③ permutation? determinant? what is this stuff?

NOTES FOR SUMMARY (PART 2, PP 10–18)

$$(1,2,3,\dots,n) \rightarrow \pi(1,2,3,\dots,n) = (\pi(1),\pi(2),\pi(3),\dots,\pi(n))$$

HINT: ($\pi \neq 3.14159\ldots$ here)

A <u>permutation</u> of n <u>ordered</u> things (elements of a set, in our case positive integers up to n, sometimes including zero) is simply a reordering of those things. Permutations of a set of n elements form a group called the symmetric group S_n.

Permutations of n integers may be represented as $2 \times n$ matrices:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & & \pi(n) \end{pmatrix} \begin{array}{l} \leftarrow \text{old order (row1)} \\ \leftarrow \text{new order (row2)} \end{array}$$

The ordering of the columns is unimportant.

Note that there are $n!$ permutations of n things.

<u>EX</u>. $(1,2,3,4) \rightarrow (4,3,2,1) = (\pi(1),\pi(2),\pi(3),\pi(4))$

or $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

A <u>transposition</u> is a permutation which interchanges 2 of the elements but leaves all others fixed.

<u>EX</u>. $P_{23} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$ interchanges 4 and 1

Note $P_{23} \circ P_{23} = \text{identity}$.

<u>FACT</u>. Any permutation can be represented (nonuniquely) as a product of either an even or odd number of transpositions. One can therefore assign a sign to each permutation by

$$\mathrm{sgn}(\pi) = (-1)^p \leftrightarrow \pi \text{ representable as } p \text{ transpositions}$$

$$\begin{cases} \mathrm{sgn}(\pi) = 1 & \text{"even permutation"} \\ \mathrm{sgn}(\pi) = -1 & \text{"odd permutation"} \end{cases}$$

$\boxed{\text{CLEARLY } \mathrm{sgn}(\pi_1 \circ \pi_2) = \mathrm{sgn}(\pi_1)\,\mathrm{sgn}(\pi_2)}$

<u>EX</u>. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ even $\leftarrow$ (these are also called cyclic permutations)

$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ odd

$\dim S_3 = 3! = 6$.

Let $A = (A^\alpha{}_\beta)$ be an $n \times n$ matrix. Define its determinant by

$$\det A = \sum_\pi \operatorname{sgn}(\pi)\, A^1{}_{\pi(1)} \cdots A^n{}_{\pi(n)}$$

note that this sum has $n!$ terms ($\dim S_n = n!$)

EX $n=1$ 1×1 matrix = real number $\det A = A$ (only 1 permutation: the identity)

$n=2$ $\det A = \underbrace{A^1{}_1 A^2{}_2}_{\text{even}} - \underbrace{A^1{}_2 A^2{}_1}_{\text{odd}}$ $(\dim S_2 = 2! = 2)$

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$n=3$ $\det A = \underbrace{A^1{}_1 A^2{}_2 A^3{}_3 + A^1{}_2 A^2{}_3 A^3{}_1 + A^1{}_3 A^2{}_1 A^3{}_2}_{\text{even}} \underbrace{- A^1{}_1 A^2{}_3 A^3{}_2 - A^1{}_2 A^2{}_1 A^3{}_3 - A^1{}_3 A^2{}_1 A^3{}_2}_{\text{odd}}$

$(\dim S_3 = 6)$

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

Now $\det A = \sum_\pi \operatorname{sgn}(\pi) \underbrace{A^1{}_{\pi(1)} \cdots A^n{}_{\pi(n)}}_{\text{reorder each term so lower indices ordered}} = \sum_\pi \operatorname{sgn}(\pi)\, A^{\pi^{-1}(1)}{}_1 \cdots A^{\pi^{-1}(n)}{}_n$

EX. $\begin{pmatrix} 1234 \\ 4321 \end{pmatrix} = \begin{pmatrix} 4321 \\ 1234 \end{pmatrix} = \begin{pmatrix} \pi^{-1}(1) & \cdots & \pi^{-1}(4) \\ 1 & \cdots & 4 \end{pmatrix}$

SINCE $\pi \circ \pi^{-1} = \mathrm{id}$ (even)
$(\operatorname{sgn}\pi)(\operatorname{sgn}\pi^{-1}) = 1$
$\operatorname{sgn}(\pi^{-1}) = \operatorname{sgn}\pi$

Since permutations form group, every element is the inverse of another element
$\sum_\pi = \sum_{\pi^{-1}}$

So $\det A = \sum_{\pi^{-1}} \operatorname{sgn}(\pi^{-1})\, A^{\pi^{-1}(1)}{}_1 \cdots A^{\pi(n)}{}_n = \sum_\pi \operatorname{sgn}(\pi)\, A^{\pi(1)}{}_1 \cdots A^{\pi(n)}{}_n$

EX. $n=2$ $A^1{}_1 A^2{}_2 - A^1{}_2 A^2{}_1 = A^1{}_1 A^2{}_2 - A^2{}_1 A^1{}_2$

$n=3$ $A^1{}_1 A^2{}_2 A^3{}_3 + A^1{}_2 A^2{}_3 A^3{}_1 + A^1{}_3 A^2{}_1 A^3{}_2 - \cdots = A^1{}_1 A^2{}_2 A^3{}_3 + A^3{}_1 A^1{}_2 A^2{}_3 + A^2{}_1 A^3{}_2 A^1{}_3 + \cdots$

Net result:

$$\det A = \sum_\pi \mathrm{sgn}(\pi)\, A^1{}_{\pi(1)} \cdots A^n{}_{\pi(n)} = \sum_\pi \mathrm{sgn}(\pi)\, A^{\pi(1)}{}_1 \cdots A^{\pi(n)}{}_n$$

<u>properties of det</u>

$\det A^T = \det A$

$\det AB = \det A \det B \longrightarrow \underbrace{\underline{1} = \det \underline{1}}$ $= \det(AA^{-1}) = (\det A)(\det A^{-1})$

$\underline{1} = (\delta^\alpha{}_\beta)$

$\det \underline{1} = \sum_\pi \mathrm{sgn}(\pi)\, \delta^1{}_{\pi(1)} \cdots \delta^n{}_{\pi(n)} = 1$

only $\pi = \mathrm{id}$ contributes

$$\det A^{-1} = \frac{1}{\det A}$$

$\det ABA^{-1} = \det A \det B \det A^{-1} = \det B$

This says if $B = (B^\alpha{}_\beta)$ is the matrix of a linear transformation with respect to a given basis, and we change the basis $(B \to ABA^{-1})$, then the determinant does not change value, ie it is independent of basis.

Now return to tensor algebra over a vectorspace V with basis e_α.
Define for a $\binom{0}{p}$ tensor with components $T_{\alpha_1 \cdots \alpha_p}$ (do the same for $\binom{p}{0}$ tensors)

$$T_{(\alpha_1 \cdots \alpha_p)} = \frac{1}{p!} \sum_\pi T_{\pi(\alpha_1) \cdots \pi(\alpha_p)} \quad (\text{= totally symmetric part "SYM(T)"})$$

$$T_{[\alpha_1 \cdots \alpha_p]} = \frac{1}{p!} \sum_\pi \mathrm{sgn}(\pi)\, T_{\pi(\alpha_1) \cdots \pi(\alpha_p)} \quad (\text{= totally antisymmetric part "ALT(T)"})$$

so we get 2 new tensors
$$\begin{cases} \mathrm{SYM}(T) = T_{(\alpha_1 \cdots \alpha_p)}\, \omega^{\alpha_1} \otimes \cdots \otimes \omega^{\alpha_p} \\ \mathrm{ALT}(T) = T_{[\alpha_1 \cdots \alpha_p]}\, \omega^{\alpha_1} \otimes \cdots \otimes \omega^{\alpha_p} \end{cases}$$

<u>EX</u> $p=2$

$$T_{(\alpha\beta)} = \tfrac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha})$$
$$T_{[\alpha\beta]} = \tfrac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha})$$

$p=3$
$$\begin{pmatrix} T_{(\alpha\beta\gamma)} \\ T_{[\alpha\beta\gamma]} \end{pmatrix} = \frac{1}{6}\Big[\underbrace{T_{\alpha\beta\gamma} + T_{\beta\gamma\alpha} + T_{\gamma\alpha\beta}}_{\text{cyclically backward}} (\pm) \big[\underbrace{T_{\gamma\beta\alpha}}_{\text{reversal}} + T_{\beta\alpha\gamma} + T_{\alpha\gamma\beta} \big] \Big]$$

reversal $\to$ cyclically backward

If $T_{\alpha_1 \cdots \alpha_p} = T_{(\alpha_1 \cdots \alpha_p)}$, T is totally symmetric, any permutation of the indices doesn't change the value of the component.

If $T_{\alpha_1 \cdots \alpha_p} = T_{[\alpha_1 \cdots \alpha_p]}$, T is totally antisymmetric, any permutation of the indices changes the value by the sign of the permutation. In particular if 2 indices are the same, the component vanishes.

EX. $T_{\alpha_1\alpha_2\alpha_3\cdots\alpha_p} = - T_{\alpha_2\alpha_1\alpha_3\cdots\alpha_p}$ (no sum, both indices at the same level)

$T_{\alpha\alpha\alpha_3\cdots\alpha_p} = - T_{\alpha\alpha\alpha_3\cdots\alpha_p} \rightarrow T_{\alpha\alpha\alpha_3\cdots\alpha_p} = 0$

Now define the generalized Kronecker deltas:

$$\boxed{\delta^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_p} = p!\,\delta^{\alpha_1}{}_{[\beta_1}\cdots\delta^{\alpha_p}{}_{\beta_p]}}$$

$$= \sum_{\pi} \mathrm{sgn}(\pi)\, \underbrace{\delta^{\alpha_1}{}_{\pi(\beta_1)}\cdots\delta^{\alpha_p}{}_{\pi(\beta_p)}}_{\substack{\text{reorder}\\ \text{lower indices}}} = \sum_{\pi \to -\pi^{-1}} \mathrm{sgn}(\pi)\, \delta^{\pi^{-1}(\alpha_1)}{}_{\beta_1}\cdots\delta^{\pi^{-1}(\alpha_p)}{}_{\beta_p}$$

$$= \sum_{\pi} \mathrm{sgn}(\pi)\, \delta^{\pi(\alpha_1)}{}_{\beta_1}\cdots\delta^{\pi(\alpha_p)}{}_{\beta_p} \boxed{= p!\,\delta^{[\alpha_1}{}_{\beta_1}\cdots\delta^{\alpha_p]}{}_{\beta_p}}$$

FACT. If you symmetrize or antisymmetrize the upper or lower indices of a tensor product of p factors of a $\binom{1}{1}$ tensor, the lower and upper indices, respectively, are automatically symmetrized or antisymmetrized.

So this $\binom{p}{p}$ tensor is antisymmetric in its contravariant and covariant indices, so a component vanishes unless every index at a given level is distinct.
To get a nonzero value, $(\beta_1\cdots\beta_p)$ must be a permutation of $(\alpha_1\cdots\alpha_p)$.
Only the identity permutation contributes in the sum, i.e. $\alpha_i = \pi(\beta_i)$, giving the result $\mathrm{sgn}\,\pi$ i.e.

$$\delta^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_p} = \begin{cases} \mathrm{sgn}\begin{pmatrix}\alpha_1\cdots\alpha_p\\ \beta_1\cdots\beta_p\end{pmatrix} & \text{if } \alpha\text{'s distinct \& } \beta\text{'s are a permutation of } \alpha\text{'s}. \\ 0 & \text{otherwise} \end{cases}$$

This tensor acts as the antisymmetrizer on p covariant (or p contravariant) indices (projects out the antisymmetric part)

$$T_{\alpha_1\cdots\alpha_p} = \delta^{\beta_1}{}_{\alpha_1}\cdots\delta^{\beta_p}{}_{\alpha_p}\, T_{\beta_1\cdots\beta_p} \qquad \text{(identity)}$$

$$T_{[\alpha_1\cdots\alpha_p]} = \underbrace{\delta^{\beta_1}{}_{[\alpha_1}\cdots\delta^{\beta_p}{}_{\alpha_p]}}_{\frac{1}{p!}\delta^{\beta_1\cdots\beta_p}{}_{\alpha_1\cdots\alpha_p}}\, T_{\beta_1\cdots\beta_p} = \frac{1}{p!}\,\delta^{\beta_1\cdots\beta_p}{}_{\alpha_1\cdots\alpha_p}\, T_{\beta_1\cdots\beta_p}$$

[same for $\binom{p}{0}$ tensors]

EX: $p=2$ $\quad \delta^{\alpha\beta}{}_{\gamma\delta} = \delta^{\alpha}{}_{\gamma}\delta^{\beta}{}_{\delta} - \delta^{\alpha}{}_{\delta}\delta^{\beta}{}_{\gamma} \qquad \delta^{12}{}_{12} = 1 \qquad \delta^{12}{}_{21} = -1$

$p=3$ $\quad \delta^{\alpha\beta\gamma}{}_{\delta\epsilon\rho} = \delta^{\alpha}{}_{\delta}\delta^{\beta}{}_{\epsilon}\delta^{\gamma}{}_{\rho} + \delta^{\alpha}{}_{\epsilon}\delta^{\beta}{}_{\rho}\delta^{\gamma}{}_{\delta} + \delta^{\alpha}{}_{\rho}\delta^{\beta}{}_{\delta}\delta^{\gamma}{}_{\epsilon} - \delta^{\alpha}{}_{\rho}\delta^{\beta}{}_{\epsilon}\delta^{\gamma}{}_{\delta} - \underbrace{\cdots}_{\text{2 other terms.}}$

Now define $\begin{cases} \epsilon^{\alpha_1\cdots\alpha_n} = \delta^{\alpha_1\cdots\alpha_n}_{1\cdots n} = \begin{cases} \operatorname{sgn}\begin{pmatrix}\alpha_1\cdots\alpha_n\\ 1\cdots n\end{pmatrix}, & \{\alpha_i\} \text{ a permutation of } (1\ldots n) \\ 0 \text{ otherwise} \end{cases} \\ \epsilon_{\alpha_1\cdots\alpha_n} = \delta^{1\cdots n}_{\alpha_1\cdots\alpha_n} = \text{same} \end{cases}$

Note $\begin{cases} (1)\ \epsilon_{1\cdots n} = \epsilon^{1\cdots n} = 1 \\ (2)\ \epsilon^{\alpha_1\cdots\alpha_n}\epsilon_{\beta_1\cdots\beta_n} = \delta^{\alpha_1\cdots\alpha_n}_{1\cdots n}\delta^{1\cdots n}_{\beta_1\cdots\beta_n} = \delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n} \end{cases}$

Space of antisymmetric tensors with p indices (covariant or contravariant) has dimension $\binom{n}{p} = \frac{n!}{p!(n-p)!}$ = # combinations of n things taken p at a time, order unimportant.

When $p=n$, get $\binom{n}{n} = 1$, so if $T_{\alpha_1\cdots\alpha_n} = T_{[\alpha_1\cdots\alpha_n]}$ then

$$T_{\alpha_1\cdots\alpha_n} = \begin{cases} T_{1\cdots n}\operatorname{sgn}\begin{pmatrix}\alpha_1\cdots\alpha_n\\ 1\cdots n\end{pmatrix} \\ 0 \text{ if } (\alpha_i) \text{ not perm of } (1\ldots n)\end{cases} = T_{1\cdots n}\,\epsilon_{\alpha_1\cdots\alpha_n} = \frac{1}{n!}T_{\beta_1\cdots\beta_n}\epsilon^{\beta_1\cdots\beta_n}\,\epsilon_{\alpha_1\cdots\alpha_n}$$

or directly:

$$T_{\alpha_1\cdots\alpha_n} = \frac{1}{n!}\delta^{\beta_1\cdots\beta_n}_{\alpha_1\cdots\alpha_n}T_{\beta_1\cdots\beta_n} = \Big(\underbrace{\frac{1}{n!}\epsilon^{\beta_1\cdots\beta_n}T_{\beta_1\cdots\beta_n}}_{T_{1\cdots n}}\Big)\epsilon_{\alpha_1\cdots\alpha_n}$$

Back to determinants

$A^\alpha{}_\beta$ components of $\binom{1}{1}$ tensor $\leftrightarrow$ matrix $A = (A^\alpha{}_\beta)$

$$\det A = \sum_\pi \operatorname{sgn}(\pi)\, A^1{}_{\pi(1)}\cdots A^n{}_{\pi(n)} = n!\, A^1{}_{[1}\cdots A^n{}_{n]}$$

(also antisymmetric in upper indices)

$$= \delta^{\alpha_1\cdots\alpha_n}_{1\cdots n}A^1{}_{\alpha_1}\cdots A^n{}_{\alpha_n} = \boxed{\epsilon^{\alpha_1\cdots\alpha_n}A^1{}_{\alpha_1}\cdots A^n{}_{\alpha_n}}$$

or

$$\det A = \cdots = \delta^{1\cdots n}_{\alpha_1\cdots\alpha_n}A^{\alpha_1}{}_1\cdots A^{\alpha_n}{}_n = \boxed{\epsilon_{\alpha_1\cdots\alpha_n}A^{\alpha_1}{}_1\cdots A^{\alpha_n}{}_n} = \det A$$

$$= \epsilon_{\alpha_1\cdots\alpha_n}A^{\alpha_1}{}_{[1}\cdots A^{\alpha_n}{}_{n]} = \frac{1}{n!}\epsilon_{\alpha_1\cdots\alpha_n}A^{\alpha_1}{}_{\beta_1}\cdots A^{\alpha_n}{}_{\beta_n}\delta^{\beta_1\cdots\beta_n}_{1\cdots n}$$

$$= \frac{1}{n!}\epsilon_{\alpha_1\cdots\alpha_n}\epsilon^{\beta_1\cdots\beta_n}A^{\alpha_1}{}_{\beta_1}\cdots A^{\alpha_n}{}_{\beta_n} = \boxed{\frac{1}{n!}\delta^{\beta_1\cdots\beta_n}_{\alpha_1\cdots\alpha_n}A^{\alpha_1}{}_{\beta_1}\cdots A^{\alpha_n}{}_{\beta_n}}$$

Suppose we write $\epsilon_{\alpha_1\ldots\alpha_n} A^{\alpha_1}{}_{\beta_1} \cdots A^{\alpha_n}{}_{\beta_n}$

This gives a nonzero result only if $(\beta_1 \cdots \beta_n)$ is a permutation of $(1 \ldots n)$. Permuting $(1 \ldots n)$ to $(\beta_1 \cdots \beta_n)$ is equivalent to permuting the α's instead which changes the expression by the sign of the permutation, namely $\epsilon_{\beta_1 \ldots \beta_n}$, ie

$$\boxed{\begin{aligned} (\det A)\,\epsilon_{\beta_1 \cdots \beta_n} &= \epsilon_{\alpha_1 \ldots \alpha_n} A^{\alpha_1}{}_{\beta_1} \cdots A^{\alpha_n}{}_{\beta_n} \\ \text{and} \quad (\det A)\,\epsilon^{\alpha_1 \cdots \alpha_n} &= \epsilon^{\beta_1 \cdots \beta_n} A^{\alpha_1}{}_{\beta_1} \cdots A^{\alpha_n}{}_{\beta_n} \end{aligned}}$$

COFACTORS

$$\det A = \frac{1}{n} \underbrace{\frac{1}{(n-1)!} \delta^{\beta_1 \cdots \beta_{n-1} \beta}_{\alpha_1 \cdots \alpha_{n-1} \alpha} A^{\alpha_1}{}_{\beta_1} \cdots A^{\alpha_{n-1}}{}_{\beta_{n-1}}} A^{\alpha}{}_{\beta}$$

$$\equiv \Delta(A)^{\beta}{}_{\alpha} \equiv \text{"}(^{\beta}_{\alpha})\text{ cofactor"}$$

In fact one can show:

$$\left.\begin{aligned} \Delta(A)^{\beta}{}_{\gamma} A^{\gamma}{}_{\alpha} &= \delta^{\beta}_{\alpha} \det A \\ A^{\beta}{}_{\gamma} \Delta(A)^{\gamma}{}_{\alpha} &= \delta^{\beta}_{\alpha} \det A \end{aligned}\right\} \text{ taking trace gives back}$$

so that if $\det A \neq 0$ can define:

$$A^{-1\,\alpha}{}_{\beta} = (\det A)^{-1} \Delta(A)^{\alpha}{}_{\beta} \qquad \text{(inverse of } A\text{)}$$

$$A^{-1} A = A A^{-1} = 1$$

Note the following special cases:

$$\begin{aligned} \det A &= \Delta(A)^{1}{}_{\gamma} A^{\gamma}{}_{1} = \text{"expansion in cofactors along } 1^{st} \text{ column"} \\ &= \Delta(A)^{\gamma}{}_{1} A^{1}{}_{\gamma} = \text{"expansion in cofactors along } 1^{st} \text{ row"} \end{aligned}$$

In fact $\Delta(A)^{\alpha}{}_{\beta}$ is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by removing the α^{th} column and β^{th} row, multiplied by $(-1)^{\alpha+\beta}$

If this has not occurred in your mathematical past, it will eventually.

Raising and lowering indices

Suppose $\mathbf{g} = g_{\alpha\beta}\,\omega^\alpha \otimes \omega^\beta$ is an inner product or metric on V

Let $\underline{g} = (g_{\alpha\beta})$ and $g = |\det(\underline{g})|$.

If we try to lower the indices on $\epsilon^{\alpha_1 \cdots \alpha_n}$:

$$g_{\alpha_1\beta_1} \cdots g_{\alpha_n\beta_n}\, \epsilon^{\beta_1 \cdots \beta_n} = \epsilon_{\beta_1 \cdots \beta_n} \det(\underline{g})$$

Similarly $$g^{\alpha_1\beta_1} \cdots g^{\alpha_n\beta_n}\, \epsilon_{\beta_1 \cdots \beta_n} = \epsilon^{\alpha_1 \cdots \alpha_n} \underbrace{\det(\underline{g}^{-1})}_{= \det(\underline{g})^{-1}},$$

so can't raise and lower indices without getting into trouble.

BUT suppose we define a tensor by

$$\eta_{\alpha_1 \cdots \alpha_n} \equiv |\det(\underline{g})|^{1/2}\, \epsilon_{\alpha_1 \cdots \alpha_n}$$

and define all other index positions to be obtained from this one by raising ~~and lowering~~ indices from the fully covariant form.

In particular

$$\eta^{\alpha_1 \cdots \alpha_n} = \underbrace{\underbrace{g^{\alpha_1\beta_1} \cdots g^{\alpha_n\beta_n}\, \epsilon_{\alpha_1 \cdots \alpha_n}}_{\epsilon^{\beta_1 \cdots \beta_n} (\det \underline{g}^{-1})} |\det(\underline{g})|^{1/2}}_{(\operatorname{sgn}(\det \underline{g})) \cdot |\det(\underline{g})|^{-1/2}}$$

$$\eta^{\alpha_1 \cdots \alpha_n} = \underbrace{(\operatorname{sgn} \det \underline{g})}_{= (-1)^{\frac{n-s}{2}}} \,|\det \underline{g}|^{-1/2}\, \epsilon^{\alpha_1 \cdots \alpha_n}:$$

can always transform $\underline{g}$ to standard form by picking an orthonormal basis;

$$\underline{g} = \underline{A}^T \underline{\eta}\, \underline{A}$$

$$\det \underline{g} = \underbrace{\det A \det A^T}_{(\det A)^2} \det \eta$$

$$\operatorname{sgn} \det \underline{g} = \operatorname{sgn} \det \eta$$

$$\eta = \operatorname{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q}) \qquad \operatorname{sgn} \det \eta = \det \eta = (-1)^q = (-1)^{\frac{n-s}{2}}$$

$$p + q = n$$
$$p - q = s$$
$$q = \frac{n-s}{2}$$

$-+++$ $s=2$ $\quad$ $+---$ $s=-2$

EX. Minkowski spacetime $\quad (-1)^q = (-1)^1 = (-1)^3 = -1$

Note that in an orthonormal basis ; $|\det \underline{g}| = 1$, so

$$\eta_{\alpha_1 \cdots \alpha_n} = \epsilon_{\alpha_1 \cdots \alpha_n}$$

$$\eta^{\alpha_1 \cdots \alpha_n} = (-1)^{\frac{n-s}{2}} \epsilon^{\alpha_1 \cdots \alpha_n}$$

(in an ON basis only)

Let $\eta = \frac{1}{n!} \eta_{\alpha_1 \cdots \alpha_n} \omega^{\alpha_1} \otimes \cdots \otimes \omega^{\alpha_n}$

$$\eta^{\#} = \frac{1}{n!} \eta^{\alpha_1 \cdots \alpha_n} e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n}$$

Then $\pm \eta$ are the only elements of the 1-dimensional space of $\binom{0}{n}$- tensors which are antisymmetric , such that their value in any orthonormal frame is ± 1.