta} = \Theta + \omega(\Theta)$ defines the connection forms, and recalling the Cartan Structural equations and Bianchi Identities, they can be written:

\[ \Theta = \Theta \quad \text{ or define } \quad \{ D(S \wedge \Theta) \} \quad \{ \Theta \wedge \omega \} \]

\[ \nabla \omega = \partial \omega + \omega(\Theta) \text{ for } (\nabla \omega)^{\Theta}_{\Theta} \]

\[ \omega \text{ is a } (1) \text{-valued one-form, but not a } (1) \text{-tensor valued form so we really shouldn't use these formulas, but even if we tried:} \]

\[ D_{\omega} = \omega + \omega(\Theta) \]

\[ \omega(\Theta) \text{ for } (\omega(\Theta))^{\Theta}_{\Theta} \]

Thus $D_{\Theta} = \Theta$ defines the connection forms, and recalling the Cartan Structural equations and Bianchi Identities, they can be written:

\[ \Theta = \Theta \quad \text{ or define } \quad \{ D(S \wedge \Theta) \} \quad \{ \Theta \wedge \omega \} \]

\[ \nabla \omega = \partial \omega + \omega(\Theta) \quad \text{ or define } \quad \{ D(S \wedge \Theta) \} \quad \{ \Theta \wedge \omega \} \]

\[ \omega \text{ is a } (1) \text{-valued one-form, but not a } (1) \text{-tensor valued form so we really shouldn't use these formulas, but even if we tried:} \]

\[ D_{\omega} = \omega + \omega(\Theta) \quad \text{ or define } \quad \{ D(S \wedge \Theta) \} \quad \{ \Theta \wedge \omega \} \]

\[ \omega \text{ is a } (1) \text{-valued one-form, but not a } (1) \text{-tensor valued form so we really shouldn't use these formulas, but even if we tried:} \]

\[ D_{\omega} = \omega + \omega(\Theta) \quad \text{ or define } \quad \{ D(S \wedge \Theta) \} \quad \{ \Theta \wedge \omega \} \]

\[ \omega \text{ is a } (1) \text{-valued one-form, but not a } (1) \text{-tensor valued form so we really shouldn't use these formulas, but even if we tried:} \]

\[ D_{\omega} = \omega + \omega(\Theta) \quad \text{ or define } \quad \{ D(S \wedge \Theta) \} \quad \{ \Theta \wedge \omega \} \]

\[ \omega \text{ is a } (1) \text{-valued one-form, but not a } (1) \text{-tensor valued form so we really shouldn't use these formulas, but even if we tried:} \]
Note that $D$ enjoys many of the properties of $d$, for instance if $f : M \rightarrow N$ and $S, \bar{S}$ are tensor-valued forms on $M, N$, respectively, such that $S = f^* \bar{S}$, then we think of $S$ as the pullback of $\bar{S}$, so for which $\bar{f}$ makes sense individually, $S^\alpha_{\mu \nu} = f^* \bar{S}^\alpha_{\mu \nu}(\bar{f})$.

Then provided that $\omega = f^* \bar{\omega}$, we have $D^{\bar{f}} S^\alpha_{\mu \nu} = f^* D^{\bar{f}} \bar{S}^\alpha_{\mu \nu}$, where $D^{\bar{f}} = f^* D^{\bar{f}}$.

If $g$ is a (3) symmetric tensor, it is a (3) tensor-valued zeroform so:

$$Dg = -\omega g - g \omega$$

If $g$ is a constant, $Dg = 0$. If $g_{\mu \nu}$ is constant, $\omega_{\mu \nu} = 0$.

**Form of Covariant Exterior Derivative** (Schild: 52, 110):

Preliminary. a) $\nabla(V \circ F) = F^* \nabla \circ (V \circ F) = F^* \nabla \circ F$.

We have $\omega = f^* \bar{\omega} = \frac{1}{f^2} \bar{\omega} f^2$,

$$\omega = \frac{1}{f^2} \bar{\omega} f^2 \text{ for } f^2 \in \mathbb{R}$$

where $f^2 \neq 0$.

We compute, given that $\omega$ is an $(\bar{f})$-tensor-valued $p$-form:

$$\nabla \omega S_{\mu \nu \lambda} = \omega^{\gamma} \nabla_\mu S_{\gamma \nu \lambda} + \omega^{\mu} \nabla_\nu S_{\lambda \gamma \phi} - \Gamma^\delta_{\mu \nu} S_{\delta \gamma \phi} - \Gamma^\delta_{\nu \lambda} S_{\delta \gamma \mu}$$

Now, $\nabla \omega = \nabla \omega f = f^2 \nabla \omega f f^{-2}$.

We have $f^2 \nabla \omega f f^{-2} = \frac{1}{2} \nabla \omega f f^{-2} + \frac{1}{2} \nabla \omega f f^{-2}$.

The Bianchi identities are:

1. $V = R^\alpha_{\beta \gamma \delta} Y^\gamma + T^\gamma_{\beta \gamma \delta} R^\alpha_{\gamma \delta} Y^\gamma = 0$
First an innocent computation:

\[
\nabla_{\alpha}(\omega_{\beta_{1}\cdots{\beta}_{a}}) = \partial_{\alpha}\omega_{\beta_{1}\cdots{\beta}_{a}} - \frac{1}{a+2} R^{\gamma}{}_{\alpha\beta_{1}\cdots{\beta}_{a}}\omega_{\gamma_{1}\cdots{\gamma}_{a}} - \frac{1}{2} T_{\alpha\beta_{1}\cdots{\beta}_{a}} \omega_{\gamma_{1}\cdots{\gamma}_{a}}
\]

\[
\frac{1}{a+1} \nabla_{\alpha}(\omega_{\beta_{1}\cdots{\beta}_{a}}) = \frac{1}{a+2} R^{\gamma}{}_{\alpha\beta_{1}\cdots{\beta}_{a}} + \frac{1}{2} T_{\alpha\beta_{1}\cdots{\beta}_{a}} \omega_{\gamma_{1}\cdots{\gamma}_{a}}
\]

\[
(\omega)_{\alpha_{1}\cdots{\alpha}_{a}}(x) = (p+1) \nabla_{\alpha_{1}}(\omega_{\beta_{1}\cdots{\beta}_{a}}(x)) + \frac{p(p+2)}{2} T^{\gamma}{}_{\alpha_{1}\beta_{1}\cdots{\beta}_{a}} \omega_{\gamma_{1}\cdots{\beta}_{a}}(x)
\]

\[
\delta\omega(X_{1}\cdots{X}_{a})(x) = \sum_{j=1}^{a} (-1)^{j+1} \nabla_{X_{j}}(\omega(X_{1}\cdots\widehat{X}_{j}\cdots{X}_{a})(x)) + \frac{1}{a+2} \sum_{j=1}^{a} (-1)^{j+1} \omega((T(X_{j}),X_{1}\cdots\widehat{X}_{j}\cdots{X}_{a})(x))
\]

\[
\delta\omega(X_{1}\cdots{X}_{a})(x) = \sum_{j=1}^{a} (-1)^{j+1} \nabla_{X_{j}}(\omega(X_{1}\cdots\widehat{X}_{j}\cdots{X}_{a})(x)) + \frac{1}{2} \sum_{j=1}^{a} (-1)^{j+1} \omega(\{X_{j},X_{1}\cdots\widehat{X}_{j}\cdots{X}_{a}\})(x)
\]

Thus we have an exactly equivalent formula for \(\delta\) but expressed in terms of the covariant derivative.

But notice that this formula, as a differential operator, cannot contain any ordinary forms but only any tensor-valued forms. We can have written down (30) immediately from the definition of \(\delta\omega(X_{1}\cdots{X}_{a})\) since \(\nabla_{X_{j}}(\omega(X_{1}\cdots\widehat{X}_{j}\cdots{X}_{a})) = X_{j}\omega(\{X_{j},X_{1}\cdots\widehat{X}_{j}\cdots{X}_{a}\})\), but the alternate expressions (30) (31) are also useful which explains our choice of action. (31) is the exterior covariant derivative, an operator which takes (\(\omega\)) tensor-valued \((p+1)\)-forms.

For a (\(\omega\)) tensor-valued p-form \(\omega\) obviously by definition \(\delta\omega = \omega\delta\omega\).

For an (\(\omega\)) tensor-valued 0-form \(S\) clearly:

\(\delta S = \nabla S\).

That \(\delta\) is the natural interpolation between \(\delta\) and \(\nabla\).

(See reverse for a derivation of the Ricci identities.)

Note: Daise annihilates n-forms.
\[ D\xi = dx \]
\[ D^2\xi - dx^2 = 0 \]
\[ 0 = \mathcal{L}\{\nabla^2\psi + \frac{1}{2} T\xi \nabla^2\psi\} \]
\[ \mathcal{L}\{\nabla^2\psi\} = -\frac{1}{2} \nabla^2\psi \]

\[ DX = dx + d^k\psi \nabla^k \]
\[ D^2X = d^k\psi \nabla^k + d^k\psi \nabla^k \psi \nabla^k + d^k\psi \nabla^k \phi \nabla^k \psi \phi \]
\[ 0 = \mathcal{L}\{\nabla^2\psi + d^k\psi \nabla^k \psi \nabla^k \} \]
\[ \mathcal{L}\{\nabla^2\psi\} = -\frac{1}{2} \nabla^2\psi \]

Suppose \( T^{\alpha\beta}\) is a symmetric tensor on \( \mathcal{M}\), define the scalar valued constant \( \mathcal{L} = \text{exp}(T) \).

Since \( D\xi = 0 \) for \( \psi = 0 \), \( D^2\psi \nabla^k \psi \nabla^k \psi \)
\[ D^2\psi = D^2\psi \nabla^k \psi \nabla^k \psi = \psi \nabla^k \psi \nabla^k \psi \]

Both the energy-momentum and Einstein tensor of this type.

Let \( F \) be a form:
\[ F = \frac{1}{3} F e^F \quad A = \frac{1}{3} F e^F + \frac{1}{2} T e^F \]
\[ D^2F = D^2F = \frac{1}{3} D^2 F + \frac{1}{2} F e^F \]

Fix \( F \) to be a multiple of \( \mathcal{L}\).

Thus, the energy-momentum form:
\[ F = e^F \mathcal{L} \]
\[ D^2F = e^F (D^2F \mathcal{L} - e^F \mathcal{L} D^2F) \]

\[ 0 = D^2\psi = -d^k\psi \nabla^k x + d^k\psi \nabla^k \psi \nabla^k \]
\[ 0 = D^2\psi = d^k\psi \nabla^k x - d^k\psi \nabla^k \psi \nabla^k \]

Ricci identities:
\[ D^2\psi = \psi \nabla^k \psi \nabla^k \psi \]
\[ D^2\psi = \psi \nabla^k \psi \nabla^k \psi \]

\[ D^2\psi = \psi \nabla^k \psi \nabla^k \psi \]
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\[ D^2\psi = \psi \nabla^k \psi \nabla^k \psi \]
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\[ D^2\psi = \psi \nabla^k \psi \nabla^k \psi \]
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\[ D^2\psi = \psi \nabla^k \psi \nabla^k \psi \]

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\[ D^2\psi = \psi \nabla^k \psi \nabla^k \psi \]
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\[ D^2\psi = \psi \nabla^k \psi \nabla^k \psi \]
\[ D^2\psi = \psi \nabla^k \psi \nabla^k \psi \]
Linear connection expressed in terms of $\Gamma^{\alpha}_{\kappa\lambda}$ and an auxiliary symmetric tensor field $g_{\kappa\lambda}$, then

\[ g_{\kappa\lambda} = g_{\lambda\kappa} \quad \text{Define} \quad (\alpha\beta\lambda) = \Gamma^{\alpha}_{\kappa\lambda} g_{\beta\lambda} + \Gamma^{\beta}_{\kappa\lambda} g_{\alpha\lambda} + \Gamma^{\lambda}_{\kappa\lambda} g_{\alpha\beta} \]

\[ \frac{d g_{\kappa\lambda}}{d\lambda} = \frac{d g_{\lambda\kappa}}{d\lambda} = \frac{d g_{\lambda\kappa}}{d\lambda} = \frac{d g_{\kappa\lambda}}{d\lambda} \]

\[ \text{Then} \quad g_{\kappa\lambda} = \frac{1}{2} \left( \frac{d g_{\kappa}\kappa}{d\lambda} + \frac{d g_{\kappa\lambda}}{d\lambda} + \frac{d g_{\lambda\kappa}}{d\lambda} + \frac{d g_{\kappa\lambda}}{d\lambda} \right) \quad \text{Recall} \quad R_{\alpha\beta\kappa\lambda} = \frac{1}{2} \left( \Gamma^{\alpha}_{\gamma\beta\lambda} + \Gamma^{\beta}_{\gamma\alpha\kappa} - \Gamma^{\gamma}_{\alpha\beta\kappa} - \Gamma^{\gamma}_{\beta\alpha\kappa} \right) \]

\[ \text{so} \quad R_{\alpha\beta\kappa\lambda} = \frac{1}{2} \left( \frac{d g_{\kappa\lambda}}{d\alpha} + \frac{d g_{\lambda\kappa}}{d\beta} - \frac{d g_{\kappa\lambda}}{d\beta} + \frac{d g_{\lambda\kappa}}{d\alpha} \right) \]

\[ \text{Finally,} \quad g_{\kappa\lambda} = \frac{1}{2} \left( \frac{d g_{\kappa\lambda}}{d\alpha} + \frac{d g_{\lambda\kappa}}{d\beta} - \frac{d g_{\kappa\lambda}}{d\beta} + \frac{d g_{\lambda\kappa}}{d\alpha} \right) \]

\[ \text{Let} \quad R_{\alpha\beta\kappa\lambda} = \frac{1}{2} \left( \frac{d g_{\kappa\lambda}}{d\alpha} + \frac{d g_{\lambda\kappa}}{d\beta} - \frac{d g_{\kappa\lambda}}{d\beta} + \frac{d g_{\lambda\kappa}}{d\alpha} \right) \quad \text{Then} \quad R_{\alpha\beta\kappa\lambda} = \frac{1}{2} \left( \frac{d g_{\kappa\lambda}}{d\alpha} + \frac{d g_{\lambda\kappa}}{d\beta} - \frac{d g_{\kappa\lambda}}{d\beta} + \frac{d g_{\lambda\kappa}}{d\alpha} \right) \]

\[ \text{This is the Ricci Tensor and if we define an Einstein Tensor it does not have zero divergence, nor be symmetric, nor positivity well-defined?} \]

\[ \text{Let} \quad \Gamma^{\alpha}_{\kappa\lambda} = \frac{1}{2} \left( \frac{d g_{\kappa\lambda}}{d\alpha} + \frac{d g_{\lambda\kappa}}{d\beta} - \frac{d g_{\kappa\lambda}}{d\beta} + \frac{d g_{\lambda\kappa}}{d\alpha} \right) \]

\[ \text{Then} \quad \Gamma^{\alpha}_{\kappa\lambda} = \frac{1}{2} \left( \frac{d g_{\kappa\lambda}}{d\alpha} + \frac{d g_{\lambda\kappa}}{d\beta} - \frac{d g_{\kappa\lambda}}{d\beta} + \frac{d g_{\lambda\kappa}}{d\alpha} \right) \]

\[ \text{Finally,} \quad \Gamma^{\alpha}_{\kappa\lambda} = \frac{1}{2} \left( \frac{d g_{\kappa\lambda}}{d\alpha} + \frac{d g_{\lambda\kappa}}{d\beta} - \frac{d g_{\kappa\lambda}}{d\beta} + \frac{d g_{\lambda\kappa}}{d\alpha} \right) \]

\[ \text{Recall} \quad R_{\alpha\beta\kappa\lambda} = \frac{1}{2} \left( \frac{d g_{\kappa\lambda}}{d\alpha} + \frac{d g_{\lambda\kappa}}{d\beta} - \frac{d g_{\kappa\lambda}}{d\beta} + \frac{d g_{\lambda\kappa}}{d\alpha} \right) \]

\[ \text{This is the Ricci Tensor and if we define an Einstein Tensor it does not have zero divergence, nor be symmetric, nor positivity well-defined?} \]

\[ \text{Let} \quad \Gamma^{\alpha}_{\kappa\lambda} = \frac{1}{2} \left( \frac{d g_{\kappa\lambda}}{d\alpha} + \frac{d g_{\lambda\kappa}}{d\beta} - \frac{d g_{\kappa\lambda}}{d\beta} + \frac{d g_{\lambda\kappa}}{d\alpha} \right) \]

\[ \text{Finally,} \quad \Gamma^{\alpha}_{\kappa\lambda} = \frac{1}{2} \left( \frac{d g_{\kappa\lambda}}{d\alpha} + \frac{d g_{\lambda\kappa}}{d\beta} - \frac{d g_{\kappa\lambda}}{d\beta} + \frac{d g_{\lambda\kappa}}{d\alpha} \right) \]

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\[ \text{Finally,} \quad \Gamma^{\alpha}_{\kappa\lambda} = \frac{1}{2} \left( \frac{d g_{\kappa\lambda}}{d\alpha} + \frac{d g_{\lambda\kappa}}{d\beta} - \frac{d g_{\kappa\lambda}}{d\beta} + \frac{d g_{\lambda\kappa}}{d\alpha} \right) \]

\[ \text{This is the Ricci Tensor and if we define an Einstein Tensor it does not have zero divergence, nor be symmetric, nor positivity well-defined?} \]
Consider a coordinate system:

$$\begin{aligned}
V' \times = \chi' \times d + \psi \times d \times \chi' \\
V' \times = \chi' \times d + \psi \times d \times \chi'
\end{aligned}$$

Torsion does not affect geodesics.
Let $\nabla$ be a connection with connection forms $\omega^a = \Gamma^a_{bc} x^b x^c$, torsion $\tau^a = \frac{1}{2} \Gamma^a_{bc} \delta^c_b$. Define the pertinent connection $\nabla^* : = \Gamma^*_{ab} x^b$. Define the cotangent frame $\Theta^a = \Gamma^a_{bc} \Theta^b \Theta^c$.

Define the transposed connection $\nabla^* \Theta^a = \Theta^b \nabla^*_{ab} \Theta^c$. Define the parallel transport $P^a_{bc} = \Gamma^a_{bc} x^{d_1} \cdots x^{d_k}$. Then a coordinate system $\nabla^* \Theta^a = \Theta^b \nabla^*_{ab} \Theta^c$, $P^a_{bc} = \Gamma^a_{bc} x^{d_1} \cdots x^{d_k}$.

Examples: 
1. Divergence: $\nabla^J = \nabla^J \Theta^a = \frac{1}{2} (\Theta^b \nabla^*_{ab} \Theta^c) = \Theta^b \nabla^*_{ab} \Theta^c$.
2. Covariance: $\Theta^a = \Theta^b \Theta^c$. This motivates a general formula.

\[ (\Theta^a)^* = \Theta^b \nabla^*_{ab} \Theta^c = \Theta^b \nabla^*_{ab} \Theta^c. \]

For a vector-valued form $\Theta^a \Theta^b \Theta^c$, define $\Theta^a \Theta^b \Theta^c$. This is a general formula.

\[ \Theta^a \Theta^b \Theta^c = \Theta^d \nabla^*_{ab} \Theta^c = \Theta^d \nabla^*_{ab} \Theta^c. \]

Notes: 
- If $\Theta^a = 0$, then $\Theta^a = \Theta^b \Theta^c$.
If we omit the second term, and $X_{1}$SA = $X_{1}$SA, we have the formula:

$$(X_{1}S_{1}S_{2}S_{3}S_{4}) = (X_{1}S_{1}S_{2}) = (X_{1}S_{1})$$

**Example:**

$$(X_{1}S_{1}S_{2}) = \begin{pmatrix}
X_{1} & S_{1} & S_{2}
\end{pmatrix}
$$

**Example:**

$$(X_{1}S_{1}) = \begin{pmatrix}
X_{1}
\end{pmatrix}
$$

In general, in a $V_{n}$ representation, we have:

$$(X_{1}S_{1}S_{2}S_{3}S_{4}) = X(S_{1}S_{2}S_{3}S_{4})$$

Note for a vector field $X_{1}$, $X_{1}Y = -X_{1}Y = \nabla_{X}Y = \nabla_{Y}X$ which is just a restatement of $[X_{1}, Y] = X_{1}Y - YX_{1} = 0$.

**Alternative trick:** We have the preliminary formula (which we skipped above):

$$(X_{1}S_{1}S_{2}) = X_{1}S_{1}S_{2} + D(X_{1}S_{2}) + C_{1}$$

**Choose a basis with** $X_{1}S_{1} = 0$, then $X_{1}S_{2} = X_{1}S_{2}$ so that we get the above formula, which is then true in any frame.

**Also in matrix form:**

$$(X_{1}S_{1}S_{2}) = X_{1}S_{1}S_{2} + D(X_{1}S_{2}) + C_{1}$$

Thus $X_{1}S_{2}$ is a symmetry if $\nabla_{X}S_{2} = 0$.

For a metric connection:

$$(X_{1}S_{1}S_{2}) + D(X_{1}S_{2}) = 0$$

**In a $V_{n}$ representation:**

$$(X_{1}S_{1}S_{2}) + D(X_{1}S_{2}) = 0$$

And

$$D(X_{1}S_{1}S_{2}) = 0$$

**Lastly:** We mention a general formula for $D$: It is basically:

$$(X_{1}S_{1}S_{2}) = D + C_{1}$$

**Finally:** We denote a differential form involving $D$ and $\omega$ by a local notation as $\sum_{\alpha}w_{\alpha} + D\omega$. Then:

$$(X_{1}S_{1}S_{2}) = \sum_{\alpha}w_{\alpha} + D\omega$$

**Note:** $\delta w_{\alpha}$ is a Frobenius form.

**Special:**

$$\delta = C + D + D\omega$$

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**Special:**

$$\delta = C + D + D\omega$$
The unit pseudoscalar and the Hodge Star operator (see also 1.4)

First: \( E_{a b c d} \) and \( E^{a b c d} \) are the Levi-Civita indicators: \( E_{a b c d} = 1 \)

Define \( T_{a b c d} = E_{a b c d}, \) alternate. \( \det \) \( (g_{a b}) \).

Then \( T_{a b c d} = \gamma_{a} g^{b c d} \)

Define also

\[
\begin{align*}
\tilde{\tau}_{a b c d} &= \frac{1}{2} T_{a b c d} g^{e f} \\
\tilde{\tau}_{a b c d} &= \frac{1}{2} T_{a b c d} g^{e f} \\
\nu_{a b c d} &= \frac{1}{2} T_{a b c d} g^{e f} \\
\nu_{a b c d} &= \frac{1}{2} T_{a b c d} g^{e f} \\
\hat{\nu}_{a} &= \frac{1}{2} T_{a b c d} g^{e f} \\
\hat{\nu}_{a} &= \frac{1}{2} T_{a b c d} g^{e f} \\
\hat{\nu}_{a} &= \frac{1}{2} T_{a b c d} g^{e f} \\
\hat{\nu}_{a} &= \frac{1}{2} T_{a b c d} g^{e f} \\
\end{align*}
\]

\( \tilde{f} \) \( D_{a} \tilde{f} = 0 \) (we used \( D_{a} g = 0 \))

\( D_{a} \nu_{a} = D_{a} (\tilde{f} \nu_{a}) = \tilde{f} D_{a} \nu_{a} + D_{a} \tilde{f} \nu_{a} = 0 \)

Using \( \tilde{f} \) Cartan eq. etc.

We define the Hodge Star operator (for motivations see Hodge-Star notes).

\[
\begin{align*}
\hat{\nu}_{a} &= \frac{1}{2} \nu_{a} \nu^{b c d} \quad \text{where we raise and lower indices on } \nu_{a}, \nu^{b c d} \text{ indiscriminately.}
\end{align*}
\]

It is almost obvious that

\[
\begin{align*}
\nu_{a} &= \frac{1}{2} \nu_{a} \nu^{b c d} \\
\nu_{a} &= \frac{1}{2} \nu_{a} \nu^{b c d} \\
\nu_{a} &= \frac{1}{2} \nu_{a} \nu^{b c d} \\
\end{align*}
\]

\begin{align*}
\text{Other useful formulas} & \\
\text{Maximally nonsingular} & \\
\text{Maximally nonsingular} & \\
\text{Maximally nonsingular} & \\
\end{align*}

\begin{align*}
\text{Average useful results} & \\
\text{Average useful results} & \\
\text{Average useful results} & \\
\end{align*}

\[
\begin{align*}
\nu_{a} \nu^{a} &= \frac{1}{2} \nu_{a} \nu^{b c d} \\
\nu_{a} \nu^{a} &= \frac{1}{2} \nu_{a} \nu^{b c d} \\
\nu_{a} \nu^{a} &= \frac{1}{2} \nu_{a} \nu^{b c d} \\
\end{align*}
\]

\[
\begin{align*}
\text{Let } M \text{ be a spacetime hypersurface in the spacetime } N = U_{a}, \text{ with unit normal } N_{a} \text{ and let } \mathbf{f}, \mathbf{c} = \mathbf{E}_{a b c d} \text{ be a normal adapted basis w. } M. \text{ Let } g \text{ be the induced metric on } M. \text{ Then Define} & \\
\text{Define } & \\
\text{Define } & \\
\text{Define } & \\
\text{Define } & \\
\text{Define } & \\
\text{Define } & \\
\text{The relation between the } 3- \text{ and } 4- \text{ tensor fields is:} & \\
\text{Another useful result is:} & \\
\text{Another useful result is:} & \\
\text{Another useful result is:} &
\end{align*}
\]

\[
\begin{align*}
\text{We shall also need a variation formula using } \delta \tilde{f} &= \frac{1}{2} \delta g_{a b} \delta \tilde{f} : \\
\delta \tilde{f}_{a} &= \delta (g_{a b} \tilde{f}^{b}) = -\frac{1}{2} \delta g_{a b} \delta \tilde{f}^{b} + \frac{1}{2} \delta \tilde{f}^{b}_{a} \delta g_{b c} = \delta \tilde{f}_{a} - \frac{1}{2} \delta \tilde{f}^{b}_{a} \delta g_{b c} \\
\delta \tilde{f}^{a}_{b} &= \delta (g_{a b} \tilde{f}^{a} - g^{a c} g^{b d} \tilde{f}_{c d}) = \delta \tilde{f}^{a}_{b} - g^{a c} g^{b d} \delta \tilde{f}_{c d}
\end{align*}
\]
\[ R_k = \pi \wedge \Lambda^2 \]
\[ \delta \Omega^r = \partial \psi \wedge \Lambda^{\mu \nu \lambda \kappa} \]
\[ \delta \Omega^r = \frac{1}{2} \left[ \left( \pi \wedge \Lambda^{\mu \nu \lambda \kappa} \right) \Lambda^{\kappa \lambda} \right] \]
\[ \delta \Omega^r = \left( \pi \wedge \Lambda^{\mu \nu \lambda \kappa} \right) \Lambda^{\kappa \lambda} \]
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\[ \delta \Omega^r = \left( \pi \wedge \Lambda^{\mu \nu \lambda \kappa} \right) \Lambda^{\kappa \lambda} \]
We start with a manifold $M_a$ equipped with an inner product $g$ and an arbitrary vector field $X$.

We define a one-parameter family of hypersurfaces by
dragging $M_a$ along the integral curves of $X$:

$$ M_b = \exp_b(t \cdot X) $$

If the integral curves of $X$ are nowhere tangent to $M_a$, then $M_b$ is a family of hypersurfaces.

A function $t$ is defined, such that $M_b$ coincides with the hypersurface $t = 0$.

An ADM frame for the family $\{M_b\}$ is a frame $\{e_a, e_b\}$ such that $e_a$ lies in the TM.

For a fixed $b$, the distribution $\{e_a\}$ is a hypersurface forming $\{e_a, e_b\} = e_a e_b = \delta_a^b$, i.e., $C_{ab} = 0$.

$[\epsilon, e_b] = [\epsilon, e_a] = 0$ is a special case of an ADM frame, which we call the adapted ADM frame.

The condition that $t$ be constant hypersurfaces agree with the integral submanifolds of the distribution $\{e_a\}$ (assumed that the $t = 0$ hypersurfaces are one of these submanifolds) is that

$$ e^a \dot{e}_a = e^a \dot{e}_b = 0 $$

In the adapted ADM frame, this just says:

$$ [\epsilon, e_b] = e^a \dot{e}_a e_b = e^a \dot{e}_b = 0. $$

Note that $t$ is the affine parameter on the integral curves of $X$, stored on the hypersurface $M_b$.

If $X^a$ are any coordinates on $M_b$, then $(t, X)$ are ADM coordinates on $M_b$, some part of $N$ (which may be $M_b$ itself). If $X$ merely generates a diffeomorphism of $M_b$, inequivalent, i.e., convex coordinates with $t$, stored by $X$, where $X^a$ are defined to be constant on the integral curves of $X$.

By definition, the integral curve through $z$: $C_{ab}(t) = 0$ satisfies:

$$ \dot{z}^b(t) e_a = 0 $$

But $C_{ab}(t) = 0$, and $e_b = 0$, implying $\dot{z}^b = 0$.

In these coordinates, $\dot{z}^b = 0$ for any geometric object with components $\partial_a$.

By a way of direct generalization, an adapted ADM frame is an adapted ADM frame in which

- the $e_a$ are dragged along the integral curves of $e_b$:

$$ C_{ab} = e^c \dot{e}_c e_a = 0 $$

- in a convex adapted ADM frame, $C_{ab} = 0, C_{ab} = 0$, i.e., $C_{ab} = 0$.

- $e^a \dot{e}_b = 0$ for any geometric objects with components $\partial_a$ in this basis.

In any adapted ADM frame we have $\dot{e}^a = dt$, where $\dot{e}^a$ is the dual frame, provided of course that $dt$ makes sense.

Two matters require discussion. First, the geometry on $N$ induces an intrinsic geometry and an extrinsic geometry on each of the elements of the family $\{M_b\}$, which must be investigated.

Secondly, we have the question of dynamics; how is the extrinsic/intrinsic geometry of the members of the family to be interrelated? Is it to be an acceptable space-time, i.e., a solution of some field equations, or an extension of some variational principle? Here the ADM method must be discussed.
Let $M_0$ be a space-like hypersurface embedded in some Riemannian space-time $(M, \langle \cdot, \cdot \rangle)$, and let $\tilde{M}_t$ be the surface obtained from $M_0$ under the action of $e^{t\xi}$ for some vector field $\xi$.

The $\xi$-metric $\langle \cdot, \cdot \rangle_\xi$ induces a metric $g_\xi$ on each $\tilde{M}_t$, and $\tilde{M}_0$ induces a metric on each $M_0$ by dragging, namely $(\exp(t\xi))^* g_\xi = \gamma_t$.

The time $t$ is an orthometry $(M_0, g_\xi) \to (M_t, \gamma_t)$.

The three geometry $\mathcal{G}_0$ of the embedded surface $M_0$ is the equivalence class of $(M_0, g_\xi)$, equivalence defined by $(X_0, g_\xi) \sim (X_1, g_\xi) \iff \exists$ diffeo $f: x \mapsto x' : g_1 = f^* g_0$.

The ADM equations:

\[ \frac{d^2 \mathcal{G}_0}{dt^2} = (\cdot) , \quad \frac{d \mathcal{G}_0}{dt} = (\cdot) \]

will give us information on $(M_0, g_\xi)$ as the induced structure on the surface as it moves through space-time. The Lie derivative enters because one wants to compare $(M_t, g_t)$ with the static geometry $(M_0, \gamma_0)$ in the limit $t \to 0$. 

[No image provided]
Let $M$ be a space-like hypersurface imbedded in $N$ via $\Sigma \rightarrow N$.

We show that $M$ has on induced $(g, \nabla, \tau = 0)$ structure, and then examine the left-over in the extrinsic geometry of $M$, not necessarily in this order:


\[ g(\nabla X, \nabla Y) = \delta(X,Y) \quad \text{(we identify $X, \nabla X, \text{etc.}$)} \]

\[ n = \text{unit (light-like)} \text{ normal to } M : \quad \xi(n) = 0, \quad \xi(\tau) = 0, \quad \tau(\tau) = 0 \quad \text{for } \tau \in TM \]

Define the extrinsic curvature tensor:

\[ X \in \mathfrak{X}(TM) \quad \nabla \tau(X) = \left\langle n, \nabla \tau X \right\rangle = -\left\langle\nabla_n X, \tau\right\rangle \]

Recall $X \in \mathfrak{X}(TM) \rightarrow [X] \in \mathfrak{X}(TM)$.

\[ K(X,Y) = \left\langle \nabla_X Y, \tau\right\rangle = \left\langle \nabla_X Y - \nabla_Y X, \tau\right\rangle = \left\langle \nabla_X (Y - \nabla_Y X), \tau\right\rangle \]

\[ \tau(\nabla X) = -\nabla X n, \quad K(X,Y) = \tau(X,Y) \quad \tau(X) = \left\langle \nabla_X Y, \tau\right\rangle n \]

Note that $\delta(\tau(X), \tau(Y)) = \tau(\nabla_X Y, \tau) = K(X,Y) = \left\langle \nabla_X Y, \tau\right\rangle n$.

L is not self-adjoint, and the projection of the torsion onto $M$ is a measure of the non-self-adjointness of $\nabla$.

\[ \text{L is self-adjoint, so all the eigenvalues (principal curvatures) are real and the eigenvectors (principal directions) may be chosen orthogonal.} \]

Along a principal direction $X$, $\nabla_X \tau = 0$.

Define the induced connection $\nabla^X$ by subtracting off the extrinsic curvature component:

\[ X \in \mathfrak{X}(TM) \quad \nabla^X X = \left\langle \nabla_X Y, \tau\right\rangle Y \quad \nabla^X Y = \mathcal{L}_Y X - K(X,Y) X \quad \nabla^X \tau = \tau(X,Y) X - \nabla^X Y \]

The induced torsion is therefore:

\[ T^X(Y,Z) = \left\langle \nabla_Y Z - \nabla_Z Y - \nabla_Y \tau X - \nabla_Z \tau X + K(Y,Z) X, \tau\right\rangle = \nabla_Y X \nabla^X Y + \nabla_Z X \nabla^X Z - \nabla^X Y \nabla^X Z - \mathcal{L}_X \tau \]

So the restriction of the 4-tensor operator to $M$ is the sum of two operators, the projection of $\nabla$ on the normal, and the induced 3-tensor operator.

\[ \text{Let } \{e_i\} \text{ be a basis on } M, \text{ then } \{e_i\} = \{e_i\}_{ij}, \text{ is a basis adapted to } M. \]

\[ \text{Let } \{e_i\} \text{ be a basis on } M, \text{ then } \{e_i\} = \{e_i\}_{ij}, \text{ is a basis adapted to } M. \]

\[ \text{We restrict ourselves to M.} \]

\[ \text{Restrict ourselves to M.} \]

\[ \text{From among the} \Omega^1 \text{forms, one forms; } \Omega^1(X) = K(X,\tau), \quad X = \text{K} \Omega^1(X) \]

\[ \text{and other torsion tensors we could define. Two useful ones are:} \]

\[ X \in \mathfrak{X}(TM), \quad X(\tau(Y,Z)) = \left\langle \nabla_X Y, \tau\right\rangle Z + \left\langle \nabla_X Z, \tau\right\rangle Y, \quad X(\tau(Z,Y)) = \left\langle \nabla_X Z, \tau\right\rangle Y + \left\langle \nabla_X Y, \tau\right\rangle Z \]

Note that in a Riemannian frame:

\[ \begin{align*}
\text{quad} (\Sigma, \mathcal{G}) \quad \mathcal{G} & \quad \text{is} \quad (\Sigma, \mathcal{G}) \\
\text{quad} (\Sigma, \mathcal{G}) \quad \mathcal{G} & \quad \text{is} \quad (\Sigma, \mathcal{G}) \\
\end{align*} \]

Some can more and lower Latin indices to Latin indices.

Define $X(\tau(Y,Z)) = \left\langle \nabla_X Y, \tau\right\rangle Z + \left\langle \nabla_X Z, \tau\right\rangle Y$ or $\nabla \tau(Y,Z) = \frac{1}{2} \left( \nabla_Y \tau Z + \nabla_Z \tau Y - \nabla_X \tau Y \right)$.

\[ X \in \mathfrak{X}(TM), \quad \tau(Y,Z) = \frac{1}{2} \left( \nabla_Y \tau Z + \nabla_Z \tau Y - \nabla_X \tau Y \right) \]

\[ \text{we define:} \quad \mathcal{G} = \mathcal{G} \quad \text{on } \Sigma, \quad \mathcal{G} = \mathcal{G} \quad \text{on } \Sigma. \]
We first decompose the Lagrangian in a $\mathcal{B}_3$-split. Note $\mathcal{B}_3 = \mathcal{B}_3 \mathcal{B}_3$. We work in a normally adapted $\mathcal{B}_3$-basis, in which we can raise and lower Latin indices to Latin indices.

We also choose the signature $(-+++)$: 

$$\eta_{\mu \nu} = 1, \eta_{\mu \nu} = -1.$$ 

We can express the field equations of motion from $\mathcal{B}$-theory as a set of equations for $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. 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We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. 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We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then use the $\mathcal{B}$-fields to calculate the $\mathcal{B}$-fields. We can then us...
First, we note that the acceleration \( \ddot{\alpha} \) is given by

\[
\ddot{\alpha} = \left[ \begin{array}{c}
\frac{d}{dt} \left( \frac{dN}{d\alpha} \right) \\
\frac{d}{dt} \left( \frac{dN}{d\beta} \right)
\end{array} \right]_{\left. \right|_{\gamma^0}} = \left( \begin{array}{c}
\frac{d^2N}{d\alpha^2} \\
\frac{d^2N}{d\beta^2}
\end{array} \right)_{\left. \right|_{\gamma^0}}
\]
Define the gravitational Hamiltonian, \( H_0 = \Pi^{ab} \dot{g}_{ab} - \frac{1}{2} \Pi_{ab} \Pi^{ab} + \nabla^a \nabla_b \Phi^c - \frac{1}{2} \nabla^a \Phi^a \nabla_b \Phi^c \), which will be written in terms of the momenta. We now work on the inversion.

1. \( \Pi^{ab} \dot{g}_{ab} = \Pi^{ab} \left( \frac{1}{2} \nabla^{(a} \Phi^{b)} - \nabla^{(a} \Phi^{b)} \right) = 2 \Pi^{ab} \nabla_a \nabla_b \Phi^c + 2 \nabla^c \left( \frac{1}{2} \nabla^a \Phi^a \nabla_b \Phi^c - \frac{1}{2} \nabla^a \Phi^a \nabla_b \Phi^c \right) \)

\[-T = N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) \]

2. \( N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) = \frac{1}{2} N \left( k_1 \nabla \Phi \right)^2 + \frac{1}{2} N \left( \dot{\Phi} \nabla \Phi \right) \)

\[-T = N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) \]

3. \( N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) = \frac{1}{2} N \left( k_1 \nabla \Phi \right)^2 + \frac{1}{2} N \left( \dot{\Phi} \nabla \Phi \right) \)

4. \( N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) = \frac{1}{2} N \left( k_1 \nabla \Phi \right)^2 + \frac{1}{2} N \left( \dot{\Phi} \nabla \Phi \right) \)

5. \( N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) = \frac{1}{2} N \left( k_1 \nabla \Phi \right)^2 + \frac{1}{2} N \left( \dot{\Phi} \nabla \Phi \right) \)

6. \( N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) = \frac{1}{2} N \left( k_1 \nabla \Phi \right)^2 + \frac{1}{2} N \left( \dot{\Phi} \nabla \Phi \right) \)

7. \( N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) = \frac{1}{2} N \left( k_1 \nabla \Phi \right)^2 + \frac{1}{2} N \left( \dot{\Phi} \nabla \Phi \right) \)

8. \( N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) = \frac{1}{2} N \left( k_1 \nabla \Phi \right)^2 + \frac{1}{2} N \left( \dot{\Phi} \nabla \Phi \right) \)

9. \( N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) = \frac{1}{2} N \left( k_1 \nabla \Phi \right)^2 + \frac{1}{2} N \left( \dot{\Phi} \nabla \Phi \right) \)

10. \( N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) = \frac{1}{2} N \left( k_1 \nabla \Phi \right)^2 + \frac{1}{2} N \left( \dot{\Phi} \nabla \Phi \right) \)

11. \( N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) = \frac{1}{2} N \left( k_1 \nabla \Phi \right)^2 + \frac{1}{2} N \left( \dot{\Phi} \nabla \Phi \right) \)

12. \( N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) = \frac{1}{2} N \left( k_1 \nabla \Phi \right)^2 + \frac{1}{2} N \left( \dot{\Phi} \nabla \Phi \right) \)

13. \( N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) = \frac{1}{2} N \left( k_1 \nabla \Phi \right)^2 + \frac{1}{2} N \left( \dot{\Phi} \nabla \Phi \right) \)

14. \( N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) = \frac{1}{2} N \left( k_1 \nabla \Phi \right)^2 + \frac{1}{2} N \left( \dot{\Phi} \nabla \Phi \right) \)

15. \( N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) = \frac{1}{2} N \left( k_1 \nabla \Phi \right)^2 + \frac{1}{2} N \left( \dot{\Phi} \nabla \Phi \right) \)

16. \( N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) = \frac{1}{2} N \left( k_1 \nabla \Phi \right)^2 + \frac{1}{2} N \left( \dot{\Phi} \nabla \Phi \right) \)

17. \( N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) = \frac{1}{2} N \left( k_1 \nabla \Phi \right)^2 + \frac{1}{2} N \left( \dot{\Phi} \nabla \Phi \right) \)

18. \( N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) = \frac{1}{2} N \left( k_1 \nabla \Phi \right)^2 + \frac{1}{2} N \left( \dot{\Phi} \nabla \Phi \right) \)

19. \( N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) = \frac{1}{2} N \left( k_1 \nabla \Phi \right)^2 + \frac{1}{2} N \left( \dot{\Phi} \nabla \Phi \right) \)

20. \( N \left( k_1 \dot{\Phi} + 2 \dot{\Phi} \nabla \Phi \right) = \frac{1}{2} N \left( k_1 \nabla \Phi \right)^2 + \frac{1}{2} N \left( \dot{\Phi} \nabla \Phi \right) \)
THE VARIATION OF THE LAGRANGIAN

The momenta conjugate to the frame \( \tau^a \) and the metric \( g_{ab} \) are not independent: \( \Pi^a = \partial \mathcal{L} / \partial \dot{\tau}^a \) (because we require the metric to be constant); so we can simultaneously vary \( \Pi^a \) and \( \Pi^{ab} \) independently.

We therefore assume a covariant frame. The variational principle will consist of independent variations of the metric coefficients \( g_{ab} \), the conjugate momentum \( \Pi^{ab} \), the lapse, the shift and the torsion fields. Before writing the Lagrangian we derive useful preliminary results.

Let \( \mathcal{L} \) be a \( 3\)-form: \( \mathcal{L} = \epsilon^{abc} \mathcal{L}_{abc} \), where \( \epsilon^{abc} \) is a volume form.

\[
\mathcal{L}_{abc} = \epsilon^{abc} \left( g_{ac} \frac{\partial \Pi^b}{\partial \tau^c} + g_{bc} \frac{\partial \Pi^a}{\partial \tau^c} \right)
\]

Using this and \( \epsilon^{abc} = \delta^{abc} \) (where \( \epsilon \) is a volume form) we get a formula for \( \mathcal{L}_{abc} \), which is a \( 3\)-form in \( \Pi^a \):

\[
\delta \mathcal{L}_{abc} = \mathcal{L}_{abc} \delta g_{ac} + \mathcal{L}_{abc} \delta g_{bc} + \mathcal{L}_{abc} \delta \Pi^a + \mathcal{L}_{abc} \delta \Pi^b + \mathcal{L}_{abc} \delta \Pi^c
\]

Traditionally, we call \( \mathcal{L}_{abc} \) the variational \( 3\)-form, and the \( \delta \mathcal{L}_{abc} \) the field variation of \( \mathcal{L}_{abc} \).

\[
\delta \mathcal{L}_{abc} = \mathcal{L}_{abc} \delta g_{ac} + \mathcal{L}_{abc} \delta g_{bc} + \mathcal{L}_{abc} \delta \Pi^a + \mathcal{L}_{abc} \delta \Pi^b + \mathcal{L}_{abc} \delta \Pi^c
\]

Recall the formula \( \delta g_{ab} = \partial \omega^{ab} / \partial \tau^c - \partial \omega^{ca} / \partial \tau^b + \partial \omega^{bc} / \partial \tau^a \). We require \( \mathcal{L}_{abc} \) to be a \( 3\)-form.

\[
\mathcal{L}_{abc} = \mathcal{L}_{abc} \delta g_{ab} + \mathcal{L}_{abc} \delta \Pi^a + \mathcal{L}_{abc} \delta \Pi^b + \mathcal{L}_{abc} \delta \Pi^c
\]

Last, recall \( \omega^{ab} = D g_{ab} \). (6)

We proceed with the variation of the Hamiltonian density \( \delta \mathcal{H} = \mathcal{H}_N + \mathcal{N}_E \).

\[
\delta \mathcal{H} = \mathcal{H}_N + \mathcal{N}_E + \mathcal{N}_E + \mathcal{N}_E
\]

The terms \( \mathcal{H}_N \) are called Hamiltonian constraints, which arise from the diffeomorphism gauge group of the theory.
We get exactly analogous formulas except for minus sign:

\[ \delta(\lambda X^a X^b) = \delta(\lambda Y^a Y^b) = \delta(\lambda X^a Y^b) = \delta(\lambda Y^a X^b) = \delta(\lambda X^a X^b) = \delta(\lambda Y^a Y^b) = \delta(\lambda X^a Y^b) = \delta(\lambda Y^a X^b) \]

Now we have to worry:

\[ \delta(\lambda X^a Y^b) - \delta(\lambda Y^a X^b) = \delta(\lambda Y^a X^b) - \delta(\lambda X^a Y^b) \]

Finally, the last term in \( \lambda \theta \):}

\[ \delta(\lambda X^a Y^b) = d[\lambda X^a Y^b] \delta(\lambda Y^a X^b) = d[\lambda Y^a X^b] \delta(\lambda X^a Y^b) \]

(7)

C + [\delta(\lambda Y^a X^b)] [\delta(\lambda X^a Y^b)] - 2N \delta(\lambda X^a Y^b) - 2N \delta(\lambda Y^a X^b) - 2N \delta(\lambda Y^a X^b) - 2N \delta(\lambda X^a Y^b)

\[ \delta(\lambda X^a Y^b) - \delta(\lambda Y^a X^b) = \delta(\lambda Y^a X^b) - \delta(\lambda X^a Y^b) \]

Define

\[ M^{\lambda \theta} = \delta(\lambda X^a Y^b) \delta(\lambda Y^a X^b) \]

But \( \delta(\lambda X^a Y^b) \delta(\lambda Y^a X^b) = \delta(\lambda X^a Y^b) \delta(\lambda Y^a X^b) \)

Using this and \( 2 \delta(\lambda X^a Y^b) \) we have:

\[ \delta(\lambda X^a Y^b) \delta(\lambda Y^a X^b) = \delta(\lambda X^a X^b) \delta(\lambda Y^a Y^b) \]
So far we have varied the action once only, as if they were all independent, but we cannot do this because the connection is metric and constrained by 18 equations $D_{\alpha\beta} \equiv \delta_{\alpha\beta}$. These constraints are introduced in the Lagrangian by a multiplier in the Hamiltonian. Thus, we may vary all the $w^{\mu}_{\nu}$ freely. The total Hamiltonian $H_{T}$ is written as $H_{T} = H_{0} + X_{\alpha\beta} D_{\alpha\beta}$ where $X^{\alpha\beta} \equiv \epsilon_{\alpha\nu}^{\beta\gamma} \epsilon_{\gamma \mu}^{\nu \lambda} \text{type 2-form.}$ The variation of this terms:

$$\delta \left[ X^{\alpha\beta} D_{\alpha\beta} \right] = \delta X^{\alpha\beta} \delta D_{\alpha\beta} + \left[ \delta w^{\mu}_{\nu} \right] A_{\nu} \left( \delta X^{\mu}_{\nu} \right) + d \left( \delta X^{\beta}_{\nu} \delta w^{\nu}_{\alpha} \right)$$

$$\delta \left[ D_{\alpha\beta} \right] = \delta \left( D_{\alpha\beta} - 2 \delta w^{\gamma}_{\nu} \omega^{\nu}_{\gamma} \right) = \delta \left( D_{\alpha\beta} \right) - 2 \delta w^{\gamma}_{\nu} \omega^{\nu}_{\gamma}$$

$$\delta \left[ X^{\alpha\beta} \delta D_{\alpha\beta} \right] = \left[ \delta X^{\alpha\beta} \right] D_{\alpha\beta} + \left( \delta X^{\alpha\beta} \right) D_{\alpha\beta}$$

$$= \delta \left( D_{\alpha\beta} \right) - 2 \left( \delta X^{\alpha\beta} \right) D_{\alpha\beta}$$

So the term involving $\delta D_{\alpha\beta}, \delta w^{\mu}_{\nu}$ are multiplied by the additional multiplier terms:

$$\delta w^{\mu}_{\nu} = \left\{ \alpha^{\nu} \right\}_{\alpha\nu} + \left\{ \beta^{\nu} \right\}_{\alpha\nu} \delta w^{\mu}_{\nu}$$

$$\delta X^{\alpha\beta} = \left\{ \gamma^{\alpha} \right\}_{\alpha\nu} \delta X^{\beta}_{\nu} + \left\{ \delta^{\beta} \right\}_{\alpha\nu} \delta X^{\alpha}_{\nu}$$

Since there is no momentum conjugate to $w^{\mu}_{\nu}$, we may set this latter expression equal to zero ($\left\{ \frac{\partial}{\partial \alpha^{\nu}} \right\} = 0$)

Taking the symmetrical part:

$$\alpha^{\nu} = \left( \alpha^{\nu} + \beta^{\nu} \right) \frac{1}{2} \left( \alpha^{\nu} - \beta^{\nu} \right) + \left\{ \gamma^{\nu} \right\}_{\alpha\nu}$$

and setting this msg $\delta g_{\mu
\nu} = \left( \alpha^{\nu} + \beta^{\nu} \right) \frac{1}{2} \left( \alpha^{\nu} - \beta^{\nu} \right) + \left\{ \gamma^{\nu} \right\}_{\alpha\nu}$

Now $\delta \left( g^{\alpha\beta} \phi \right) = \left( \delta g^{\alpha\beta} \phi \right) + \left( \delta \phi \right)^{\alpha\beta}$ so considering the antisymmetric part of $\alpha$ same:

$$\delta g^{\alpha\beta} \phi = \left( \delta g^{\alpha\beta} \phi \right) + \left( \delta \phi \right)^{\alpha\beta}$$

$$\delta \left( g^{\alpha\beta} \phi \right)^{\alpha\beta} = g^{\alpha\beta} \left( \delta \phi \right)^{\alpha\beta} + \left( \delta g^{\alpha\beta} \phi \right)$$

$$\delta \left( N^{\lambda\nu} \phi \right)^{\lambda\nu} = N^{\lambda\nu} \left( \delta \phi \right)^{\lambda\nu} + \left( \delta N^{\lambda\nu} \phi \right)$$

$$\delta \left( N^{\lambda\nu} \phi \right)^{\lambda\nu} = -2 \lambda^{\lambda} \left( \delta \phi \right)^{\lambda\nu} + \lambda^{\lambda} \left( \delta \phi \right)^{\lambda\nu}$$

$$\delta \left( N^{\lambda\nu} \phi \right)^{\lambda\nu} = 0$$

This is a constraint equation.

Weather altogether the constraint equations, these are obtained by setting to zero the coefficients of the variations of all variables not having conjugate momenta:

$$\left[ \delta g_{\mu\nu} \right] = \left\{ \begin{array}{c}
0 = 2N \varphi^{\alpha\beta} \phi^{\alpha\beta} \\
0 = 2N \left( \varphi^{\alpha\beta} \phi^{\alpha\beta} + \varphi^{\mu\nu} \phi^{\mu\nu} \right) \\
0 = 2N \left( \varphi^{\mu\nu} \phi^{\mu\nu} - \varphi^{\nu\mu} \phi^{\nu\mu} \right) \\
0 = N \left( \varphi^{\mu\nu} \phi^{\mu\nu} + \varphi^{\nu\mu} \phi^{\nu\mu} \right) \\
0 = -2N \varphi^{\mu\nu} \phi^{\mu\nu} - N \left( \varphi^{\mu\nu} \phi^{\mu\nu} - \varphi^{\nu\mu} \phi^{\nu\mu} \right)
\end{array} \right. \right.$$
\[
\begin{align*}
\phi &= 2N\left( \pi_{ab} - \frac{1}{3} \text{tr}(\pi) \gamma^{ab} \right) + 9\omega \vec{e}_b N^c + 9\omega \vec{e}_c N^b \\
-\frac{1}{16} &= N G^{ac} \gamma^{ab} \left[ R_{bc} - 2\gamma_{bc} \theta^a \right] + 2N \left[ \pi^{(a} \gamma^b \gamma^{(c} - \frac{1}{2} \pi^{ab} \gamma^{bc} \right) + \\
&\quad - \frac{1}{2} N G^{ab} \left( \pi_c^{(d} \gamma^e \pi^{b)}_{de} - \frac{1}{2} \pi^{ab} \gamma^{cd} \right) - 2N \left[ \pi^{(a} \gamma^b \gamma^{(c} - \frac{1}{2} \pi^{ab} \gamma^{cd} \right) + \\
&\quad + \frac{1}{2} N G^{ab} \left( \pi_c^{(d} \gamma^e \pi^{b)}_{de} - \frac{1}{2} \pi^{ab} \gamma^{cd} \right) \right] - D \left[ \left( N \pi + N \gamma \right) \delta^c \left( \gamma^{ab} \right) \right] - \\
&\quad - \left[ D \left( \gamma^{ab} \right) \right] \pi^{bc} \vec{e}_b N^c - \pi^{bc} \vec{e}_c N^b \\
Note that where the partial derivatives of the lapse appear one can replace \( N \pi + N \gamma \) by \( N \pi \).
\end{align*}
\]

The strategy is to solve the algebraic constraints for the torsion variables, and find \((\gamma^{ab}, \pi^{(a})\) such that the Hamiltonian constraints are satisfied. At this stage one has a good initial data set. The physicist now decides how to continue the surface by choosing the lapse, and then shifts. Using the dynamical equations he can compute the values of \((\gamma^{ab}, \pi^{(a})\) on the next surface.

Set \( N = 0 \):
\[
\begin{align*}
\delta_\omega &= 9\omega \vec{e}_b N^c + 9\omega \vec{e}_c N^b - \left( \mathcal{L}_Y g \right)_{ab} \\
\tau^{bc} &= D \left( N \pi^{(a} \right) + \pi^{bc} \vec{e}_b N^c + \pi^{bc} \vec{e}_c N^b - \left( \mathcal{L}_Y g \right)_{ab}
\end{align*}
\]

This is repeated once the surface is past being dragged along the shift vector \( \vec{e}_b \).

Note that the choice of \( N, N' \) amounts to a choice of the generator \( \vec{e}_b \) of the surface deformation.
SOLUTION TO THE VACUUM ALGEBRAIC CONSTRAINTS. RETURN TO \( V \) THEORY.

The \( V \) theory is a matter theory, i.e. in vacuum it reduces to \( V_0 \) theory as will be shown. But the solution of the vacuum algebraic constraints is instructive currently because it gives this result but because with matter terms present the reduction occurs less elegantly the same lines.

Dilaton by \( N \) is allowed even if matter is present since \( N \) would appear as a factor.

Constraint equations (a) \( \partial_\tau h^{\mu\nu} - \partial_\nu h^{\mu\tau} = 0 \), \( \partial_\nu h^{\mu\tau} = 0 \), \( \partial_\mu h = 0 \).

\( (c) \quad 0 = T^{\underline{\underline{a}}b}_{\underline{\underline{a}}b} = \left( g^{\underline{\underline{a}}b} - 2g^{\underline{\underline{a}}c}g^{\underline{\underline{c}}b}/T^{\underline{\underline{c}}c} \right) T^{\underline{\underline{c}}c} = 2g^{\underline{\underline{a}}c}T^{\underline{\underline{c}}c} \), \( 0 = T^{\underline{\underline{a}}b} \).

Now

\[ 0 = T^{\underline{\underline{a}}b} = \frac{1}{2} T^{\underline{\underline{a}}b} \left( g^{\underline{\underline{a}}b} - \frac{2g^{\underline{\underline{a}}c}g^{\underline{\underline{c}}b}}{T^{\underline{\underline{c}}c}} \right) \]

\[ = \frac{1}{2} T^{\underline{\underline{a}}b} \left( g^{\underline{\underline{a}}b} + g^{\underline{\underline{b}}c}g^{\underline{\underline{a}}c}/T^{\underline{\underline{c}}c} \right) \]

\[ = T^{\underline{\underline{a}}b} \left( g^{\underline{\underline{a}}b} + g^{\underline{\underline{c}}d}g^{\underline{\underline{d}}b}/T^{\underline{\underline{c}}c} \right) \]

\[ = T^{\underline{\underline{a}}b} \left( g^{\underline{\underline{a}}b} + g^{\underline{\underline{c}}d}g^{\underline{\underline{d}}b}/T^{\underline{\underline{c}}c} \right) \]

\[ = T^{\underline{\underline{a}}b} \quad \text{for all } \underline{\underline{a}}b \]

New

\( \underline{\underline{a}}^{\underline{\underline{b}}}c = \frac{1}{2} T^{\underline{\underline{c}}c} \underline{\underline{a}}^{\underline{\underline{b}}}c = \frac{1}{2} T^{\underline{\underline{c}}c} \left( g^{\underline{\underline{a}}b} + g^{\underline{\underline{c}}d}g^{\underline{\underline{d}}b}/T^{\underline{\underline{c}}c} \right) \]

\[ \rightarrow \underline{\underline{a}}^{\underline{\underline{b}}}c = \frac{1}{2} T^{\underline{\underline{c}}c} \left( g^{\underline{\underline{a}}b} + g^{\underline{\underline{c}}d}g^{\underline{\underline{d}}b}/T^{\underline{\underline{c}}c} \right) \]

\( \rightarrow \underline{\underline{a}}^{\underline{\underline{b}}}c = \frac{1}{2} T^{\underline{\underline{c}}c} \left( g^{\underline{\underline{a}}b} + g^{\underline{\underline{c}}d}g^{\underline{\underline{d}}b}/T^{\underline{\underline{c}}c} \right) \)

\[ \frac{1}{2} T^{\underline{\underline{a}}b} \left( g^{\underline{\underline{a}}b} + g^{\underline{\underline{c}}d}g^{\underline{\underline{d}}b}/T^{\underline{\underline{c}}c} \right) \]

\[ \rightarrow T^{\underline{\underline{a}}b} = T^{\underline{\underline{c}}c} \left( g^{\underline{\underline{a}}b} + g^{\underline{\underline{c}}d}g^{\underline{\underline{d}}b}/T^{\underline{\underline{c}}c} \right) \]

Hence

\( e) \quad 0 = Q^{\underline{\underline{a}}}Q^{\underline{\underline{b}}} - Q^{\underline{\underline{a}}}Q^{\underline{\underline{b}}} + Q^{\underline{\underline{a}}}Q^{\underline{\underline{b}}} + Q^{\underline{\underline{a}}}Q^{\underline{\underline{b}}} \)

\( \rightarrow Q^{\underline{\underline{a}}}Q^{\underline{\underline{b}}} = 0 \)

Now use \( \partial_\tau T^{\underline{\underline{a}}b} = 0 \)

\[ \partial_\tau Q^{\underline{\underline{a}}}Q^{\underline{\underline{b}}} = 0 \quad \text{Set } \partial_\tau T^{\underline{\underline{a}}b} = 0 \text{ to get:} \]

\[ -2Q^{\underline{\underline{a}}} = 0 \]

\[ \text{which inserted back into gives} \]

\[ T^{\underline{\underline{a}}b} = 0 \]

Thus in vacuum all the logarithms are zero. If matter were present, essentially the same manipulations would yield the solution.

STRUCTURE OF THE ALGEBRAIC CONSTRAINTS, arising from the form of the Hamiltonian).

\( \underline{\underline{a}}b \) is determined by \( \frac{d\underline{\underline{a}}b}{d\underline{\underline{a}}b} \). \( \underline{\underline{b}}b \) is the leftover of \( \underline{\underline{a}}b \), when one subtracts off \( \underline{\underline{a}}b \).

The trace of \( T \) is determined by \( \frac{d\underline{\underline{a}}b}{d\underline{\underline{a}}b} \)

\[ \underline{\underline{a}}b = \frac{1}{2} T^{\underline{\underline{a}}b} \]

\[ \underline{\underline{c}}b = \frac{1}{2} T^{\underline{\underline{c}}b} \]

\[ \underline{\underline{c}}b = \frac{1}{2} T^{\underline{\underline{c}}b} \]

\( \underline{\underline{c}}b \) is just the trace of \( T \) plus the trace of a matter term if present. (Recall \( \underline{\underline{c}}b = \frac{1}{2} T^{\underline{\underline{c}}b} \))

The trace of \( T \) is then the leftover when one subtracts \( Q \) away from it the Lagrange multiplier constraint.

Interestingly enough, \( \partial_\tau Q^{\underline{\underline{a}}b} = T^{\underline{\underline{a}}b} \) is not coupled to all \( \partial_\tau T^{\underline{\underline{a}}b} = T^{\underline{\underline{a}}b} \).

(\( \text{Q=1} \))
\[
\begin{align*}
\mathcal{L}_\phi &= -\frac{1}{2} \left( \frac{1}{N^2} \nabla^2 \phi \right) \phi + \frac{1}{2} \left( \frac{1}{N^2} \nabla^2 \phi \right) \phi + \frac{1}{2} \nabla^2 \phi \phi + \frac{1}{2} \left( \frac{2}{N^2} \nabla^2 \phi \right) \phi \\
\Pi &= \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} N \phi - \frac{1}{2} N^\prime \phi N^\prime - \frac{1}{2} N^\prime \phi N^\prime - \frac{1}{2} \left( \frac{1}{N^2} \nabla^2 \phi \right) \phi \\
H &= \Pi \dot{\phi} - \mathcal{L}_\phi \\
\Pi &= \Pi \frac{1}{2} N \phi - \frac{1}{2} \left( \frac{1}{N^2} \nabla^2 \phi \right) \phi \\
H &= \Pi \frac{1}{2} N \phi - \frac{1}{2} \left( \frac{1}{N^2} \nabla^2 \phi \right) \phi \\
\end{align*}
\]