⑧ ORIENTATION, INTEGRATION OF P-FORMS, NORMALS, VOLUME ELEMENTS

I've already fooled you into thinking you can integrate n-forms on n-manifolds BUT one little detail was missing on p. 61. The "change of variable" formula involves the absolute value of the Jacobian determinant:

$$\int_{F(A)} f\, du^1\cdots du^n = \int_A f\circ F\ |\det(\partial_i F^j)|\ du^1\cdots du^n .$$

Then

$$\int_V \bar{\mathcal{L}}\, d\bar{x}^{1\cdots n} = \int_{\bar{\Phi}(V)} \bar{\mathcal{L}}\circ\bar{\Phi}^{-1}\, du^1\cdots du^n = \int_{\phi(V)} \underbrace{\bar{\mathcal{L}}\circ\phi^{-1}\ |\mathcal{J}\circ\phi^{-1}|}_{\operatorname{sgn}\mathcal{J}\quad \mathcal{L}\circ\phi^{-1}}\, du^1\cdots du^n$$

$$= \operatorname{sgn}\mathcal{J} \int_V \mathcal{L}\, dx^{1\cdots n}$$

In order to get a result which is sign independent, one has to pick out a subset of coordinate charts related to each other by positive Jacobians. Then $\int_V \mathcal{L}\, dx^{1\cdots n}$ is well defined in that class of coordinate charts, called "positively oriented". Its value in a "negatively oriented" chart (one related to the first class by a negative Jacobian) is then given by the same formula with a minus sign.

If $\{x^i\}$ is a positively oriented chart then the local n-form $\pm f\, dx^{1\cdots n}$, $f>0$, is called positively (+) or negatively (−) oriented.

If the manifold M has an everywhere nonvanishing n-form then one can speak of a global orientation for M and one can consistently pick an orientation for each local coordinate chart of a covering of M without getting into trouble (the Jacobian of each overlapping coordinate transformation will always be positive). M is then called orientable. Otherwise M is called nonorientable.

For example, if σ is an everywhere nonvanishing n-form on M, then given a covering of M by local coordinate charts one can always permute the local coordinate functions in each chart (if necessary) so that the component $\sigma_{1\cdots n}$ is positive in each local chart. All such local charts will then be called positively oriented.

The space of everywhere nonvanishing n-forms on an orientable manifold has 2 discrete components. An orientation for M is simply a choice of one of these components as positively oriented and the other as negatively oriented.

Local coordinates $\{x^i\}$ or a local frame $\{e_i\}$ are said to be positively (negatively) oriented if $dx^{1\cdots n}$ or $\omega^{1\cdots n}$ are positively (negatively oriented).

The "natural" orientation on R^n calls the cartesian coordinates associated with the natural basis $\{e_i\}$ positively oriented. The orientation of cartesian coordinates associated with any other basis of R^n then depends on the sign of the determinant of the matrix of the basis transformation.

On R^3, for example, we have "righthanded frames" (+ orientation) and "left handed frames" (− orientation)

EX. $\{x^1, x^2\} = \{u^1, u^2\}$ on R^2, du^{12} specifies the usual orientation of R^2.

$U = \{ (r^1, r^2) \mid 0 < r^1 < 1,\ 0 < r_2 < 1 \}$ is an open submanifold of R^2.

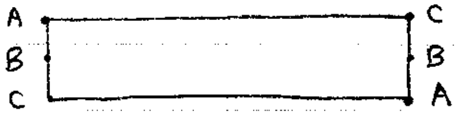
Let $\omega = dx^1 \wedge dx^2$. Then $\int_U \omega = \int_U dx^1 \wedge dx^2 = \int_0^1 \int_0^1 du^1 du^2 = 1.$

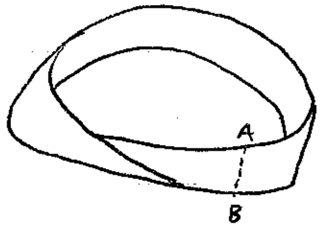
But $\{y^1, y^2\} = \{u^2, u^1\}$ are negatively oriented so

$$\int_U \omega = \int_U dy^2 \wedge dy^1 = \int_U - dy^1 \wedge dy^2 \equiv \underset{\overset{\nearrow}{\text{orientation}}}{-} \int_0 \int_0^1 - dy^1 dy^2 = 1$$

Without the extra sign we wouldn't get the same result in any coordinate system.

The Mobius strip is a nonorientable 2-manifold.
It is just a rectangle with one pair of edges identified in a twisted fashion.

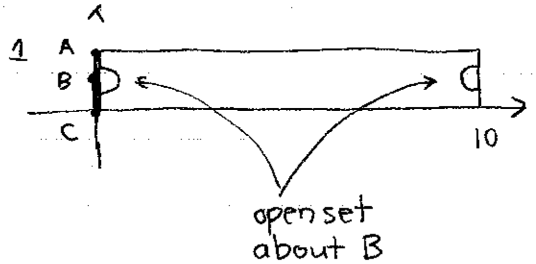




visualized as a subset of R^3
(not a submanifold since it has an edge)

To be more precise let

"1×10" Mobius strip $= \{ (x,y) \in R^2 \mid x \in [0,10) , y \in (0,1) \}$



But let the topology correspond to identifying the points $(0, y_0)$ with $(10, 1-y_0)$ for all $y_0 \in (0,1)$.

In order to discuss orientation for submanifolds, needed to integrate p-forms on p-submanifolds, we first clarify dual conventions, then discuss orientation on vector spaces and linear subspaces, and then transfer this to manifolds via the tangent spaces.

METRIC DUAL CONVENTIONS

One can introduce four different definitions of the Hodge star operator $*$ in pseudo-Riemannian geometry (2 different definitions in Riemannian geometry) all differing only in sign. For each possible definition we list the action of the star operation on the frame basis p-forms, the component definition, and the invariant definition.

Recall $\quad \eta_{i_1\cdots i_p} = \eta_{i_1\cdots i_p i_{p+1}\cdots i_n}\,\omega^{|i_{p+1}\cdots i_n|}$

$$\langle \underset{p}{\alpha}, \underset{p}{\beta} \rangle = \frac{1}{p!}\,\alpha\cdot\beta = \frac{1}{p!}\,\alpha_{i_1\cdots i_p}\beta^{i_1\cdots i_p} = \alpha_{i_1\cdots i_p}\beta^{|i_1\cdots i_p|}.$$

$(-1)^{\frac{n-s}{2}}$ $\downarrow$

right dual $*_R$ $*_R\,\omega^{i_1\cdots i_p} = \eta^{i_1\cdots i_p}$ $*_R\,\sigma_{i_1\cdots i_{n-p}} = \frac{1}{p!}\,\sigma_{i_{n-p+1}\cdots i_n}\,\eta^{i_{n-p+1}\cdots i_n}{}_{i_1\cdots i_{n-p}}$ $\underset{p}{\alpha}\wedge *_R\underset{p}{\beta} = \langle\alpha,\beta\rangle\,\eta$	left dual $*_L$ $*_L\,\omega^{i_1\cdots i_p} = (-1)^{p(n-p)}\,\eta^{i_1\cdots i_p}$ $*_L\,\sigma_{i_1\cdots i_{n-p}} = \frac{1}{p!}\,\eta_{i_1\cdots i_{n-p}}{}^{i_{n-p+1}\cdots i_n}\,\sigma_{i_{n-p+1}\cdots i_n}$ $*_L\underset{p}{\alpha}\wedge\underset{p}{\beta} = \langle\alpha,\beta\rangle\,\eta$
Flanders right dual $*_{FR}$ $*_{FR}\,\omega^{i_1\cdots i_p} = (-1)^{\frac{n-s}{2}}\,\eta^{i_1\cdots i_p}$ $*_{FR}\,\sigma_{i_1\cdots i_{n-p}} = (-1)^{\frac{n-s}{2}}\,*_R\,\sigma_{i_1\cdots i_{n-p}}$ $\underset{p}{\alpha}\wedge\underset{n-p}{\beta} = \langle *_{FR}\alpha,\beta\rangle\,\eta$	Flanders left dual $*_{FL}$ $*_{FL}\,\omega^{i_1\cdots i_p} = (-1)^{\frac{n-s}{2}+p(n-p)}\,\eta^{i_1\cdots i_p}$ $*_{FL}\,\sigma_{i_1\cdots i_{n-p}} = (-1)^{\frac{n-s}{2}}\,*_L\,\sigma_{i_1\cdots i_{n-p}}$ $\underset{p}{\alpha}\wedge\underset{n-p}{\beta} = \langle\alpha, *_{FL}\beta\rangle\,\eta$

$\longrightarrow (-1)^{p(n-p)}$

All definitions satisfy:

(1) $\quad ** = (-1)^{\frac{n-s}{2}+p(n-p)}$

(2) $\quad \langle *\underset{n-p}{\alpha}, \underset{p}{\beta}\rangle = (-1)^{p(n-p)}\langle\alpha, *\beta\rangle$

(3) $\quad \langle *\alpha, *\beta\rangle = (-1)^{\frac{n-s}{2}}\langle\alpha,\beta\rangle$

Different people choose to put the signs in different places. We use the MTW (= Misner, Thorne, Wheeler, GRAVITATION) convention of right dual.

For Riemannian geometry, the signature s is n and the Flanders change is irrelevant (he uses the Flanders right dual). Mathematicians never discuss pseudo-Riemannian geometry, but invariably choose the left dual.

For $n=3$, $(-1)^{p(n-p)} = 1$. For $n=4$, $(-1)^{p(n-p)} = (-1)^{p^2} = (-1)^{p-1}$

For Riemannian 3-manifolds, all definitions are equivalent.

These conventions also change the definition of <u>induced orientation</u> for a linear subspace of an oriented vector space.

NATURAL DUAL CONVENTIONS

The natural dual (denote it now by $\circledast$) was introduced on p. 39. It maps p-forms to (n-p)-vectors and p-vectors to (n-p)-forms. There are left and right natural duals:

Let $\mathbb{X} = \mathbb{X}^{|i_1\cdots i_p|} e_{i_1\cdots i_p}$ be a p-vector and $\sigma = \sigma_{|i_1\cdots i_p|}\omega^{i_1\cdots i_p}$ a p-form and define their natural contraction by

$$\mathbb{X} \lrcorner \sigma = \mathbb{X}^{i_1\cdots i_p}\sigma_{i_1\cdots i_p} = p!\ \mathbb{X}^{|i_1\cdots i_p|}\sigma_{i_1\cdots i_p}.$$

right dual $\circledast_R$	left dual $\circledast_L$
$\circledast_R \mathbb{X}_{i_{p+1}\cdots i_n} = \mathbb{X}^{|i_1\cdots i_p|}\epsilon_{i_1\cdots i_p i_{p+1}\cdots i_n}$	$\circledast_L \mathbb{X}_{i_{p+1}\cdots i_n} = \epsilon_{i_{p+1}\cdots i_n i_1\cdots i_p}\mathbb{X}^{|i_1\cdots i_p|}$
$\circledast_R \sigma^{i_{p+1}\cdots i_n} = \sigma_{|i_1\cdots i_p|}\epsilon^{i_1\cdots i_p i_{p+1}\cdots i_n}$	$\circledast_L \sigma^{i_{p+1}\cdots i_n} = \epsilon^{i_{p+1}\cdots i_n i_1\cdots i_p}\sigma_{|i_1\cdots i_p|}$
$\mathbb{X}\wedge\circledast_R\sigma = (p!)^{-1}(\mathbb{X}\lrcorner\sigma)\, e_{1\cdots n}$	$\circledast_L\sigma\wedge\mathbb{X} = (p!)^{-1}(\mathbb{X}\lrcorner\sigma)\, e_{1\cdots n}$
$\sigma\wedge\circledast_R\mathbb{X} = (p!)^{-1}(\mathbb{X}\lrcorner\sigma)\, \omega^{1\cdots n}$	$\circledast_L\mathbb{X}\wedge\sigma = (p!)^{-1}(\mathbb{X}\lrcorner\sigma)\, \omega^{1\cdots n}$

$$\boxed{\circledast_R = (-1)^{p(n-p)}\circledast_L}$$

Both satisfy (1) $\circledast\circledast = (-1)^{p(n-p)}$

(2)

(3) $\frac{1}{(n-p)!}\circledast\sigma\lrcorner\circledast\mathbb{X} = \frac{1}{p!}\mathbb{X}\lrcorner\sigma$

Unfortunately I first introduced the natural dual on p. 39 as a right dual but then without thinking used the left dual on p. 64. Lets stick with the right dual. Again for 3-manifolds these are identical.

For the right dual we have:

$$\circledast e_{i_1\cdots i_p} = \epsilon_{|i_1\cdots i_p||i_{p+1}\cdots i_n|}\,\omega^{i_{p+1}\cdots i_n}$$

$$\circledast \omega^{i_1\cdots i_p} = \epsilon^{|i_1\cdots i_p||i_{p+1}\cdots i_n|}\, e_{i_{p+1}\cdots i_n}$$

SWITCH BACK TO GREEK INDICES
and recall the index
conventions of pp. 17-18

$\alpha, \beta, \ldots \in \{1, \ldots, n\}$
$i, j, k, \ldots \in \{1, \ldots, p\}$
$a, b, c, \ldots \in \{p+1, \ldots, n\}$

ORIENTATION OF A VECTOR SPACE V, $\dim V = n$

The 1-dim space $\Lambda^n(V^*) - \{0\}$ of nonzero n-forms over V has two disjoint components. An orientation (inner orientation) for V is a choice of one component as the positively oriented component and the other as the negatively oriented component.

A basis $\{e_\alpha\}$ of V is called positively (negatively) oriented if $\omega^{1\cdots n}$ $(-\omega^{1\cdots n})$ belongs to the positively oriented component.

$\Lambda^n(V) - \{0\}$ also has two disjoint components which are called positively (negatively) oriented if $\{e_\alpha\}$ is a positively oriented basis of V and $e_{1\cdots n}$ $(-e_{1\cdots n})$ belongs to that component.

A nonzero n-form or n-vector is called positively or negatively oriented depending on which of the two components it belongs to.

In other words an ordered basis picks out an orientation for the disjoint spaces of nonzero n-forms and n-vectors $e_{1\cdots n}$ and $\omega^{1\cdots n}$ are positively (negatively) oriented if $\{e_\alpha\}$ is positively (negatively) oriented.

ADAPTED BASES OF A LINEAR SUBSPACE $W \subset V$, $\dim W = p$

Let $\{e_\alpha\} = \{e_i, e_a\}$ be a basis of V adapted to the linear subspace $W = \mathrm{span}\{e_i\}$. Any two such adapted bases are related by the following linear transformation

$$e_{\alpha'} = e_\beta A^{-1\beta}{}_\alpha \qquad \omega^{\alpha'} = A^\alpha{}_\beta \omega^\beta$$

$$(A^\alpha{}_\beta) = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \qquad (A^{-1\alpha}{}_\beta) = \begin{pmatrix} A^{-1} & D \\ 0 & B^{-1} \end{pmatrix} \qquad D = -A^{-1}CB^{-1}$$

$$e_{i'} = e_j A^{-1j}{}_i \qquad \omega^{i'} = A^i{}_j \omega^j + C^i{}_a \omega^a$$

$$\omega^{a'} = B^a{}_b \omega^b \qquad e_{a'} = e_b B^{-1b}{}_a + e_i D^i{}_a$$

$$0 = \omega^{i'}(e_{a'}) = A^i{}_j D^j{}_a + C^i{}_b B^{-1b}{}_a \rightarrow D^j{}_a = -A^{-1j}{}_i C^i{}_b B^{-1b}{}_a$$

Notice that span$\{\omega^i\}$ is not invariant under such a transformation so the dual space W^* to W cannot <u>naturally</u> be identified with a subspace of the dual space V^*.

However, ω^i is a linear function on V so we can restrict its domain to W and get a linear function on W, i.e. a 1-form on W. In other words if we denote the restriction of a 1-form σ to W by $\sigma|_W$, then $\{\omega^i|_W\}$ is the basis of W^* dual to the basis $\{e_i\}$ of W.

Note that $\omega^a|_W = 0$ since $\omega^a(X) = X^i\omega^a(e_i) = 0$ for any $X \in W$.

Similarly any covariant tensor over V can be restricted to W

$$(\sigma_{\alpha_1\cdots\alpha_q}\,\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_q})|_W = \sigma_{i_1\cdots i_q}\,\omega\;|_W\otimes\cdots\otimes\omega^{i_q}|_W .$$

On the other hand, the "star-dual" space $^*W = \text{span}\,\{\omega^a\} \subset V^*$ is invariantly defined and consists of all 1-forms over V which annihilate W:

$$\omega^a(X) = X^i\omega^a(e_i) = 0 \qquad X \in W$$

In an arbitrary basis $\{e_\alpha\}$

$$\sigma \in {}^*W \;\rightarrow\; \sigma_\alpha X^\alpha = 0 \qquad X \in W.$$

One might be tempted to call $^*W$ the space of covariant normals to W.

If we have a metric $g = g_{\alpha\beta}\,\omega^\alpha\otimes\omega^\beta$ on V, then the space $W^\perp = {}^*W^\#$ obtained from $^*W$ by "raising its index" in component language

$$(\sigma^\#)^\alpha = g^{\alpha\beta}\sigma_\beta \;, \qquad (X^\flat)_\alpha = g_{\alpha\beta}X^\beta$$

is exactly the subspace of V orthogonal to W (orthogonal complement):

$$g_{\alpha\beta}(\sigma^\#)^\alpha X^\beta = \sigma_\beta X^\beta = 0 \qquad \text{if } X \in W,\ \sigma \in {}^*W.$$

We can restrict g to an inner product on W:

$g|_W = g_{ij}\,\omega^i\otimes\omega^j$, but when g has an indefinite signature $g|_W$ can be degenerate if W contains null directions

$$X \quad \text{such that} \quad g(X,X) = 0.$$

EX. $V = M^4 = R^4$ with positively oriented natural basis $\{e_0, e_1, e_2, e_3\}$

$W = R^3 \subset R^4$, namely span $\{e_1, e_2, e_3\}$.

Note $\{e_0, e_1, e_2, e_3\}$ is not a basis adapted to W since e_0 appears first rather than last. This is an important remark. If instead we used the natural basis $\{e_1, e_2, e_3, e_4\}$ of R^4 with e_4 identified with the timelike direction, we would have an adapted basis.

The natural orientation of R^4, namely $e_{0123} \sim +$ orientation or $e_{1234} \sim +$ orientation give different orientations for Minkowski space since $e_{0123} = -e_{1230}$.

This causes some convention problems.

$$g = \eta_{\alpha\beta}\,\omega^\alpha \otimes \omega^\beta = -\omega^0 \otimes \omega^0 + \delta_{ij}\,\omega^i \otimes \omega^j$$

$$g|_W = \delta_{ij}\,\omega^i|_W \otimes \omega^j|_W$$

$$^*W = \text{span}\{\omega^0\} \qquad W^\perp = \text{span}\{e_0\}$$

If $\{e_{1'}, e_{2'}, e_+, e_-\} = \{e_1, e_2, \frac{1}{\sqrt{2}}(e_0+e_3), \frac{1}{\sqrt{2}}(e_0-e_3)\}$ is another basis, then $e_{12+-} = e_{12} \wedge e_+ \wedge e_- = e_{1230} = -e_{0123}$ tells us it is negatively oriented (it would be + oriented if we used the indices 1234 instead) and adapted to the null subspace

$$W' = \text{span}\{e_1, e_2, e_+\} \qquad ^*W' = \text{span}\{\omega^-\}$$

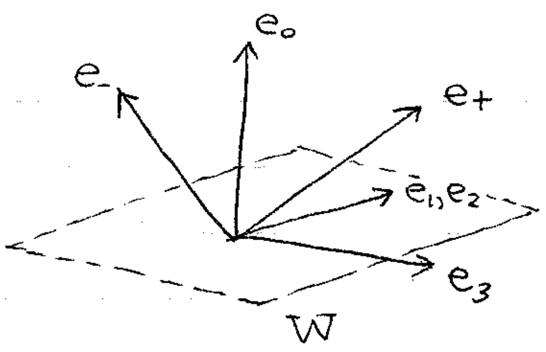
$$g = -(\omega^+ \otimes \omega^- + \omega^- \otimes \omega^+) + \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2$$

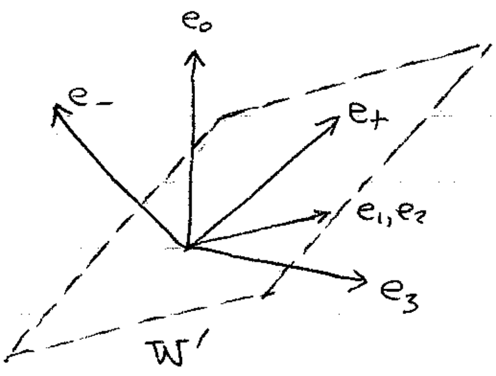
$$\omega^+ = \frac{1}{\sqrt{2}}(\omega^0 + \omega^3) \qquad \omega^- = \frac{1}{\sqrt{2}}(\omega^0 - \omega^3)$$

$$(\omega^+)^\# = \frac{1}{\sqrt{2}}(-e_0 + e_3) = -e_- \,, \quad (\omega^-)^\# = \frac{1}{\sqrt{2}}(-e_0 - e_3) = -e_+$$

$$W'^\perp = \text{span}\{e_+\} \subset W'$$

The space of vectors orthogonal to W' is contained in W'!





e_0 is the future pointing unit normal to W

e_+ is a future pointing normal to W′ (null vectors cannot be normalized).

$g|_{W}$ is degenerate along e_+.

INNER AND OUTER ORIENTATIONS FOR A SUBSPACE

The subspace W is completely determined by the p-vector $e_{1\cdots p} = (-1)^{p(n-p)} \circledast \omega^{p+1\cdots n}$ or the (n-p)-form $\omega^{p+1\cdots n} = \circledast e_{1\cdots p}$ for any adapted basis $\{e_\alpha\}$. Here $\circledast$ is the natural dual discussed on p. .

Such p-vectors or (n-p)-forms are called decomposable since they can be written as an exterior product of linearly independent vectors or 1-forms respectively.

Any (nonzero) decomposable p-vector Y or (n-p)-form σ determines a linear subspace W

$$W = \{X \in V \mid X \wedge Y = 0\} \quad \text{or} \quad W = \{X \in V \mid X \lrcorner \sigma = 0\}$$

If $Y = Y_1 \wedge \dots \wedge Y_p$, this just says $\{X, Y_1, \dots, Y_p\}$ are linearly dependent, i.e. $X \in \operatorname{span}\{Y_1, \dots, Y_p\}$

If $\sigma = \sigma_{p+1} \wedge \dots \wedge \sigma_n$, this just says $\sigma_a(X) = 0$ for all $a = p+1, \dots, n$.

EXERCISE. Use the component definitions of $\wedge$, $\circledast$ and $\lrcorner$ to show

$$X \lrcorner \circledast Y = \circledast (Y \wedge X)$$

so Y and $\circledast Y$ both determine the same space.

Every nonzero decomposable p-vector Y determines a linear subspace with an inner orientation. First let W be the p-dimensional subspace it determines and let $\{e_\alpha\}$ be a basis of V adapted to W. Then the basis $\{e_i\}$ of W is positively (negatively) oriented if

$$Y_{1\dots p} = Y(e_1,\dots,e_p) > 0 \quad (<0).$$

An inner orientation for the star-dual space $^*W$ is called an outer orientation for W. If the (n-p)-form $\omega^{p+1\dots n}$ is positively oriented with respect to this inner orientation of $^*W$, then any decomposable (n-p)-vector $Z = Z_{p+1}\wedge\dots\wedge Z_n$ is called positively oriented if $\omega^{p+1\dots n}(Z_{p+1},\dots,Z_n) > 0$. $e_{p+1\dots n}$ is positively oriented.

An outer orientation for W can therefore also be specified by an ordered set of (n-p) linearly independent vectors not contained in W.

Thus every nonzero decomposable n-p)- form σ determines a linear subspace W with an outer orientation

$$\sigma(\omega^{p+1},\dots,\omega^n) > 0$$ means $\{\omega^a\}$ is a positively oriented basis of $^*W$.

INDUCED ORIENTATION

If V has an inner orientation, then an inner orientation for W induces an outer orientation for $^*W$ and vice versa.

If $Y = Y_1\wedge\dots\wedge Y_p$ and $Z = Z_{p+1}\wedge\dots\wedge Z_n$ determine corresponding inner and outer orientations for W, then $Y\wedge Z$ is a positively oriented n-form on V.

(n-p) vector normal to W

When V has a metric g, its orientation is specified by a choice η from the set of 2 unit n-forms $\{\sigma,-\sigma\}$ satisfying $<\pm\sigma,\pm\sigma> = 1$

Note that $(-1)^{\frac{n-s}{2}}\eta^\#$ is the positively oriented unit n-vector.

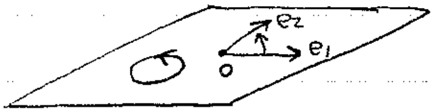
A subspace W is also determined by a nonzero (n-p)-vector n, called a normal (n-p)-vector for W:

$$W = \{X\in V \mid 0 = X\cdot n \equiv X_\alpha n^{\alpha\alpha_{p+2}\dots\alpha_n} e_{|\alpha_{p+2}\dots\alpha_n|}\}$$

n determines an oriented subspace W only if $<n,n> = \epsilon \neq 0$ (n nonnull), in which case one can assume $|\epsilon| = 1$ by rescaling n to obtain a unit normal. A basis $\{e_i\}$ for W is then positively oriented if $e_{1\dots p}\wedge n = c(-1)^{\frac{n-s}{2}}\eta^\#$

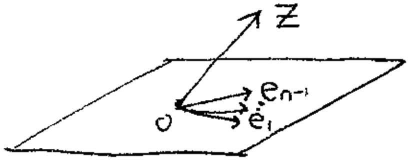
In this case one can normalize Y and σ such that $Y\wedge n = (-1)^{\frac{n-s}{2}}\eta^\#$ and $n \lrcorner \sigma \equiv \sigma_{\alpha_{p+1}\dots\alpha_n} n^{\alpha_{p+1}\dots\alpha_n} = (n-p)!$

EX. $p=2$



An inner orientation for a 2-plane in any dimension can be thought of as a directed or oriented loop in the 2-plane which tells which direction to rotate e_1 to get to e_2 if $\{e_1, e_2\}$ is a positively oriented basis.

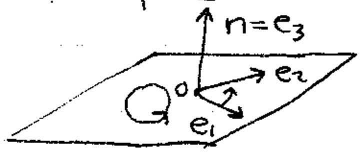
$p=n-1$



An outer orientation for an $(n-1)$-hyperplane W can be given by any vector Z not contained in the plane. The orientation it induces for W is such that $e_{1\cdots p} \wedge Z$ being positively oriented in V implies $\{e_1, \ldots, e_p\}$ is a positively oriented basis of W.

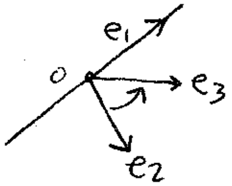
E^3 $n=3$

$p=2$



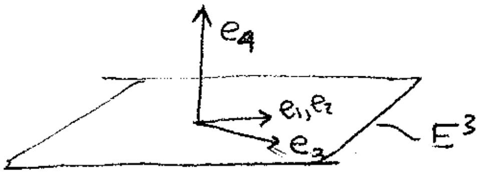
Using orthonormal vectors $\{e_1, e_2, e_3\}$ the inner and outer orientations of a 2-plane are related by the right hand rule (p.26). The oriented loop gives the screw sense.

$p=1$



For a line $(p=1)$, one must specify 2 "normals" e_2 and e_3, with the right hand rule again relating the inner orientation (a direction) to the outer orientation.

M^4 $p=3$



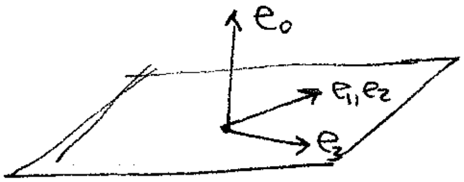
e_{1234} + oriented

$Y = e_{123}$ + inner orientation for E^3

$n = e_4$ corresponding outer orientation

since $e_{123} \wedge e_4 = e_{1234}$.

$\sigma = \omega^4$



e_{0123} + oriented

$Y = e_{123}$ + inner orientation for E^3

$n = -e_0$ corresponding outer orientation

since $e_{123} \wedge (-e_0) = e_{0123}$.

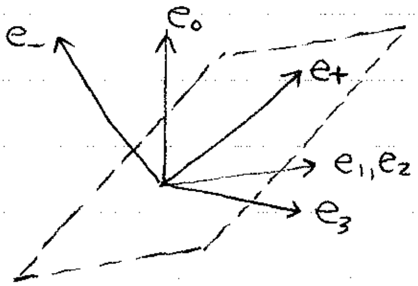
So outer orientation is not the future directed normal here but the past directed normal (YUCK!)

However, the 1-form $\sigma = -\omega^0$
is such that $\sigma^{\#} = e_0$ is the
future directed unit normal (that's nice?),

whereas $\sigma = \omega^4$ is such that
$\sigma^{\#} = -e_4$ is the past directed
unit normal. (yuck?)

Both signs can't work out unless we change the signature,
but then we triple our trouble with the spacelike directions.

For the null example on pages 78–79:



$Y = e_{12} \wedge e_+$ + oriented

$Z = -e_-$ ~ + outer orientation

since $Y \wedge Z = e_{0123}$

$\sigma = -\omega^-$ since $\{e_1, e_2, e_+, -e_-\}$ is + oriented

$n = -e_+$ past directed normal

$n \cdot n = 0$

$$\sigma(Z) = 1 = -n \cdot Z$$

Given Y, σ, Z and n for the null case in general, one can always rescale them and if necessary change the sign of n such that the following relations hold

$$Y \wedge Z = (-1)^{\frac{n-s}{2}} n^{\#} \quad , \quad \frac{1}{(n-p)!} Z \lrcorner \sigma = 1 = -\langle n, Z \rangle \ .$$

In the nonnull case when $\langle n,n \rangle = \epsilon \neq 0$, then

$$*n \wedge n = (-1)^{p(n-p)} n \wedge *n = (-1)^{p(n-p)} \underbrace{\langle n,n \rangle}_{\epsilon} n^{\#}$$

so $$Y \equiv \epsilon^{-1} *^{-1} n \quad \text{satisfies} \quad Y \wedge n = (-1)^{\frac{n-s}{2}} n^{\#}$$

while $\boxed{\sigma = \epsilon^{-1} n^{\flat}}$ satisfies $\frac{1}{(n-p)!} n \lrcorner \sigma = \frac{1}{(n-p)!} \epsilon^{-1} n \lrcorner n^{\flat} = \epsilon^{-1} \langle n,n \rangle = 1.$

where $\boxed{*^{-1} \equiv (-1)^{p(n-p) + \frac{n-s}{2}} *}$.

VOLUME p-form ON W

Introduce the p-form $(n-p)!\,\eta_W = \begin{cases} \eta \llcorner Z & \text{in the null case } \langle n,n\rangle = 0 \\ \eta \llcorner n & \text{in the nonnull case } \langle n,n\rangle \neq 0 \end{cases}$.

$\eta_W|_W$ is the positively oriented unit p-form on W when $\langle n,n\rangle \neq 0$ and a positively oriented p-form on W when $\langle n,n\rangle = 0$

Let $\{e_\alpha\}$ be a positively oriented basis of V adapted to W with $\{e_i\}$ positively oriented. and assume $\eta = \omega^{1\dots n}$, i.e. $g=1$ in this basis.
One can then assume

$$Y = e_{1\cdots p}\ ,\ Z = e_{p+1\dots n}\ ,\quad \sigma = \omega^{p+1\dots n}\ ,\ \eta_W = \omega^{1\cdots p}$$

When $\langle n,n\rangle \neq 0$ one can furtherassume $\{e_\alpha\}$ is an orthonormal basis with $n = e_{p+1\cdots n}$, and one can then write

$$\boxed{\eta_W = (*^{-1} n)^\flat}\ .$$

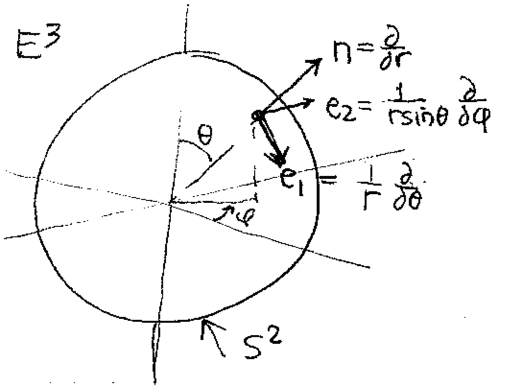
EXERCIZE. Verify this last statement.

FINALLY INTEGRATION OF P-FORMS

So we've taken a long detour and now we return to the question of integrating p-forms on p-submanifolds, which we will discuss only locally for open submanifolds of p-submanifolds contained in a single adapted coordinate patch.

We have begun this section (week) by stating that a manifold M is orientable if we can continuously specify an orientation for each of the tangent spaces, equivalent to giving an everywhere nonvanishing n-form. A submanifold is orientable if it is orientable as a manifold. Our entire discussion of a linear subspace W of a vector space V can be carried over to the tangent space TM_q at each point q of a submanifold N, with the tangent space $TN_q \subset TM_q$ playing the role of the linear subspace $W \subset V$. If $\dim N = p$, each of the p-vectors and p-forms of that discussion become p-vector fields and p-form fields on N.

For example, the orientation of an orientable submanifold of an orientable manifold can be specified by an outer orientation exactly as in the vector space case, but now for each tangent space.



For a closed hypersurface $(p = n-1)$ $N \subset M$, the outward direction picks out an outer orientation for N and induces an inner orientation.

If N is not closed we have to choose a vector field on one side or the other for an outer orientation. For Riemannian geometry, this means a choice of one of the 2 unit normals.

For example $\{\theta, \varphi\}$ and $\left\{\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}\right\}$ have the canonical orientation for a closed surface in E^3. Again its just the right hand rule.

Suppose β is a p-form on an n-dimensional manifold M. Then β can be restricted to a p-form on a p-dimensional manifold by restricting its domain (the arguments it accepts) to the tangent space to N at each point of N. Denote this by $\beta|_N$.

If $\{x^\alpha\} = \{x^i, x^a\}$ are local coordinates associated with a local chart (ϕ, U), ie $x^\alpha = u^\alpha \circ \phi$, which is adapted to the submanifold N, i.e. $x^a = 0$ at points of N where $\{x^i\}$ are local coordinates on N,

then
$$\beta|_N = (\beta_{\alpha_1\cdots\alpha_p} dx^{\alpha_1\cdots\alpha_p})|_N = \beta_{1\cdots p}|_N dx^{1\cdots p}.$$

This is true since $dx^a|_N = 0$, since vectors tangent to N are annihilated by dx^a; this is just a property of duality: if $X = X^i \frac{\partial}{\partial x^i} \in TN_q$ then $dx^a(X) = X^i dx^a(\frac{\partial}{\partial x^i}) = 0$.

Here the restriction of the function (0-form) $\beta_{1\cdots p}$ to N means simply restricting its domain to N. Strictly speaking, we should write $dx^{1\cdots p}|_N$ but since we are identifying the coordinate functions x^i on M with coordinate functions on N, this will be omitted but implicitly understood.

Suppose $C \subset N \cap U$ is an open submanifold of N. Assume that N is orientable and $\{x^i\}$ is a positively oriented local coordinate chart. Then define

$$\int_C \beta \equiv \int_C \beta|_N$$

as the integral over C of the restriction of β to N, which is the integral of a p-form on an oriented p-manifold N, already defined at the beginning of this section of notes. Namely

$$\int_C \beta = \int_C \beta|_N = \int_C \beta_{1\cdots p} dx^{1\cdots p}$$
$$= \int_{\phi(C)} \beta_{1\cdots p} \circ \phi^{-1} du^1 \cdots du^p.$$

If $\{x^i\}$ was a negatively oriented coordinate system, this would have to be multiplied by -1.

That's it. No metric ever enters into the game. So why all the junk about metrics, normals and volume forms? Because the integral of a p-form on a p-submanifold may be rewritten in terms of a metric when available so that it resembles the kind of integrals physicists are used to throwing around.

Before the metric butts into this business, let's first introduce another way of integrating p-forms that relies on a parametrization of N rather than an adapted coordinate system.

Suppose $f: M \rightarrow N$ is a differentiable map from an m-dimensional manifold M into an n-dimensional manifold N. Then if F is a function on N, we get a function $f^* F \equiv F \circ f$ on M called the pullback of F by f. We can also "pull back" its differential from N to M by defining $f^* dF \equiv d(f^* F) = d(F \circ f)$.

If $\{y^i\}$ and $\{x^\alpha\}$ are positively oriented coordinates on M and N respectively, then $f^* x^\alpha = x^\alpha \circ f$ are functions on M so

$$f^* dx^\alpha \equiv d(f^* x^\alpha) = d(x^\alpha \circ f) = \frac{\partial(x^\alpha \circ f)}{\partial y^i} dy^i$$

If β is a p-form on N and $\beta = \beta_{\alpha_1 \dots \alpha_p} dx^{|\alpha_1 \dots \alpha_p|} = \beta_{\alpha_1 \dots \alpha_p} dx^{\alpha_1} \otimes \dots \otimes dx^{\alpha_p}$ define its pullback to M by

$$\begin{aligned} f^* \beta &= f^* \beta_{\alpha_1 \dots \alpha_p} \; (f^* dx^{\alpha_1}) \otimes \dots \otimes (f^* dx^{\alpha_p}) \\ &= \beta_{\alpha_1 \dots \alpha_p} \circ f \; \frac{\partial(x^{\alpha_1} \circ f)}{\partial y^{i_1}} \dots \frac{\partial(x^{\alpha_p} \circ f)}{\partial y^{i_p}} \; dy^{i_1} \otimes \dots \otimes dy^{i_p} \\ &= \underbrace{\beta_{\alpha_1 \dots \alpha_p} \circ f \; \frac{\partial(x^{\alpha_1} \circ f)}{\partial y^{i_1}} \dots \frac{\partial(x^{\alpha_p} \circ f)}{\partial y^{i_p}}}_{(f^* \beta)_{i_1 \dots i_p}} \; dy^{|i_1 \dots i_p|} \end{aligned}$$

In other words just plug in the transformation and differentiate.

Now suppose $f: V \subset R^p \rightarrow C \subset N \subset M$ is a diffeomorphism from an open submanifold V of R^p onto an open submanifold C of an orientable p-dimensional submanifold N of an n-dimensional manifold M.

If β is a p-form on M, then $f^* \beta$ is a p-form on $V \subset R^p$ and we can therefore integrate it. One can show

$$\int_C \beta = \int_V f^* \beta ,$$

assuming that f is an orientation preserving diffeomorphism, namely that if α is a positively oriented p-form on N, then

$f^*\beta$ is a positively oriented p-form on $V \subset R^p$ with the natural orientation. $\left[\det\left(\frac{\partial x^i}{\partial u^j}\right) > 0 \text{ in adapted coordinates} \right]$

Otherwise one must introduce a minus sign to compensate for the reversal of orientation.

f is called a <u>parametrization</u> of C. As a map it goes from R^p into N instead of from N into R^p like a coordinate chart.

For example, if $p=1$, V is an interval of the real line R and f is a parametrized curve in M, while N is the image of f, namely the curve of points without a parametrization:

$$f^*\beta = \beta_\alpha \circ f(t)\, d(x^\alpha \circ f(t)) = \beta_\alpha \circ f(t)\, \frac{d(x^\alpha \circ f)(t)}{dt}\, dt$$

$$\int_C \beta = \int_V \beta_\alpha \circ f(t)\, \frac{d(x^\alpha \circ f)(t)}{dt}\, dt.$$

This is called a line integral. We have assumed that the curve is oriented by the direction of the tangent to the parametrized curve.

<u>EX</u>. Let C be the unit circle at the origin in the x-y plane in E^3, oriented by the counterclockwise direction. Let $\{x^\alpha\} = \{x, y, z\}$.

Let $\{f^* x^\alpha\} = \{\cos t, \sin t, 0\}$ and $V = (0, 2\pi) \subset R$.

Let $\beta = x\,dy - y\,dx + xy\,dz$. Note $f^* dz = 0$

Then $\int_C \beta = \int_0^{2\pi} [\cos t\,(\cos t\,dt) - \sin t\,(-\sin t)]\,dt = \int_0^{2\pi} dt = 2\pi.$

Even though $f(V) = C - \{(1,0,0)\}$, the loss of a single point doesn't affect the integral of a smooth differential form.

<u>EX</u>. Let C be $S^2 \subset E^3$ with the inner orientation induced by the outward unit normal n. The usual spherical coordinates $\{\theta, \varphi\}$ are positively oriented with this orientation. Let $\{x^\alpha\} = \{x, y, z\}$.

Let $\{f^* x^\alpha\} = \{\sin u^1 \cos u^2, \sin u^1 \sin u^2, \cos u^1\}$

$V = (0, \pi) \times (0, 2\pi) \subset R^2$

Let $\beta = z\, dx \wedge dy \Rightarrow f^*\beta = \cos u^1\, d(\sin u^1 \cos u^2) \wedge d(\sin u^1 \cos u^2)$

$= \cos^2 u^1 \sin u^1\, du^1 \wedge du^2$

Then $\int_q \beta = \int_V f^*\beta = \int_0^\pi \int_0^{2\pi} \cos^2 u^1 \sin u^1 \, du^1 du^2$

$= -2\pi \frac{\cos^3 u^1}{3}\Big|_0^\pi = \frac{4\pi}{3}$

Note that $f^*(V) = S^2 - \{(\sin t, 0, \cos t) \mid t \in [0,\pi]\}$ but again we've missed only a set of measure zero so the integral doesn't care.

Note that this parametrization is very simple in terms of spherical coordinates $\{\theta, \varphi\}$, namely $\{f^*\theta, f^*\varphi\} = \{u^1, u^2\}$.

In other words we can usually get away with integrating a p-form locally using coordinates or a parametrization which covers almost all of what we are integrating over, therefore avoiding the complication of discussing how to do it globally which is quite involved.

Now we rewrite the integral of a p-form when a metric g is available on our manifold M. Let N be an oriented p-submanifold of M and let Y be a p-vector field, σ an $(n-p)$-form and Z an $(n-p)$-vector field on M such that at each point q of N they pick out the oriented tangent space to N within TM_q as described above for a linear subspace W of a vector space V.

Let n be an $(n-p)$-vector field on M such that on N it is a normal to N and let η_N be the restriction to N of the p-form $\frac{1}{(n-p)!}\eta \lrcorner Z$ (null case) or $\frac{1}{(n-p)!}\eta \lrcorner n$ (nonnull case).

η_N is the volume form on N when n is a unit normal in the nonnull case which has a positive outer orientation relative to N. Its integral then gives the measure of N relative to g ($\underset{p=1}{\text{length}}, \ldots, \underset{p=n-1}{\text{area}}, \underset{p=n}{\text{volume}}$)

If β is a p-form on M we can paint in the metric in two steps to achieve a more complicated but geometric expression for the integral of β.

First, we turn the page.

Next, as an <u>EXERCI~~Z~~SE</u> we can derive the formula:

(1) $$\beta = (-1)^{\frac{n-s}{2}} \frac{\eta \llcorner ({*}\beta)^{\#}}{(n-p)!}$$ (just application of definitions)

Already we've succeeded in making an enormously complicated formula.
Next:

(2) $$\beta\big|_N = (-1)^{\frac{n-s}{2}} \frac{\langle ({*}\beta)^{\#}, n\rangle}{\langle n, Z\rangle} \frac{\eta \llcorner Z}{(n-p)!}\Big|_N = (-1)^{\frac{n-s}{2}} \frac{\langle ({*}\beta)^{\#}, n\rangle}{\langle n, Z\rangle}\Big|_N \eta_N$$

(3) $$\underset{\text{case}}{\overset{\text{nonnull}}{=\!=}} (-1)^{\frac{n-s}{2}} \frac{\langle ({*}\beta)^{\#}, n\rangle}{\langle n, n\rangle} \frac{\eta \llcorner n}{(n-p)!}\Big|_N \underset{\text{normal}}{\overset{\text{unit}}{=\!=}} (-1)^{\frac{n-s}{2}} \frac{\langle ({*}\beta)^{\#}, n\rangle}{\langle n, n\rangle}\Big|_N \eta_N$$

Note first that if we set $\alpha = \frac{1}{(n-p)!} \eta \llcorner Z$, then $Z = ({*}\alpha)^{\#} (-1)^{\frac{n-s}{2}}$ follows from formula (1) and by (2), $\alpha|_N$ reduces to $\frac{\eta \llcorner Z}{(n-p)!}\Big|_N$ so they are at least consistent.

By introducing the p-vector field: $\mathbb{X} \equiv \frac{(-1)^{\frac{n-s}{2}} ({*}\beta)^{\#}}{\langle n, n\rangle}$

we can finally write:

$$\int_N \beta = \int_N \underbrace{\langle \mathbb{X}, n\rangle\big|_N}_{\substack{\text{normal}\\ \text{component}}} \underbrace{\eta_N}_{\substack{\text{volume}\\ \text{element}}}$$

For $p = n-1$ this is

$$\int_N \beta = \int_N \beta^{\alpha} n_{\beta}\, \eta_N \equiv \int_N \mathbb{X}^{\beta} \text{"}dS_{\beta}\text{"} = \text{"}\int \mathbb{X} \cdot dS\text{"}$$

$$\text{where } dS_{\beta} \equiv n_{\beta}\, \eta_N = \left(n_{\beta} Z^{\beta_1} \eta_{\beta_1 \beta_2 \ldots \beta_n} dx^{|\beta_2 \ldots \beta_n|}\right)\Big|_N$$
$$= \left(\eta_{\beta_1 \cdots \beta_n}\, dx^{|\beta_2 \cdots \beta_n|}\right)\Big|_N$$

This has cost me a week. You will not find this discussion in any book.

EX. N = sphere of radius r_0 in E_3, usual orientation

parametrization by spherical coords (or evaluation in adapted coords)

$$x^1 = r_0 \sin\theta\cos\varphi \qquad dx^1|_N = r_0\left(\cos\theta\cos\varphi\, d\theta - \sin\theta\sin\varphi\, d\varphi\right)$$
$$x^2 = r_0 \sin\theta\sin\varphi \qquad dx^2|_N = r_0\left(\cos\theta\sin\varphi\, d\theta + \sin\theta\cos\varphi\, d\varphi\right)$$
$$x^3 = r_0 \cos\theta \qquad dx^3|_N = r_0\left(\sin\theta\, d\theta\right)$$

$$(dx^{23}, dx^{31}, dx^{12})|_N \underset{\substack{\uparrow \\ \text{omitted step}}}{=} \underbrace{(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)}_{(n^1,n^2,n^3)} \underbrace{r_0^2\sin\theta\, d\theta\wedge d\varphi}_{dS}$$

Integrate the 2-form $\beta = \beta_{23}\,dx^{23} + \beta_{31}\,dx^{31} + \beta_{12}\,dx^{12} \equiv E_1\,dx^{23} + E_2\,dx^{31} + E_3\,dx^{12}$

$$\int_N \beta = \int_N \left(E_1 dx^{23} + E_2 dx^{31} + E_3 dx^{12}\right)$$
$$= \int_N E_i n^i\, dS = \text{“} \int_N E\cdot d\vec{S} \text{”}$$

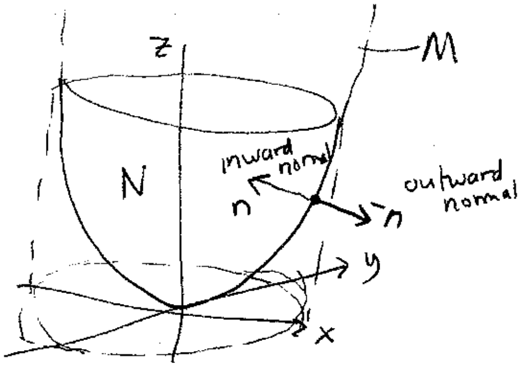
Pick $E_i = \frac{Q}{r_0^2} n_i$

$$\int_N \beta = \int_N \frac{Q}{r_0^2}\, dS = Q\int_N dS = \int_0^{\pi}\int_0^{2\pi} \sin\theta\, d\theta\, d\varphi = 4\pi Q$$

Exercize Repeat for $\beta = xz^2\, dx\wedge dy$

Exercize $M = \{(x,y,z)\in R^3 \mid z - (x^2+y^2) = 0\}$ parabola of revolution

$N = \{(x,y,z)\in M \mid 0 \le z < 1\}$



Introduce coordinates $\{x^1, x^2\}$ on M

$(x^1,x^2)(x,y,z) = (x,y)$. [Choose orientation so these are oriented (+) coords.]

Evaluate $\int_N x^2\, dy\wedge dz$.

↑ Question: which normal induces this orientation?

Helpful hint: use the parametrization (this preserves the orientation)

$x^1 = \rho\cos\varphi$ $\rho\in(0,1)$

$x^2 = \rho\sin\varphi$ $\theta\in(0,2\pi)$

~~What~~ Show that $n = \frac{\partial}{\partial z} - 2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y} = (df)^{\#}$ where $f = z - (x^2+y^2)$ is the inward normal.

What is dS for this case. (in terms of ρ, φ)

Out of time

Next we will do some more examples, and ~~prove~~ discuss Stoke's theorem and how to rewrite it in terms of the metric and then maybe some examples like the integral formulation of the Maxwell equations.
Then we will do covariant differentiation & pseudoRiemannian geometry quickly.

Students, come see me Monday – Wednesday. Make an appointment if necessary to find me, if you wish to have the extra explanation you probably need.