9 STOKES' THEOREM WITHOUT METRIC, WITH METRIC

ERRATA On p.75 I mentioned that I mistakenly used the left natural dual on p. 64. If we redo that derivation with the right natural dual, a sign appears in the divergence formula.

$$\circledast\sigma^i = \frac{1}{(n-1)!}\sigma_{i_1\cdots i_{n-1}}\epsilon^{i_1\cdots i_{n-1}i} \qquad \sigma_{i_1\cdots i_{n-1}} = \epsilon_{i_1\cdots i_{n-1}i}\circledast\sigma^i$$

$$\sigma = \frac{1}{(n-1)!}\sigma_{i_1\cdots i_{n-1}}dx^{i_1\cdots i_{n-1}} = \frac{1}{(n-1)!}\epsilon_{i_1\cdots i_{n-1}i}\circledast\sigma^i dx^{i_1\cdots i_{n-1}} = \circledast\sigma^i d\hat{x}_i$$

$$d\hat{x}_i \equiv \frac{1}{(n-1)!}\epsilon_{i_1\cdots i_{n-1}i}\,dx^{i_1\cdots i_{n-1}} = \epsilon_{|i_1\cdots i_{n-1}|i}\,dx^{|i_1\cdots i_{n-1}|}$$

$$= (-1)^{n-i}\underbrace{\epsilon_{1\cdots i-1\,i\,i+1\cdots n}}_{1}\,dx^{1\cdots i-1\,i+1\cdots n}$$

$$dx^j\wedge d\hat{x}_i = \delta^j{}_i(-1)^{n-i+i-1}dx^{1\cdots n} = (-1)^{n-1}\delta^j{}_i\,dx^{1\cdots n}$$

$$d\sigma = \partial_j(\circledast\sigma^i)dx^j\wedge d\hat{x}_i = (-1)^{n-1}(\partial_j\circledast\sigma^j)\,dx^{1\cdots n}$$

This sign $(-1)^p = (-1)^{n-1}$ appears here because of the right dual. It will reappear below.

[Another option for the natural dual is to use a right dual for p-forms and a left dual for p-vectors so that $\circledast\circledast = 1$
Synge and Schild, TENSOR CALCULUS]

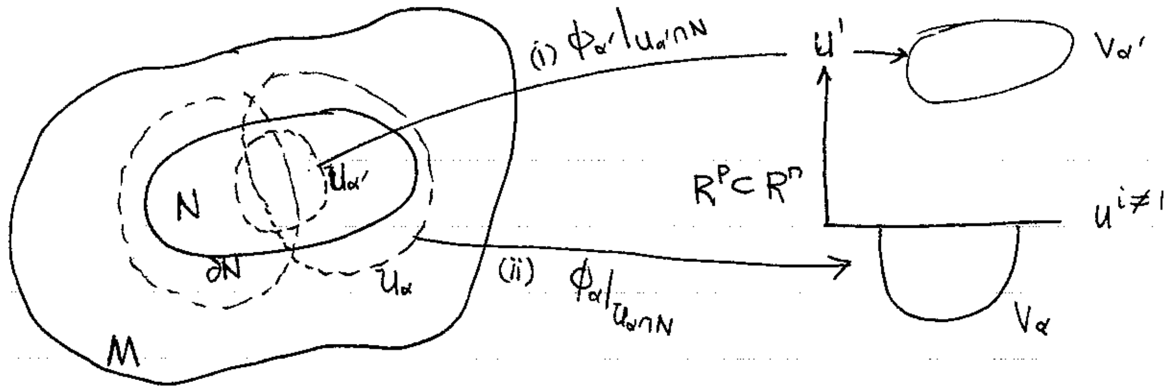
The incorrect spelling of exercise in previous notes is blamed on bad vibes. Speaking of exercises, students, where are they? I haven't seen but 2 faces in my office.

To discuss STOKES' theorem we first generalize the notion of a p-dimensional submanifold. First introduce the space $H^p = \{(r_1,\ldots,r_p)\in R^p \mid r_1\le 0\}$
and let $H^p\times\{0^{n-p}\} = \{(r_1,\ldots,r_p,0,\ldots,0)\in R^n \mid (r_1,\ldots,r_p)\in H_p\}$.
H stands for half space.

A subset N of an n-dimensional manifold M is called a
P-DIMENSIONAL SUBMANIFOLD WITH BOUNDARY
if we can find a set of local coordinate charts $\{U_\alpha,\phi_\alpha\}$ of M (not necessarily all of M) such that $N\subset\bigcup_\alpha U_\alpha$ is covered by these charts and either

(i) $\phi_\alpha(U_\alpha\cap N) = V_\alpha$ is an open set in $R^p\subset R^n$, described by the vanishing of the last n-p coordinates: $x^i_\alpha(q)\equiv u^i\circ\phi_\alpha(q) = 0$ for $q\in U_\alpha\cap N$ and $i = p+1,\ldots,n$ [This is identical with the submanifold condition]

or (ii) $\phi_\alpha(U_\alpha\cap N) = \phi(U_\alpha)\cap(H^p\times\{0^{n-p}\}) = V_\alpha$, i.e. not only do the last n-p coordinates vanish: $x^i_\alpha(q) = u^i\circ\phi_\alpha(q) = 0$, $i = p+1,\ldots,n$
but also $x^1_\alpha(q)\le 0$ for $q\in U_\alpha\cap N$.



In other words either the image of $U_\alpha \cap N$ is described by the vanishing of the last $n-p$ coordinates, in which case it consists entirely of INTERIOR POINTS or the image satisfies the additional condition that the first coordinate be nonpositive.

Again such local coordinates are said to be adapted to N.

For an adapted coordinate chart of the type (ii), points of N satisfying $X^1(a) = 0$ are called BOUNDARY POINTS.

The set of all such points is called the BOUNDARY ∂N OF N.

A covering of ∂N by local coordinate charts adapted to N makes ∂N a $(p-1)$-dimensional submanifold of M, with the restrictions of $\{X^2, \dots, X^p\}$ serving as local coordinates on ∂N.

QUESTION. Why not make the definition so that $\{X^1, \dots, X^{p-1}\}$ are local coordinates?
ANSWER. A new complication called INDUCED ORIENTATION OF ∂N needed to hide a sign in STOKE'S THEOREM.

With this definition (R^p, id) is a global coordinate chart of R^p adapted to H^p which is a p-dimensional submanifold with boundary: $R^{p-1} = \{(r_1, \dots, r_p) \in R^p \mid r_1 = 0\}$.

If N is an oriented submanifold of an oriented manifold M, an "INDUCED ORIENTATION" of ∂N can be defined

Let $\{X^1, \dots, X^n\}$ be positively oriented adapted coordinates of type (ii) such that $\{X^1, \dots, X^p\}$ are positively oriented with respect to the inner orientation of N. The local coordinates $\{X^2, \dots, X^p\}$ on ∂N are defined to be positively oriented with respect to the induced orientation.

But in the subspace $V = \text{span}\{\frac{\partial}{\partial X^1}, \dots, \frac{\partial}{\partial X^p}\}$ of the full tangent space at a point of ∂N, the induced orientation of $W = \text{span}\{\frac{\partial}{\partial X^2}, \dots, \frac{\partial}{\partial X^p}\}$ is $(-1)^{p-1}$ times the inner orientation of this subspace $W \subset V$ induced by the outer orientation specified by $\partial/\partial X^1$ since:

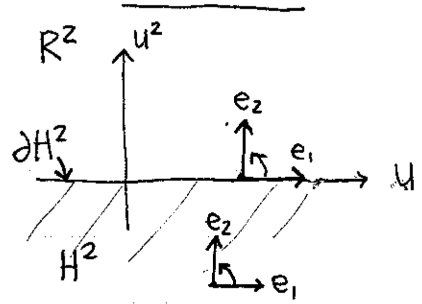
$$\left(\frac{\partial}{\partial x^2}\wedge\dots\wedge\frac{\partial}{\partial x^p}\right)\wedge\frac{\partial}{\partial x^1} = (-1)^{p-1}\Big(\underbrace{\frac{\partial}{\partial x^1}\wedge\dots\wedge\frac{\partial}{\partial x^p}}\Big)$$

positively oriented with respect to inner orientation of N

Alternatively, the $(n-p)$-vector $Z_N = \frac{\partial}{\partial x^{p+1}}\wedge\dots\wedge\frac{\partial}{\partial x^n}$ specifies the inner orientation of N but the inner orientation of ∂N determined by $Z_{\partial N} = \frac{\partial}{\partial x^1}\wedge Z_N$ is $(-1)^{p-1}$ times the "induced orientation"

We could have defined adapted coordinates more naturally so that $x^p(q)=0$ for $q\in\partial N$ in which case $Z_{\partial N} = \frac{\partial}{\partial x^p}\wedge Z_N$ induces an inner orientation of ∂N making $\{x^1,\dots,x^{p-1}\}$ positively oriented. Then we could have defined the "induced orientation" to be $(-1)^{p-1}$ times this natural orientation.

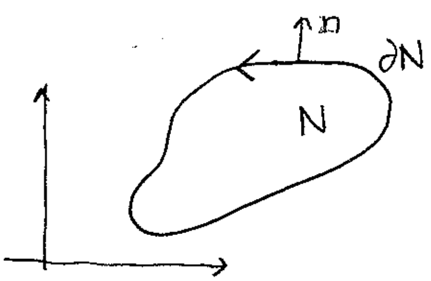
EXAMPLES



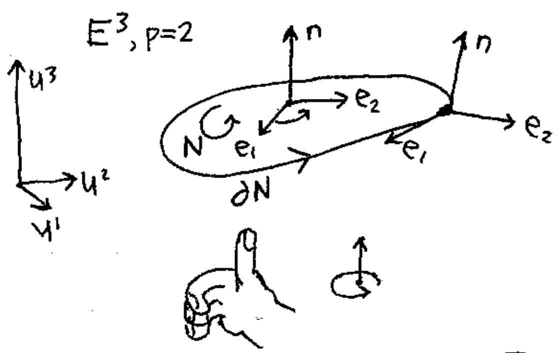
$H^2\subset R^2$, $p=2$: $(-1)^{p-1}=-1$ Let $\{e_i\}=\{\partial/\partial u^i\}$.

$\{e_1,e_2\}$ is positively oriented on H^2 and e_2 is the unit outward normal on ∂H^2

Since $e_1\wedge e_2$ is positively oriented, e_1 is positively oriented with respect to the corresponding inner orientation by negatively oriented with respect to the induced orientation.



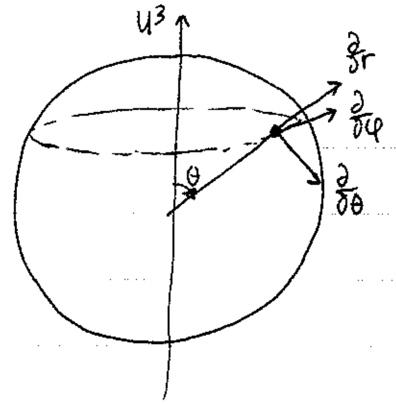
In fact the induced orientation for any open submanifold $N\subset R^2$ is just the counterclockwise orientation for the curve ∂N (opposite to the orientation corresponding to the outward pointing normal)



Let $\{e_1,e_2,n\}$ be an ON positively oriented frame with n a unit normal to N on N and e_1 tangent to ∂N on ∂N.

Then $n_N = n$ and $n_{\partial N} = e_2\wedge n$ but although $e_1\wedge n_{\partial N}$ is positively oriented, $-e_1$ specifies the induced orientation of ∂N.

This is exactly the circulation sense associated with the inner orientation of N and following from the right hand rule.



Let $N = S^2 \cup \text{interior } S^2 \subset E^3$

$\partial N = S^2$.

Since $p = 3$, $(-1)^{p-1} = 1$ so the induced orientation of S^2 is exactly that corresponding to the outward normal $n = \partial/\partial r$.

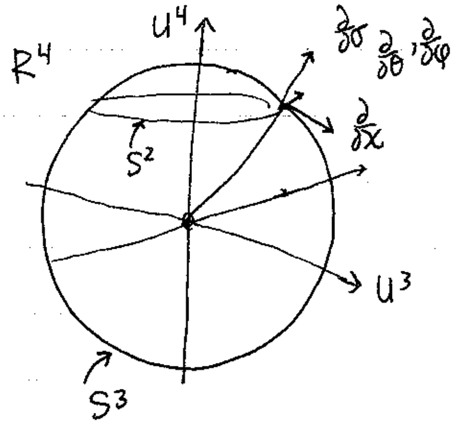
Note $\{r-1, \theta, \varphi\}$ are adapted coordinates and $\{\theta, \varphi\}$ are positively oriented, period.

The outward direction determines the induced orientation for any 3-submanifold of a 3-dimensional manifold. [Right hand rule applies]

Now let $N = \{x \in S^2 \mid \theta(x) \in [0, \theta_0]\}$ $\qquad p = 2, (-1)^p = -1$

$\partial N = \{x \in S^2 \mid \theta(x) = \theta_0\}$

Then $\{\theta - \theta_0, \varphi, r-1\}$ are adapted coordinates and $\frac{\partial}{\partial \varphi}$ is positively oriented as discussed on the previous page.



Now go to R^4 and let N be S^3 plus its interior, with $\partial N = S^3$, so $p = 4$, $(-1)^{p-1} = -1$

Spherical coordinates $\{\chi, \theta, \varphi, \sigma\}$ are positively oriented with respect to the natural orientation:

$u^1 = \sigma \sin\chi \sin\theta \cos\varphi$
$u^2 = \sigma \sin\chi \sin\theta \sin\varphi$
$u^3 = \sigma \sin\chi \cos\theta$
$u^4 = \sigma \cos\chi$

$\{\chi, \theta, \varphi\}$ are positively oriented with respect to the inner orientation of S^2 determined by the unit outward normal $\partial/\partial\sigma$, but $\{\sigma - 1, \chi, \varphi, \theta\}$ are adapted coordinates positively oriented on R^4, so $\{\chi, \varphi, \theta\}$ are positively oriented with respect to the induced orientation (opposite orientation)

Now let $N = \{x \in S^3 \mid \chi(x) \in [0, \chi_0]\}$, $\partial N \sim S^2$, $p = 3$, $(-1)^{p-1} = 1$

In this case $\{\chi - \chi_0, \theta, \varphi, \sigma - 1\}$ are positively oriented adapted coordinates and $\{\theta, \varphi\}$ are positively oriented on ∂N, period.

We'll save the spacetime examples till after Stokes' Theorem with metric.

Note that in an n-dimensional manifold, the outward direction at the boundary of an n-dimensional submanifold with boundary determines an inner orientation which is $(-1)^{n-1}$ times the induced orientation.

STOKES' THEOREM

If β is a p-form on M and N is an oriented $(p+1)$-submanifold with boundary ∂N having the induced orientation, then

$$\int_{\partial N} \beta = \int_N d\beta .$$

Note $\dim \partial N = p$ now so the induced orientation sign is $(-1)^p$.
When $p = n-1$, N has the natural orientation of M itself while the induced orientation of ∂N is $(-1)^{n-1}$ times that determined by the outward direction at the boundary. When $p=1$, the curve ∂N has the same orientation as the circulation sense of the inner orientation of N.

ERRATA. $*F$ has the wrong sign on pages 38 and 69. How come nobody caught me?
Correct signs:
$$F = E_i\, dx^i \wedge dt + \tfrac{1}{2} B^i \epsilon_{ijk}\, dx^{jk}$$
$$*F = -B_i\, dx^i \wedge dt + \tfrac{1}{2} E^i \epsilon_{ijk}\, dx^{jk}$$
Then on page 69: $*d*F = 4\pi J$ or $d*F = 4\pi * J$.

EXAMPLES in spacetime with electromagnetism : $M = M^4$

$p=1$: Let N be a 2-surface with boundary at constant time t and ∂N its boundary with unit tangent vector $n^i \frac{\partial}{\partial x^i}$ of positive orientation:

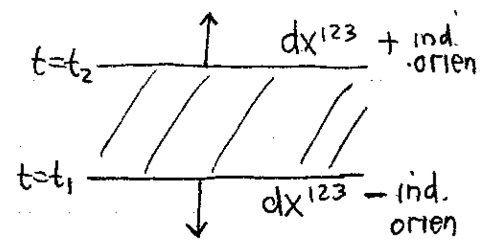
$$\int_N dA = \int_N F = \int_N \tfrac{1}{2} B^i \epsilon_{ijk}\, dx^{jk}\Big|_N = \int_N B^i n_i\, da \qquad (\partial N \neq \phi)$$
$$\int_{\partial N} A = \int_{\partial N} A_i dx^i\Big|_{\partial N} = \int_{\partial N} A_i n^i\, d\ell$$

$p=2$: Let N be a 3-manifold with boundary at fixed time t with the natural orientation (past directed normal) and let ∂N have the natural orientation (outward normal at fixed time).

$$0 = \int_N dF = \int_{\partial N} F = \int_{\partial N} \tfrac{1}{2} B^i \epsilon_{ijk}\, dx^{jk}\Big|_{\partial N} = \int_{\partial N} B^i n_i\, da \qquad (\partial\partial N = \phi)$$

$$\int_N d*F = 4\pi \int_N *J = 4\pi \int_N \rho\, dx^{123}\Big|_N = 4\pi Q \qquad (\partial N \neq \phi)$$
$$\int_{\partial N} *F = \int_{\partial N} \tfrac{1}{2} E^i \epsilon_{ijk}\, dx^{jk}\Big|_{\partial N} = \int_{\partial N} E^i n_i\, da$$

$p=3$: Let N be the region between $t=t_1$ and $t=t_2 > t_1$. $(-1)^3 = -1$.
Then $d*J = d^2 * F = 0$ so



$$\int_N d^*J = 0$$

$$\int_{\partial N} {}^*J = \int_{\partial N} \rho\, dx^{123}\Big|_{\partial N} = \int_{t_2} \rho\,(+dx^1dx^2dx^3) + \int_{t_1} \rho\,(-dx^1dx^2dx^3)$$

$$= Q(t_2) - Q(t_1)$$

i.e. the total charge is independent of time; "charge is conserved"

In all of these examples, the restriction to the submanifold essentially makes the metric appear since the coordinates are orthonormal cartesian coordinates and natural and metric duals coincide up to sign and index raising & lowering (more signs). Similar results occur for orthogonal coordinates.

However, on an arbitrary pseudo-Riemannian manifold M, Stokes' Theorem can be explicitly rewritten in terms of the metric in coordinate free notation. Instead of a metric independent theorem for integrating p-forms, it becomes a metric dependent theorem for integrating p-vector fields.

PRELIMINARY STEP: Integration by parts with metric, divergence

Let α be a $(p-1)$-form, β a p-form, then

$$d(\alpha\wedge{}^*\beta) = d\alpha\wedge{}^*\beta + (-1)^{p-1}\alpha\wedge d^*\beta = \langle d\alpha,\beta\rangle\eta + \langle\alpha, \underbrace{(-1)^{p-1} {}^{*-1}\overbrace{d^*\beta}^{n-p+1\text{-form}}}_{\equiv -\delta\beta}\rangle\eta$$

$$\delta\beta = \underbrace{(-1)^{\frac{n-s}{2} + (n-p+1)(n-[n-p+1]) + p}}_{(-1)^{\frac{n-s}{2}}(-1)^{(n-p+1)(p-1)+p}} {}^*d^*\beta$$

$$n(p-1) + \underbrace{p - p^2}_{-p(p-1)\ \text{even}} - 1 + \overbrace{2p}^{\text{even}}$$

$$\delta\beta \equiv (-1)^{\frac{n-s}{2}}(-1)^{n(p-1)+1}\, {}^*d^*\beta \equiv -\operatorname{div}\beta \qquad (p-1)\text{-form}$$

$$\operatorname{div}\beta \equiv (-1)^{\frac{n-s}{2}}(-1)^{n(p-1)}\, {}^*d^*\beta$$

Then

$$\langle d\alpha,\beta\rangle\eta = d(\alpha\wedge{}^*\beta) + \langle\alpha,\delta\beta\rangle\eta$$

$$\int_C \langle d\alpha,\beta\rangle\eta = \int_C \langle\alpha,\delta\beta\rangle\eta + \underbrace{\int_{\partial C}\alpha\wedge{}^*\beta}$$

If ∂C vanishes (compact manifold without boundary) or α and β belong to a function space so that this integral vanishes, then δ is the adjoint of d with respect to the inner product $\int_C \langle\ ,\ \rangle\eta$ [like in quantum mechanics]

δ is called the codifferential.

coordinate version:

$$\frac{1}{p!}\int_c p\,\partial_{[i_1}\alpha_{i_2\cdots i_p]}\,\mathcal{B}^{i_1\cdots i_p}\,\eta_{1\cdots n}\,dx^{1\cdots n} = \frac{1}{(p-1)!}\int_c \partial_{i_1}\alpha_{i_2\cdots i_p}\,\mathcal{B}^{i_1\cdots i_p}\,g^{1/2}\,dx^{1\cdots n}$$

$$= \frac{1}{(p-1)!}\int_c \partial_{i_1}(\alpha_{i_2\cdots i_p}\mathcal{B}^{i_1\cdots i_p}g^{1/2})\,dx^{1\cdots n} - \underbrace{\frac{1}{(p-1)!}\int_c \alpha_{i_2\cdots i_p}\underbrace{\partial_{i_1}(\mathcal{B}^{i_1\cdots i_p}g^{1/2})}_{\equiv(\mathrm{Div}\,\mathfrak{B})^{i_1\cdots i_p}}\,dx^{1\cdots n}}_{\frac{1}{(p-1)!}\int_c \alpha_{i_2\cdots i_p}\underbrace{[g^{-1/2}\partial_{i_1}(\mathcal{B}^{i_1\cdots i_p}g^{1/2})]}_{\equiv(\mathrm{div}\,\mathcal{B})^{i_1\cdots i_p}}\eta}$$

$$\mathfrak{B}^{i_1\cdots i_p} \equiv g^{1/2}\,\mathcal{B}^{i_1\cdots i_p} \qquad (p-1) \text{ oriented vector density}$$

$$\underbrace{(\mathrm{Div}\,\mathfrak{B})^{i_2\cdots i_p}}\,dx^{1\cdots n} = \underbrace{(\mathrm{div}\,\mathcal{B})^{i_2\cdots i_p}}\,\eta$$

natural divergence of $\mathfrak{B}$: $\partial_{i_1}\mathfrak{B}^{i_1\cdots i_p}$

contravariant form of metric divergence of $\mathcal{B}$: $g^{-1/2}\partial_{i_1}[g^{1/2}\mathcal{B}^{i_1\cdots i_p}]$

Define the metric divergence of a p-vector field $\mathbb{X}$ by

$$\mathrm{div}\,\mathbb{X} \equiv (\mathrm{div}\,\mathbb{X}^{\flat})^{\#} = g^{-1/2}(\partial_{i_1}(g^{1/2}\mathbb{X}^{i_1\cdots i_p})\,\frac{\partial}{\partial x^{i_2}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_p}}$$

and let $\delta\mathbb{X} \equiv -\mathrm{div}\,\mathbb{X}$.

<u>EXAMPLE</u> Electromagnetism on M^4.

$\mathrm{div}\,F = -{*d*F} = -4\pi J$ or in cartesian coordinates

$-4\pi J^{\alpha} = \partial_\beta F^{\beta\alpha} = -\partial_\beta F^{\alpha\beta} \equiv -F^{\alpha\beta}{}_{,\beta}$ or $F^{\alpha\beta}{}_{,\beta} = 4\pi J^{\beta}$

$\mathrm{div}\,{*F} = -{*d**F} = {*dF} = 0$ or ${*F}^{\alpha\beta}{}_{,\beta} = 0$

Now we're ready to rewrite STOKES' THEOREM.

Let's turn the page for this.

Using page 89: defining $\displaystyle \mathbb{X} = \frac{(-1)^{\frac{n-s}{2}} (*\beta)^{\#}}{\langle n_{\partial N}, Z_{\partial N}\rangle}$ (note mistake on p. 89 here $\langle n, n\rangle \to \langle n, Z\rangle$)

and $\displaystyle Q = \frac{(-1)^{\frac{n-s}{2}} (*d\beta)^{\#}}{\langle n_N, Z_N\rangle}$

$\mathbb{X}$ = $(n-p)$-vector field
Q = $(n-p-1)$-vector field

$$\int_{\partial N} \langle \mathbb{X}, n_{\partial N}\rangle \eta_{\partial N} = \int_{\partial N} \beta = \int_N d\beta = \int_N \langle Q, n_N\rangle \eta_N$$

But $*\beta = (-1)^{\frac{n-s}{2}} \langle n_{\partial N}, Z_{\partial N}\rangle \mathbb{X}^\flat$

$$\beta = (-1)^{\frac{n-s}{2}} \langle n_{\partial N}, Z_{\partial N}\rangle *^{-1} \mathbb{X}^\flat = (-1)^{p(n-p)} \langle n_{\partial N}, Z_{\partial N}\rangle * \mathbb{X}^\flat$$

$$(*d\beta)^{\#} = (-1)^{p(n-p)} \underbrace{\langle n_{\partial N}, Z_{\partial N}\rangle}_{\text{constant}} \underbrace{(*d*\mathbb{X}^\flat)^{\#}}_{(-1)^{\frac{n-s}{2}} \cdot n(n-p-1)} \operatorname{div}\mathbb{X}$$

$$Q = \frac{\langle n_{\partial N}, Z_{\partial N}\rangle}{\langle n_N, Z_N\rangle} \underbrace{(-1)^{p(n-p) - n(n-p-1)}}_{(-1)^{-(n-p)^2 + n} = (-1)^{-(n-p)+n} = (-1)^p} \operatorname{div}\mathbb{X}$$

But $\eta^{ind}_{\partial N} = (-1)^p \eta_{\partial N}$ since the induced orientation on ∂N is $(-1)^p$ times the inner orientation induced by the outer orientation specified by $Z_{\partial N}$. So

$$\boxed{\int_{\partial N} \frac{\langle \mathbb{X}, n_{\partial N}\rangle \eta^{ind}_{\partial N}}{\langle n_{\partial N}, Z_{\partial N}\rangle} = \int_N \frac{\langle \operatorname{div}\mathbb{X}, n_N\rangle \eta_N}{\langle n_N, Z_N\rangle}}$$

$\mathbb{X}$ an $(n-p)$-vector field, N a $(p+1)$-submanifold with boundary ∂N having the induced orientation

For the usual application: $n-p=1$, $p=n-1$ [$n_N = 1 = Z_N$ unit 0-vector]
this reduces to

GAUSS'S LAW

$$\boxed{\int_{\partial N} \frac{\mathbb{X}\cdot n_{\partial N}}{n_{\partial N}\cdot Z_{\partial N}} \eta^{ind}_{\partial N} = \int_N \operatorname{div}\mathbb{X}\, \eta}$$

The other usual case is $n-p = n-1$, $p = 1$

In this case $*\mathbf{X}$ is a vector field. Since $\langle *A, *B\rangle = (-1)^{\frac{n-s}{2}} \langle A, B\rangle$, we can star everything in the inner products

$$\int_{\partial N} \frac{\langle *\mathbf{X}, *n_{\partial N}\rangle}{\langle *n_{\partial N}, Z_{\partial N}\rangle} \, n^{\text{ind}}_{\partial N} = \int_N \frac{\langle * \operatorname{div} *^{-1} * \mathbf{X}, *n_N\rangle}{\langle *n_N, *Z_N\rangle} \, n_N$$

Now replace $*\mathbf{X}$ by $\mathbf{X}$, still a p-vector field, and define

$$\operatorname{Curl} \mathbf{X} = * \operatorname{div} *^{-1} \mathbf{X} = * \left[(-1)^{n-p-1} *^{-1} d * (*^{-1} \mathbf{X}^\flat) \right]^\# = (-1)^{n-p-1} (d\mathbf{X}^\flat)^\#$$

$$\int_{\partial N} \frac{\langle \mathbf{X}, *n_{\partial N}\rangle}{\langle *n_{\partial N}, *Z_{\partial N}\rangle} \, n^{\text{ind}}_{\partial N} = \int_N \frac{\langle \operatorname{Curl} \mathbf{X}, *n_N\rangle}{\langle *n_N, *Z_N\rangle} \, n_N$$

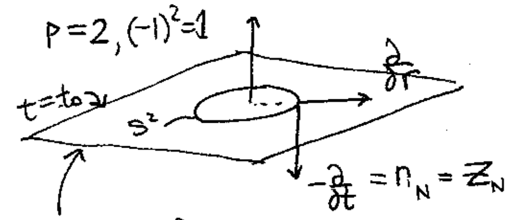
For $p = 1$
$$\int_{\partial N} \frac{\mathbf{X} \cdot \hat{n} \, d\ell}{\hat{n} \cdot Z_{\partial N}} = \int_N \frac{\langle \operatorname{Curl} \mathbf{X}, *n_N\rangle}{\langle *n_N, *Z_N\rangle} \, n_N$$
$\operatorname{Curl} \mathbf{X} = (-1)^n (d\mathbf{X}^\flat)^\#$ = 2-vector field.

STUDENTS: I think I got the sign wrong on the definition of Curl. Compare p. 67 for $n = 3$ where one can take the dual of the 2-vector field $\operatorname{Curl} \mathbf{X}$ to get $\operatorname{curl} \mathbf{X} = * \operatorname{Curl} \mathbf{X}$. Think about this.

We want $\displaystyle\int_{\partial N} \mathbf{X} \cdot \hat{n} \, d\ell = \int_N \operatorname{curl} \mathbf{X} \cdot \hat{n}_N \, da$.

<u>EXAMPLES</u> electromagnetism again.

$p = 2, (-1)^2 = 1$



$$n = \frac{\partial}{\partial r} \wedge \left(-\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial r} = Z_{\partial N}$$

Let $N = S^2 \cup \text{Interior } S^2$ at fixed time $t = t_0$

$\partial N = S^2$

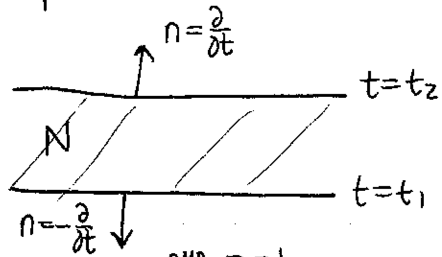
$\langle n_N, Z_N\rangle = -1 = \langle n_{\partial N}, Z_{\partial N}\rangle$

$$\int_{\partial N} \underbrace{\tfrac{1}{2} F^{\mu\nu} n_{\mu\nu}}_{\substack{-F^{0r} \\ -E^r}} \, n_{\partial N} = \int_N \underbrace{\underbrace{\partial_\mu F^{\mu\nu}}_{-4\pi J^\nu} n_\nu}_{-4\pi J^0 = -4\pi\rho} \, n_N$$

$$\int_{\partial N} E^r \, da = 4\pi \int_N \rho \, dv$$

Repeat for the 2-vector with components $*F^{\mu\nu}$ and get $\displaystyle\int_{\partial N} B^r \, da = 0$

$p=3$



$n^\mu n_\mu = -1$

$(-1)^p = -1$

$J^0 = \rho$

Choosing the outer normal gives the induced orientation times $(-1)^3 = -1$, so as before dx^{123} + oriented at $t=t_2$ and $-dx^{123}$ is + oriented at $t=t_1$:

$$\int_{\partial N} \underbrace{\frac{J^\mu n_\mu}{n^\nu n_\nu}}\, \eta^{ind}_{\partial N} = \int_N \partial_\mu J^\mu \eta = 0$$

$$= \int_{\partial N} (J^\mu n_\mu)(\eta \llcorner n)$$

$$= \int_{t_2} (-J^0)(-dx^{123}) + \int_{t_1} (+J^0)(dx^{123})$$

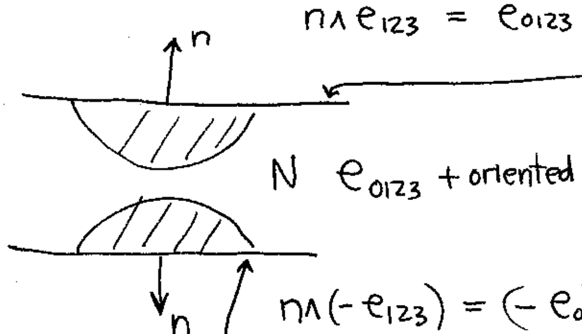
$$= \int_{t_2} \rho\, dx^1 dx^2 dx^3 - \int_{t_1} \rho\, dx^1 dx^2 dx^3$$

$$= Q(t_2) - Q(t_1)$$

charge conserved.

Incase it wasn't perfectly clear lets go over the inner orientation for this case again: INDUCED ORIENTATION wedge from left

$n \wedge e_{123} = e_{0123}$ so e_{123} + oriented



$n \wedge (-e_{123}) = (-e_0) \wedge (-e_{123}) = e_{0123}$ so e_{123} − oriented on

The inner orientation corresponding to the outward normal would be determined by wedging from the right, giving $(-1)^3 = -1$ difference in sign for the orientation relative to the induced orientation.