A permutation of $n$ ordered things (elements of a set, in our case positive integers up to $n$, sometimes including zero) is simply a reordering of those things. Permutations of a set of $n$ elements form a group called the symmetric group $S_n$.

Permutations of $n$ integers may be represented as $2 \times n$ matrices:

\[
\begin{pmatrix}
1 & 2 & 3 & \ldots & n
\end{pmatrix}
\begin{pmatrix}
\pi(1)
\pi(2)
\pi(3)
\ldots
\pi(n)
\end{pmatrix}
\]

The ordering of the columns is unimportant.

Note that there are $n!$ permutations of $n$ things.

**Ex.** $(1,2,3,4) \rightarrow (4,3,2,1) = (\pi(1), \pi(2), \pi(3), \pi(4))$

or $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

A transposition is a permutation which interchanges 2 of the elements but leaves all others fixed.

**Ex.** $P_{23} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$ interchanges 4 and 1

Note $P_{23} \circ P_{23} = \text{identity}$.

**Fact.** Any permutation can be represented (nonuniquely) as a product of either an even or odd number of transpositions. One can therefore assign a sign to each permutation by

\[
\text{sgn}(\pi) = (-1)^p \iff \text{\pi representable as } p \text{ transpositions}
\]

\[
\begin{cases}
\text{sgn}(\pi) = 1 & \text{"even permutation"} \\
\text{sgn}(\pi) = -1 & \text{"odd permutation"}
\end{cases}
\]

**Ex.** $(1 \ 2 \ 3 \ 2 \ 1) = (1 \ 2 \ 3)$ even $\quad \dim S_3 = 3! = 6$

$(1 \ 2 \ 3 \ 3 \ 2 \ 1) = (1 \ 2 \ 3)$ odd

These are also called cyclic permutations.

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 2 & 1
\end{pmatrix}
\]
Let \( A = (A^i_j) \) be an \( nxn \) matrix. Define its determinant by

\[
\det A = \sum_{\pi} \text{sgn}(\pi) \prod_{i=1}^{n} A^i_{\pi(i)}
\]

\[
\text{note that this sum has } n! \text{ terms (dim } S_n = n! \text{)}
\]

\[
\begin{align*}
\text{EX} & \quad n=1 \quad 1x1 \text{ matrix } = \text{ real number} \quad \det A = A \quad (\text{ only 1 permutation: the identity}) \\
\text{n=2} & \quad \det A = A^1_1A^2_2 - A^1_2A^2_1 \quad (\dim S_2 = 2! = 2)
\end{align*}
\]

\[
\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc
\]

\[
\text{n=3} \quad \det A = A^1_1A^2_2A^3_3 + A^1_2A^2_3A^3_1 + A^1_3A^2_1A^3_2 - A^1_1A^2_3A^3_2 - A^1_2A^2_1A^3_3 - A^1_3A^2_2A^3_1
\]

\[
(\dim S_3 = 6)
\]

Now \( \det A = \sum_{\pi} \text{sgn}(\pi) \prod_{i=1}^{n} A^i_{\pi(i)} \)

\[
\text{reorder each term so lower indices ordered}
\]

\[
\text{EX. } (1234) = (4321) = (\Pi^{-1}(1) \ldots \Pi^{-1}(4))
\]

\[
\text{SINCE } \Pi \circ \Pi^{-1} = \text{Id (even)}
\]

\[
\text{sgn} \Pi \circ \text{sgn} \Pi^{-1} = 1
\]

\[
\text{sgn} \Pi^{-1} = \text{sgn} \Pi
\]

So \( \det A = \sum \text{sgn}(\pi^{-1}) \prod_{i=1}^{n} A^i_{\pi^{-1}(i)} \)

\[
\text{Since permutations form group, every element is the inverse of another element:}
\]

\[
\sum_{\Pi^{-1}} \text{sgn}(\pi) \prod_{i=1}^{n} A^i_{\pi(i)} = \sum_{\pi} \text{sgn}(\pi) \prod_{i=1}^{n} A^i_{\pi(i)}
\]

\[
\text{EX. } n=2 \quad A^1_1A^2_2 - A^1_2A^2_1 = A^1_1A^2_2 - A^1_2A^2_1
\]

\[
\text{n=3} \quad A^1_1A^2_2A^3_3 + A^1_2A^2_3A^3_1 + A^1_3A^2_1A^3_2 - \cdots = A^1_1A^2_2A^3_3 + A^1_2A^2_3A^3_1 + A^1_3A^2_1A^3_2 - \cdots
\]
Net result:
\[
\det A = \sum_{\pi} \text{sgn}(\pi) \ A_{\pi(1)}^{1} \cdots A_{\pi(n)}^{n} = \sum_{\pi} \text{sgn}(\pi) \ A_{\pi(1)}^{1} \cdots A_{\pi(n)}^{n}
\]

Properties of \(\det\):

\[
\det A^T = \det A
\]
\[
\det AB = \det A \det B \rightarrow 1 = \det 1 = \det (AA^{-1}) = (\det A)(\det A^{-1})
\]

\[
\det A^{-1} = \frac{1}{\det A}
\]

Only \(\pi = 1\text{d}\) contributes

\[
\det ABA^{-1} = \det A \det B \det A^{-1} = \det B
\]

This says if \(B = (B^\nu_\sigma)\) is the matrix of a linear transformation with respect to a given basis, and we change the basis \((B \rightarrow ABA^{-1})\), then the determinant does not change value, i.e. it is independent of basis.

Now return to tensor algebra over a vector space \(V\) with basis \(e_\alpha\).

Define for a \((\rho)_p\) tensor with components \(T_{\alpha_1 \cdots \alpha_p}\) (do the same for \((\rho)_p\) tensors)

\[
T(\alpha_1 \cdots \alpha_p) = \frac{1}{p!} \sum_{\pi} T_{\pi(\alpha_1) \cdots \pi(\alpha_p)} \quad (\text{totally symmetric part "SYM(T)"})
\]

\[
T[\alpha_1 \cdots \alpha_p] = \frac{1}{p!} \sum_{\pi} \text{sgn}(\pi) T_{\pi(\alpha_1) \cdots \pi(\alpha_p)} \quad (\text{totally antisymmetric part "ALT(T)"})
\]

so we get 2 new tensors

\[
\begin{align*}
\text{SYM}(T) &= T(\alpha_1 \cdots \alpha_p) \omega^{\alpha_1} \cdots \omega^{\alpha_p} \\
\text{ALT}(T) &= T[\alpha_1 \cdots \alpha_p] \omega_{\alpha_1} \cdots \omega_{\alpha_p}
\end{align*}
\]

\[
\text{EX} \quad p=2 \quad T(\alpha \beta) = \frac{1}{2} (T_{\alpha \beta} + T_{\beta \alpha})
\]

\[
T[\alpha \beta] = \frac{1}{2} (T_{\alpha \beta} - T_{\beta \alpha})
\]

\[
p=3 \quad \left(\frac{T(\alpha \beta \gamma)}{T[\alpha \beta \gamma]}\right) = \frac{1}{6} \left[ \frac{T_{\alpha \beta \gamma} + T_{\beta \alpha \gamma} + T_{\gamma \alpha \beta}}{\text{cyclically backward}} \right] \left[ T_{\gamma \alpha \beta} + T_{\beta \gamma \alpha} + T_{\alpha \beta \gamma}} \right] \quad (\text{reversal} \rightarrow \text{cyclically backward})
\]

If \(T_{\alpha_1 \cdots \alpha_p} = T(\alpha_1 \cdots \alpha_p)\), \(T\) is totally symmetric, any permutation of the indices does not change the value of the component.

If \(T[\alpha_1 \cdots \alpha_p] = T[\alpha_1 \cdots \alpha_p]\), \(T\) is totally antisymmetric, any permutation of the indices changes the value by the sign of the permutation. In particular if 2 indices are the same, the component vanishes.
EX: \[T_{\alpha_1\alpha_2\ldots\alpha_p} = -T_{\alpha_p\alpha_1\alpha_2\ldots\alpha_{p-1}} \quad (\text{no sum, both indices at the same level})\]

Now define the generalized Kronecker deltas:

\[
\delta^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_p} = p! \left[ \delta^{\alpha_1}_{\beta_1} \ldots \delta^{\alpha_p}_{\beta_p} \right] = \frac{\prod \text{sgn}(\pi)}{\prod \text{sgn}(\pi')} \delta^{\pi(\alpha_1)}_{\beta_1} \ldots \delta^{\pi(\alpha_p)}_{\beta_p}
\]

FACT. If you symmetrize or antisymmetrize the upper or lower indices of a product of \(p\) factors of a \((p)\) tensor, the lower and upper indices, respectively, are automatically symmetrized or antisymmetrized.

So this \((p)\) tensor is antisymmetric in its contravariant and covariant indices, so a component vanishes unless every index at a given level is distinct.

To get a nonzero value, \((\beta_1\ldots\beta_p)\) must be a permutation of \((\alpha_1\ldots\alpha_p)\). Only the identity permutation contributes in the sum, i.e. \(\alpha_i = \pi(\beta_i)\), giving the result \(\text{sgn} \pi \) i.e.

\[
\delta^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_p} = \begin{cases} 
\text{sgn} \left( \begin{array}{c} \alpha_1 \ldots \alpha_p \\ \beta_1 \ldots \beta_p \end{array} \right) & \text{if } \alpha_i \text{'s distinct & } \beta_i \text{'s are a permutation of } \alpha_i \text{'s.} \\
0 & \text{otherwise}
\end{cases}
\]

This tensor acts as the antisymmetrizer on \(p\) covariant (or \(p\) contravariant) indices (projects out the antisymmetric part)

\[
T_{\alpha_1\ldots\alpha_p} = \delta^{\beta_1\ldots\beta_p}_{\alpha_1\ldots\alpha_p} T_{\beta_1\ldots\beta_p} \quad \text{(identity)}
\]

\[
T_{[\alpha_1\ldots\alpha_p]} = \delta^{\beta_1\ldots\beta_p}_{[\alpha_1\ldots\alpha_p]} T_{\beta_1\ldots\beta_p} = \frac{1}{p!} \delta^{\beta_1\ldots\beta_p}_{\alpha_1\ldots\alpha_p} T_{\beta_1\ldots\beta_p} \quad \text{[same for \((p)\) tensors]}
\]

EX: \(p=2\) \[\delta^{\alpha\beta}_{\gamma\delta} = \delta^{\alpha\gamma} \delta_{\beta\delta} - \delta^{\alpha\delta} \delta_{\beta\gamma} \quad \delta^{12}_{12} = 1 \quad \delta^{12}_{21} = -1\]

\(p=3\) \[\delta^{\alpha\beta\gamma}_{\delta\varepsilon\rho} = \delta^{\alpha\delta} \delta^{\beta\varepsilon} \delta_{\gamma\rho} + \delta^{\alpha\varepsilon} \delta^{\beta\delta} \delta_{\gamma\rho} - \delta^{\alpha\delta} \delta^{\beta\delta} \delta_{\gamma\varepsilon} + \delta^{\alpha\varepsilon} \delta^{\beta\delta} \delta_{\gamma\rho} - \delta^{\alpha\varepsilon} \delta^{\beta\rho} \delta_{\gamma\delta} - \ldots\]

\(\text{other terms.}\)
Now define \( \epsilon_{\alpha_1 \ldots \alpha_n} = \delta_{\alpha_1 \ldots \alpha_n} \) if \( (\alpha_i) \) a permutation of \((1 \ldots n)\)

\[
\begin{cases}
\epsilon_{\alpha_1 \ldots \alpha_n} = \delta_{\alpha_1 \ldots \alpha_n} = \text{same} & \\
0 & \text{otherwise}
\end{cases}
\]

Note:
1. \( \epsilon_{\alpha_1 \ldots \alpha_n} = \epsilon_{\beta_1 \ldots \beta_n} = 1 \)
2. \( \epsilon_{\alpha_1 \ldots \alpha_n} \epsilon_{\beta_1 \ldots \beta_n} = \delta_{\alpha_1 \ldots \alpha_n} \delta_{\beta_1 \ldots \beta_n} = \delta_{\alpha_1 \ldots \alpha_n} \)

Space of antisymmetric tensors with \( p \) indices (covariant or contravariant) has dimension \( \binom{n}{p} = \frac{n!}{p!(n-p)!} = \) \# combinations of \( n \) things taken \( p \) at a time, order unimportant.

When \( p = n \), get \( \binom{n}{n} = 1 \), so if \( T_{\alpha_1 \ldots \alpha_n} = T_{\alpha_i \ldots \alpha_n} \) then

\[
T_{\alpha_1 \ldots \alpha_n} = \frac{1}{n!} \sum_{\text{perm of } (1 \ldots n)} \epsilon_{\alpha_1 \ldots \alpha_n} \text{sgn}(\text{perm of } (1 \ldots n)) = \frac{1}{n!} \sum_{\text{perm of } (1 \ldots n)} \epsilon_{\alpha_1 \ldots \alpha_n} \delta_{\beta_1 \ldots \beta_n} \epsilon_{\beta_1 \ldots \beta_n}
\]

or directly:

\[
T_{\alpha_1 \ldots \alpha_n} = \frac{1}{n!} \epsilon_{\beta_1 \ldots \beta_n} \epsilon_{\alpha_1 \ldots \alpha_n} T_{\beta_1 \ldots \beta_n} = \frac{1}{n!} \epsilon_{\beta_1 \ldots \beta_n} \epsilon_{\alpha_1 \ldots \alpha_n} T_{\beta_1 \ldots \beta_n}
\]

Back to determinants

Components of \( \epsilon \) tensor

\[
\begin{align*}
\det A &= \epsilon_{\alpha_1 \ldots \alpha_n} A_{\alpha_1 \ldots \alpha_n} = \frac{1}{n!} A_{\alpha_1 \ldots \alpha_n} A_{\alpha_1 \ldots \alpha_n} = \frac{1}{n!} A_{\alpha_1 \ldots \alpha_n} A_{\alpha_1 \ldots \alpha_n} \\
\det A &= \epsilon_{\alpha_1 \ldots \alpha_n} A_{\alpha_1 \ldots \alpha_n} = \frac{1}{n!} A_{\alpha_1 \ldots \alpha_n} A_{\alpha_1 \ldots \alpha_n} = \frac{1}{n!} A_{\alpha_1 \ldots \alpha_n} A_{\alpha_1 \ldots \alpha_n}
\end{align*}
\]

(Also antisymmetric in upper indices)
Suppose we write \[ \varepsilon_{\alpha_1 \ldots \alpha_n} A^{\alpha_1}_{\beta_1} \ldots A^{\alpha_n}_{\beta_n} \]
This gives a nonzero result only if \((\beta_1 \ldots \beta_n)\) is a permutation of \((1 \ldots n)\).
Permuting \((1 \ldots n)\) to \((\beta_1 \ldots \beta_n)\) is equivalent to permuting the \(\alpha_i\)s
instead which changes the expression by the sign of the permutation, namely \(\varepsilon_{\beta_1 \ldots \beta_n}\), i.e.
\[
(\det A) \varepsilon_{\beta_1 \ldots \beta_n} = \varepsilon_{\alpha_1 \ldots \alpha_n} A^{\alpha_1}_{\beta_1} \ldots A^{\alpha_n}_{\beta_n}
\]
and
\[
(\det A) \varepsilon_{\alpha_1 \ldots \alpha_n} = \varepsilon_{\beta_1 \ldots \beta_n} A^{\alpha_1}_{\beta_1} \ldots A^{\alpha_n}_{\beta_n}
\]

**COFACTORS**

\[
\det A = \frac{1}{n!} \frac{1}{(n-1)!} \delta_{\alpha_1 \ldots \alpha_{n-1}, \beta} A^{\alpha_1}_{\beta_1} \ldots A^{\alpha_{n-1}}_{\beta_{n-1}} A^{\alpha_n}_{\beta_n}
\]
\[= \Delta(A)_{\beta \alpha} \equiv "(\beta)\ cofactor" \]

In fact one can show:
\[
\Delta(A)_{\beta \alpha} A^{\alpha}_{\chi} = \delta_{\beta \chi} \det A
\]
\[
A^{\beta}_{\chi} \Delta(A)_{\chi \alpha} = \delta_{\alpha \beta} \det A
\]
so that if \(\det A \neq 0\) can define:
\[
A^{-1}_{\beta \alpha} = (\det A)^{-1} \Delta(A)_{\beta \alpha} \quad \text{inverse of } A
\]
\[A^{-1} A = AA^{-1} = I \]

Note the following special cases:

\[
\det A = \Delta(A)^{1 \chi} A^{\chi \chi} = \text{"expansion in cofactors along 1st column"}
\]
\[= \Delta(A)^{\chi 1} A^{\chi \chi} = \text{"expansion in cofactors along 1st row"}
\]

In fact \(\Delta(A)^{\chi \beta}\) is the determinant of the \((n-1) \times (n-1)\)
matrix obtained from \(A\) by removing the \(\alpha\)th column
and \(\beta\)th row, multiplied by \((-1)^{\alpha+\beta}\).

If this has not occurred in your mathematical past, it will eventually.
Suppose $\mathcal{g} = g_{\alpha\beta} \omega^\alpha \otimes \omega^\beta$ is an inner product or metric on $\mathcal{V}$. Let $\mathcal{g} = (g_{\alpha\beta})$ and $\mathcal{g} = |\det(\mathcal{g})|$. If we try to lower the indices on $\varepsilon^{\alpha_1 \ldots \alpha_n}$:

$$g_{\alpha_1 \beta_1} \ldots g_{\alpha_n \beta_n} \varepsilon^{\beta_1 \ldots \beta_n} = \varepsilon^{\alpha_1 \ldots \alpha_n} \det(\mathcal{g})$$

Similarly, $g^{\alpha_1 \beta_1} \ldots g^{\alpha_n \beta_n} \varepsilon_{\beta_1 \ldots \beta_n} = \varepsilon^{\alpha_1 \ldots \alpha_n} \det(\mathcal{g})$

$$= \det(\mathcal{g})^{-1}$$

so can't raise and lower indices without getting into trouble.

BUT suppose we define a tensor by

$$\eta_{\alpha_1 \ldots \alpha_n} = |\det(\mathcal{g})|^{1/2} \varepsilon_{\alpha_1 \ldots \alpha_n}$$

and define all other index positions to be obtained from this one by raising and lowering indices from the fully covariant form. In particular

$$\eta^{\alpha_1 \ldots \alpha_n} = g^{\alpha_1 \beta_1} \ldots g^{\alpha_n \beta_n} \varepsilon_{\beta_1 \ldots \beta_n} |\det(\mathcal{g})|^{1/2}$$

$$\varepsilon^{\beta_1 \ldots \beta_n} (\det g^{-1})$$

$$= (\text{sgn}(\det g)) \cdot |\det(\mathcal{g})|^{-1/2}$$

$$\eta^{\alpha_1 \ldots \alpha_n} = (\text{sgn} \det g) |\det g|^{-1/2} \varepsilon^{\alpha_1 \ldots \alpha_n}$$

can always transform $\mathcal{g}$ to standard form by picking an orthonormal basis:

$$\mathcal{g} = A^T \eta A$$

$$\det \mathcal{g} = \det A \det A^T \det \eta$$

$$= (\det A)^2$$

$$\text{sgn} \det \mathcal{g} = \text{sgn} \det \eta$$

$$\eta = \text{diag}(1 \ldots 1, -1 \ldots -1)$$

$$\text{sgn} \det \eta = (-1)^{\frac{n-5}{2}}$$

$$\det \eta = (-1)^{\frac{n-5}{2}}$$

EX. Minkowski spacetime

$$(-1)^q = (-1)^{\frac{1}{4}} = (-1)^{\frac{3}{2}} = -1$$
Note that in an orthonormal basis, $|\det g| = 1$, so

$$\eta_{\alpha_1 \ldots \alpha_n} = E_{\alpha_1 \ldots \alpha_n}$$

$$\eta^{\alpha_1 \ldots \alpha_n} = (-1)^{\frac{n-s}{2}} \epsilon_{\alpha_1 \ldots \alpha_n}$$

(only in an ON basis)

Let

$$\mathcal{N} = \frac{1}{n!} \eta_{\alpha_1 \ldots \alpha_n} \omega^{\alpha_1} \otimes \cdots \otimes \omega^{\alpha_n}$$

$$\mathcal{N}^\# = \frac{1}{n!} \eta^{\alpha_1 \ldots \alpha_n} E_{\alpha_1} \otimes \cdots \otimes E_{\alpha_n}$$

Then $\pm \mathcal{N}$ are the only elements of the 1-dimensional space of $(n)$-tensors which are antisymmetric, such that their value in any orthonormal frame is $\pm 1$. 