7 THE EXTERIOR DERIVATIVE d

We have already defined the differential of a 0-form (namely a function)

$$f \to df \qquad \text{where} \qquad df(X) \equiv Xf$$

(0-form) (1-form) (value on vector field)

In a local coordinate frame $\{ \partial/\partial x^i \}$:

$$df = \frac{\partial f}{\partial x^i} dx^i$$

In a local frame $\{e_i\}$: $\quad df = (e_i f)\, \omega^i$

NOTATION: $\quad e_i f \equiv \partial_i f \equiv f_{,i}$ in a coordinate or noncoordinate frame

This derivative operation is easily extended to any p-form by the local coordinate definition

$$\sigma = \frac{1}{p!} \sigma_{i_1 \cdots i_p} dx^{i_1 \cdots i_p} \xrightarrow{d} d\sigma = \frac{1}{p!} d\sigma_{i_1 \cdots i_p} \wedge dx^{i_1 \cdots i_p}$$

(p-form) ((p+1)-form)

Simplifying

$$d\sigma = \frac{1}{p!} \partial_{i_{p+1}} \sigma_{i_1 \cdots i_p} \underbrace{dx^{i_{p+1}} \wedge dx^{i_1 \cdots i_p}}_{dx^{i_{p+1} i_1 \cdots i_p}}$$

$$= \frac{1}{(p+1)!} \underbrace{\underbrace{\frac{(p+1)!}{p!}}_{=(p+1)} \partial_{[i_{p+1}} \sigma_{i_1 \cdots i_p]}}\; dx^{i_{p+1} i_1 \cdots i_p}$$

$$d\sigma_{i_{p+1} i_1 \cdots i_p} = (p+1)\, \partial_{[i_{p+1}} \sigma_{i_1 \cdots i_p]} \qquad \left(\text{components of } d\sigma\right)$$

This can also be written

$$d\sigma_{i_1 \cdots i_{p+1}} = \underbrace{(p+1) \frac{1}{(p+1)!}}_{\frac{1}{p!}} \delta^{j_1 \cdots j_{p+1}}_{i_1 \cdots i_{p+1}} \partial_{j_1} \sigma_{j_2 \cdots j_{p+1}} = \delta^{j_1 | j_2 \cdots j_{p+1}|}_{i_1 \cdots\cdots i_{p+1}} \partial_{j_1} \sigma_{j_2 \cdots j_{p+1}}$$

$$= \partial_{i_1} \sigma_{i_2 \cdots i_{p+1}} - \partial_{i_2} \sigma_{i_1 i_3 \cdots i_p} + \partial_{i_3} \sigma_{i_1 i_2 i_4 \cdots i_{p+1}} + \cdots$$

$$+ (-1)^{j+1} \partial_{i_j} \sigma_{i_1 \cdots i_{j-1} i_{j+1} \cdots i_{p+1}} + \cdots$$

$$= \sum_{j=1}^{p+1} (-1)^{j+1} \partial_{i_j} \sigma_{i_1 \cdots i_{j-1} i_{j+1} \cdots i_{p+1}}$$

no i_j index

EX. $p=1$ $\quad d\sigma_{ij} = \partial_i \sigma_j - \partial_j \sigma_i$

$p=2$ $\quad d\sigma_{ijk} = \partial_i \sigma_{jk} \underbrace{- \partial_j \sigma_{ik}}_{+\partial_j \sigma_{ki}} + \partial_k \sigma_{ij} = \underbrace{\partial_i \sigma_{jk} + \partial_j \sigma_{ki} + \partial_k \sigma_{ij}}$

$\equiv \partial_{\{i} \sigma_{jk\}}$ cyclic sum

$p=3$ $\quad d\sigma_{ijkl} = \partial_i \sigma_{jkl} - \partial_j \sigma_{ikl} + \partial_k \sigma_{ijl} - \partial_l \sigma_{ijk}.$

For $p = n-1$ it is useful to introduce the natural dual

$$\circledast\sigma^i = \frac{1}{(n-1)!} \epsilon^{i i_1 \cdots i_{n-1}} \sigma_{i_1 \cdots i_{n-1}} \quad \text{"vector density"}$$

($\circledast$ to differentiate from a metric dual $*$)

$$\sigma_{i_1 \cdots i_{n-1}} = \circledast\sigma^i \epsilon_{i i_1 \cdots i_{n-1}}$$

Then $\quad \sigma = \circledast\sigma^i \cdot \underbrace{\frac{1}{(n-1)!} \epsilon_{i i_1 \cdots i_{n-1}} dx^{i_1 \cdots i_{n-1}}}$

$$\equiv d\hat{x}_i = (-1)^{i+1} dx^{1 \cdots i-1\, i+1 \cdots n}$$

(since $d\hat{x}_i = \epsilon_{i i_1 \cdots i_{n-1}} dx^{|i_1 \cdots i_{n-1}|}$)

Note $\quad dx^j \wedge d\hat{x}_i = \delta^j{}_i \, dx^{1 \cdots n}$

And $\quad d\sigma = \partial_j \circledast\sigma^i \, dx^j \wedge d\hat{x}_i = \partial_j \circledast\sigma^i \delta^j{}_i \, dx^{1 \cdots n} = \underbrace{(\partial_i \sigma^{\circledast i})}_{\text{div } \sigma^*} dx^{1 \cdots n}$

[Alternatively: $\sigma = \frac{1}{(n-1)!} \sigma_{i_1 \cdots i_{n-1}} dx^{i_1 \cdots i_{n-1}}$

$d\sigma = \frac{1}{(n-1)!} \partial_i \sigma_{i_1 \cdots i_{n-1}} \underbrace{dx^i \wedge dx^{i_1 \cdots i_{n-1}}}_{\epsilon^{i i_1 \cdots i_{n-1}} dx^{1 \cdots n}} = \partial_i \underbrace{\left(\frac{1}{(n-1)!} \epsilon^{i i_1 \cdots i_{n-1}} \sigma_{i_1 \cdots i_{n-1}} \right)}_{\circledast\sigma^i} dx^{1 \cdots n}$]

Thus the exterior derivative of an $(n-1)$-form is just the divergence of the corresponding vector density times the coordinate basis n-form.

Some properties of d

$$d\sigma = \frac{1}{p!}\,\partial_{i_{p+1}}\sigma_{i_1\cdots i_p}\; dx^{i_{p+1} i_1\cdots i_p}$$

$$d^2\sigma = \frac{1}{p!}\,\partial_{i_{p+1}}\partial_{i_{p+1}}\sigma_{i_1\cdots i_p}\;\underbrace{dx^{i_{p+1}}\wedge dx^{i_{p+2} i_1\cdots i_p}}_{dx^{i_{p+1} i_{p+2} i_1\cdots i_p}} = 0$$

but $\partial_{[i}\partial_{j]}f \equiv 0$ — partial derivatives commute.

So $\boxed{d^2 \equiv 0}$

$$d(\alpha\wedge\beta) = d\Big(\frac{1}{p!\,q!}\alpha_{i_1\cdots i_p}\beta_{j_1\cdots j_q}\underbrace{dx^{i_1\cdots i_p}\wedge dx^{j_1\cdots j_p}}_{dx^{i_1\cdots i_p j_1\cdots j_p}}\Big)$$

$$= \frac{1}{p!\,q!}\underbrace{\partial_k(\alpha_{i_1\cdots i_p}\beta_{j_1\cdots j_q})}_{\left[\begin{array}{l}(\partial_k\alpha_{i_1\cdots i_p})\beta_{j_1\cdots j_q}\\ +\,\alpha_{i_1\cdots i_p}\,\partial_k\beta_{j_1\cdots j_q}\end{array}\right]} dx^k\wedge dx^{i_1\cdots i_p j_1\cdots j_p}$$

$$= \frac{1}{p!}\partial_k\alpha_{i_1\cdots i_p}\,dx^k\wedge dx^{i_1\cdots i_p}\wedge\frac{1}{q!}\beta_{j_1\cdots j_q}\,dx^{j_1\cdots j_p}$$

$$+ \frac{1}{p!}\alpha_{i_1\cdots i_p}\underbrace{dx^k\wedge dx^{i_1\cdots i_p}}_{(-1)^p dx^{i_1\cdots i_p}\wedge dx^k}\wedge\frac{1}{q!}\partial_k\beta_{j_1\cdots j_q}\,dx^{j_1\cdots j_p}$$

$$= \boxed{d\alpha\wedge\beta + (-1)^p\alpha\wedge d\beta = d(\underset{p}{\alpha}\wedge\underset{q}{\beta})}$$

EXERCISE The Lie bracket of 2 vector fields X and Y is a vector field $[X,Y]$ defined by

$$[X,Y]f \equiv (XY - YX)f .$$

a) Derive the coordinate expression

$$[X,Y] = (X^i \partial_i Y^j - Y^i \partial_i X^j) \frac{\partial}{\partial x^j} .$$

b) In a frame $\{e_i\}$, $[e_i,e_j]$ is a vector field which can be expressed in terms of the frame: $[e_i,e_j] \equiv C^k{}_{ij} e_k$

$$C^k{}_{ij} = \omega^k([e_i,e_j]).$$

Using the fact that if $e_i = e^j{}_i \frac{\partial}{\partial x^j}$ and $\omega^i = \omega^i{}_j dx^j$, then $(e^j{}_i)$ and $(\omega^j{}_i)$ are inverse matrices, together with the coordinate definition of $d\omega^i$ to prove:

$$d\omega^i = -\tfrac{1}{2} C^i{}_{jk} \omega^{jk}$$

($d\omega^i$ is a 2-form and $-C^i{}_{jk}$ are its components in this frame)

$C^k{}_{ij}$ are called the structure functions for the frame $\{e_i\}$.

$C^k{}_{ij} = 0$ for a coordinate frame since partial derivatives commute.

FRAME VERSION OF d

$p=1$ $\quad \sigma = \sigma_i \omega^i$

$$d\sigma = \underbrace{d\sigma_i \wedge \omega^i}_{\partial_j \sigma_i \, \omega^{ji}} - \underbrace{\sigma_i \wedge d\omega^i}_{-\frac{1}{2}\sigma_i C^i{}_{jk}\omega^{jk}} = (\partial_{[j}\sigma_{k]} + \tfrac{1}{2}\sigma_i C^i{}_{jk})\omega^{jk}$$

$$= \tfrac{1}{2} \underbrace{(2\partial_{[j}\sigma_{k]} + \sigma_i C^i{}_{jk})}_{d\sigma_{jk}} \omega^{jk}$$

$$d\sigma_{jk} = 2\partial_{[j}\sigma_{k]} + \sigma_i C^i{}_{jk} .$$

(the frame 1-forms now contribute to the derivative)

What does this mean? It is shorthand for

$$d\sigma(e_j,e_k) = e_j\sigma_k - e_k\sigma_j + \sigma_i C^i{}_{jk}$$

$$= e_j\,\sigma(e_k) - e_k\,\sigma(e_j) + \underbrace{\sigma_i\omega^i}_{\sigma}([e_i,e_j])$$

$$\underset{X,\,Y}{d\sigma(e_j,e_k)} = e_j\,\sigma(e_k) - e_k\,\sigma(e_j) + \sigma([e_i,e_j])$$

$$d\sigma(X,Y) = X\sigma(Y) - Y\sigma(X) + \sigma([X,Y]).$$

The last formula holds for any linearly independent vector fields X and Y and hence for all vector fields X and Y.

It is a coordinate free definition of the exterior derivative of a 1-form.

One can derive a general expression for a p-form.

EXERCISE. Derive the frame formula for d of a 2-form.

EXAMPLES Euclidean space (R^3, δ_{ij}) cartesian coords $\{x^i\}$

"Vector analysis" deals only with vector fields, but using the metric we can translate vector fields into 1-forms by lowering the index and 2-forms $(2=n-1)$ into vector fields by taking the metric dual ($\rightarrow$ 1-form) and raising the index (this is equivalent to the natural dual in cartesian coordinates)($\eta = dx^{123}$, $\sqrt{g}=1$)

Let f be a function, $X = X^i \frac{\partial}{\partial x^i}$ a vector field.

$$\text{"}\nabla f\text{"} = \text{grad } f = \partial_i f\, \delta^{ij} \frac{\partial}{\partial x^j} = (df)^\#$$

$$\text{"}\nabla \times X\text{"} = \text{curl } X = \epsilon^{ijk} \partial_j X_k \frac{\partial}{\partial x^i}$$

$$\text{"}\nabla \cdot X\text{"} = \text{div } X = \partial_i X^i$$

($\#$ means raise index on 1-form
$\flat$ means lower index on vector field)

Arbitrary 1-forms and 2-forms may be written in the form

$$X^\flat = X_i dx^i, \quad {}^*X^\flat = X^i d\hat{x}_i = X^{\{1} dx^{23\}} \quad \text{(cyclic sum)}$$

The above objects are then simply related to the $p=0$, $p=1$, and $p=2$ versions of d:

$p=0$ $\quad \text{grad } f = (df)^\#$ (above)

$p=1$ $\quad dX^\flat = d(X_i dx^i) = \partial_{[j} X_{i]} dx^{ji} = \underbrace{(\epsilon^{ijk}\partial_j X_k)}_{(\text{curl } X)^i} d\hat{x}_i$

$$({}^* dX^\flat)^\# = \text{curl } X$$

$p=2$ $\quad d\,{}^*X^\flat = (\partial_i X^i)\, dx^{123}$
$=n-1$

$$^* d\, {}^* X^\flat = \text{div } X$$

EXERCISE. Use $d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^p \alpha\wedge d\beta$ to prove some of the following (some = at least two).

$$\text{grad}\,(fg) = g\,\text{grad}\,f + f\,\text{grad}\,g$$

$$\text{curl}\,(fV) = \text{grad}\,f \times V + f\,\text{curl}\,V$$

$$\text{div}\,(fV) = (\text{grad}\,f)\cdot V + f\,\text{div}\,V$$

$$\text{div}\,(V\times U) = U\cdot\text{curl}\,V - V\cdot\text{curl}\,U$$

Recall $(V\times U)^i = \epsilon_{ijk}V^jU^k$.

<u>EXAMPLE</u> Minkowski spacetime $(R^4, \eta_{\alpha\beta})$ cartesian coords $\{x^\alpha\}_{\alpha=0,1,2,3}$

$x^0 \equiv t$, $(\eta_{\alpha\beta}) = \text{diag}(-1,1,1,1)$

$p=1$ $\quad A = A_\alpha dx^\alpha = A_0 dt + \underbrace{A_i dx^i}_{{}^3A}$ $\qquad$ Let $A^0 = -A_0 = \phi$

$${}^3A^\# = A^i \frac{\partial}{\partial x^i}$$

$$dA = \partial_i A_0\, dx^i\wedge dt + \underbrace{\partial_j A_i\, dx^{ji}}_{{}^3d\,{}^3A = \frac{1}{2}(\text{curl}\,{}^3A^\#)^i \epsilon_{ijk}dx^{jk}} + \partial_0 A_i\, dt\wedge dx^i$$

$$= \underbrace{(\partial_i A_0 - \partial_0 A_i)}_{-\partial_i\phi - \frac{\partial}{\partial t}A_i \;\equiv E_i} dx^i\wedge dt + \underbrace{(\text{curl}\,{}^3A^\#)^i}_{\equiv B^i} d\Omega_i$$

If we let $F = dA$, then $F = E_i dx^i\wedge dt + \frac{1}{2}B^i\epsilon_{ijk}dx^{jk}$.

$$E_i = F_{i0}, \quad B^i = \tfrac{1}{2}\epsilon^{ijk}F_{jk}.$$

If $(\phi, A^i\frac{\partial}{\partial x^i})$ are the scalar and vector potentials, $E^i\frac{\partial}{\partial x^i}$ and $B^i\frac{\partial}{\partial x^i}$ are the electric and magnetic fields and $F = \frac{1}{2}F_{\alpha\beta}dx^{\alpha\beta}$ is the electromagnetic field or Maxwell tensor

Since $F = dA$, $\quad dF = 0$ so

$p=2$

$$dF = \underbrace{\underbrace{\partial_j E_i\, dx^{ji}\wedge dt}_{(\text{curl}\,E)^i \frac{\epsilon_{ijk}dx^{jk}}{2}\wedge dt} + \tfrac{1}{2}\partial_0 B^i\epsilon_{ijk}dx^{0jk}}_{(\text{curl}\,E + \frac{\partial B}{\partial t})^i \frac{1}{2}\epsilon_{ijk}dx^{jk}\wedge dt} + \underbrace{\tfrac{1}{2}\partial_l B^i\epsilon_{ijk}dx^{ljk}}_{\underbrace{\partial_i B^i}_{\text{div}\,B} dx^{123}}$$

Recall page 41 : $\begin{cases} * \ \omega^{0ij} = -\omega^k \\ * \ \omega^{ijk} = -\omega^0 \end{cases}$ $\quad (i,j,k) = \sigma^+(1,2,3)$

set $\omega^i = dx^i$

So $-*\,dF = (\text{div}\, B)\, dt + (\text{curl}\, E + \frac{\partial B}{\partial t})_i\, dx^i$

Half of Maxwell's equations are therefore just

$$dF = d^2 A = 0. \quad \leftrightarrow \quad \boxed{\begin{array}{l} \text{div}\, B = 0 \\ \text{curl}\, E + \frac{\partial B}{\partial t} = 0 \end{array}}$$

From page 38 :

$$*F = -B_i\, dx^i \wedge dt + \tfrac{1}{2} E^i \epsilon_{ijk}\, dx^{jk}$$

We get $d\,{*F}$ from dF by replacing (E, B) by $(-B, E)$.

$$-*\,d\,{*F} = (\text{div} E)dt + (\text{curl}\, B + \tfrac{\partial E}{\partial t})_i\, dx^i .$$

Let $J^\# = J^\alpha \frac{\partial}{\partial x^\alpha} = \rho \frac{\partial}{\partial t} + J^i \frac{\partial}{\partial x^i}$ be the 4-current vector field.

and $J = -\rho\, dt + J_i dx^i$ the 4-current 1-form

Setting $\underbrace{-*d*}_{=-\delta} F = 4\pi J \quad \leftrightarrow \quad \boxed{\begin{array}{l} \text{div}\, E = 4\pi\rho \\ \text{curl}\, B - \frac{\partial E}{\partial t} = 4\pi\, {}^3J \end{array}}$

<u>Exercise</u> Rederive the vacuum Maxwell equations using the formula $d\left(\tfrac{1}{2} G_{\alpha\beta}\, dx^{\alpha\beta}\right) = 3\ \partial_{\{\gamma} G_{\alpha\beta\}}\, dx^{\alpha\beta\gamma}$ and

using $F_{i0} = E_i \quad F_{jk} = B_i \qquad (ijk) = \sigma^+(1,2,3)$ (cyclic permutation)

$*F_{i0} = B_i \quad *F_{jk} = -E_i$

ie reexpress $\boxed{dF = 0 = d\,{*F}.}$

Note that the metric enters these equations only in the $*$ operation, so once a nonzero 4-form is specified, one has the generalization of Maxwell's equations to any 4-manifold.

EXERCISE

A 1-form σ is "hypersurface forming" if there exists functions f and α such that $\sigma = \alpha\, df$ (then $d(\alpha^{-1}\sigma)=0$).

If X is a vector field satisfying $Xf=0$ (ie X is tangent to the level surfaces of f) then $\sigma(X) = \sigma_i X^i = \alpha\, df(X) = \alpha X f = 0$

so σ kills all vectors tangent to surfaces $f=f_0$.

Show that $\sigma \wedge d\sigma = 0$ if σ is hypersurface forming.

If we have a metric g then $\sigma^\#$ is a normal vector field to the level hypersurfaces of the function f:

$$g(\sigma^\#, X) = \sigma(X) = 0 \quad \text{if } X \text{ tangent to } f=f_0$$

Gauge transformations

Since $F = dA$, if $A \to A + d\varphi$ then

$$F \to d(A+d\varphi) = dA + d^2\varphi = F \quad \text{is unchanged.}$$

Changing the vector potential by the addition of the differential of a function gives the same electromagnetic field.

Suppose ψ is the wavefunction of a particle.
Since only probabilities and relative phases matter in QM we are free to redefine all fields by a position dependent phase factor

$$\psi \to e^{-iq\theta}\psi \qquad (\text{a } U(1) \text{ transformation})$$

Then $d\psi \to e^{-iq\theta}(d\psi - iq\, d\theta\, \psi)$ or

$$D_A\psi = d\psi - iqA\psi \to e^{-iq\theta}(d\psi - iq(A+d\theta))\psi = e^{-iq} D_{A+d\theta}\psi.$$

Then $D_A\psi$ not $d\psi$ behaves "covariantly" under this phase change and is called the "gauge covariant derivative" of ψ, provided we also change A by $A \to A + d\theta$.

More of this later.