① I. SOME LINEAR ALGEBRA: familiar (or not so familiar) ideas summarized

References: A. Trautmann, *Foundations and Current Problems of General Relativity*, in Brandeis Lectures on General Relativity 1964

C. Misner, K. Thorne, J. Wheeler, *GRAVITATION*

M. Spivak, *Calculus on Manifolds*

H. Flanders, *Differential Forms*

A REAL VECTOR SPACE V

A real vector space V is a space (whose elements are called vectors), together with an additive operation $+$ $(X, Y \in V \to X+Y \in V)$ such that multiplication of vectors by real numbers is defined $(X \to aX,\ a \in R)$ and "all the usual rules of addition and multiplication apply."

In particular V contains the zero vector (written simply $0 \in V$) which results from multiplying any vector by $0 \in R$; $0 \in V$ is the additive identity: $X + 0 = X$.

EX (i) $R^n = \underbrace{R \times \cdots \times R}_{n \text{ times}} = \{(x^1, \dots, x^n) \mid x^i \in R\} =$ n-tuples of real numbers

(ii) the nm-dimensional space of $n \times m$ real matrices $\begin{pmatrix} n \leftrightarrow \text{rows} \\ m \leftrightarrow \text{columns} \end{pmatrix}$:

R^n can be identified either with the space of $1 \times n$ matrices (row vectors; convenient for inserting in paragraphs) or $n \times 1$ matrices (column vectors; convenient for matrix multiplication)

(iii) $gl(n, R) =$ n^2-dimensional space of $n \times n$ real matrices

BASES AND COMPONENTS

k vectors $V_1, \dots V_k$ are linearly independent if $\sum_{\alpha=1}^{k} a^\alpha V_\alpha = 0 \longrightarrow a^\alpha = 0$. A finite dimensional real vector space V of dimension $n = \dim V$ has at most n linearly independent vectors. An ordered set $\{e_\alpha\}_{\alpha=1,\dots,n} \equiv \{e_1, \dots, e_n\}$ of n linearly independent vectors of V is then called a basis of V. (Write simply $\{e_\alpha\}$.)

Then if $X \in V$, $\{X, e_\alpha\}$ is a linearly dependent set, so there exist real numbers X^α such that $X - X^\alpha e_\alpha = 0$ or $\boxed{X = X^\alpha e_\alpha}$

SUMMATION CONVENTION: When an upper and lower index are repeated without a summation symbol, a summation over all allowed values of the index is implied.

So every vector can be written as a linear combination of the basis vectors $\{e_\alpha\}$. The real numbers X^α are called the components of X with respect to the basis $\{e_\alpha\}$.

In fact the map $\quad X = X^\alpha e_\alpha \to (X^\alpha) = (X^1, \dots, X^n) \in R^n$

is a vector space isomorphism from V to R^n determined by the basis $\{e_\alpha\}$. Each basis determines such an isomorphism.

EX (i) The natural basis of R^n consists of the n vectors

$$e_1 = (1, 0, \dots, 0) \;, \; \dots \;, \; e_n = (0, \dots, 0, 1)$$

(ii) The natural basis of $gl(n, R)$ is $\{e^\beta{}_\alpha\}$, where the matrix $e^\beta{}_\alpha$ has a 1 in the α^{th} row and β^{th} column and zeros elsewhere, so a matrix can be written $\quad A = (A^\alpha{}_\beta) = A^\alpha{}_\beta e^\beta{}_\alpha$

[upper left component index = row, lower right component index = column]

CHANGE OF BASIS

Suppose $\{e_{\alpha'}\}$ is another basis of V. Then one can expand this basis in terms of the original basis:

$$e_{\alpha'} = e_\beta A^\beta{}_{\alpha'}$$

and vice versa:

$$e_\alpha = e_{\beta'} A^{\beta'}{}_\alpha = e_\gamma \underbrace{A^\gamma{}_{\beta'} A^{\beta'}{}_\alpha}_{= \delta^\gamma{}_\alpha \text{ Kronecker delta}}$$

i.e. these are inverse matrices:

$$A = (A^{\beta'}{}_\alpha) \qquad A^{-1} = (A^\beta{}_{\alpha'})$$

; we will also write $A = (A^\alpha{}_\beta)$ and $A^{-1} = (A^{-1\alpha}{}_\beta)$.

Any vector X has components: $\; X = X^\alpha e_\alpha = X^\alpha A^{\beta'}{}_\alpha e_{\beta'} = X^{\beta'} e_{\beta'}$

i.e. $\quad \boxed{X^{\beta'} = A^{\beta'}{}_\alpha X^\alpha} \quad$ or $\quad (X^{\beta'}) = A(X^{\beta'})$

column vectors

so the corresponding points of R^n undergo the linear transformation by the matrix A.

This is called a change of basis, or passive transformation, as opposed to an active linear transformation of the points of V: $\quad X = X^\alpha e_\alpha \longrightarrow BX = B^\alpha{}_\beta X^\beta e_\alpha$,

where $\quad B e_\beta = e_\alpha B^\alpha{}_\beta$.

THE DUAL SPACE V^*

A real valued function $X \to \sigma(X) \in R$ on V which is linear
$\sigma(aX+bY) = a\sigma(X) + b\sigma(Y)$ is called a linear form on V.
Let V^* be the space of linear forms on V.

If $\sigma, \rho \in V^*$ then $(a\sigma + b\rho)(X) = a\sigma(X) + b\rho(X)$ converts V^* into a real vector space called the dual space of V.

DUAL BASIS

In fact $\dim V^* = \dim V$ since a linear form σ is entirely determined by the n numbers $\sigma_\alpha = \sigma(e_\alpha)$ for a basis $\{e_\alpha\}$, called the components of σ with respect to $\{e_\alpha\}$.

Introducing the n linear forms $\{\omega^\alpha\}$ such that $\boxed{\omega^\alpha(e_\beta) = \delta^\alpha{}_\beta}$, one sees that they pick out the components of a vector with respect to $\{e_\alpha\}$:

$$X = X^\alpha e_\alpha \longrightarrow \quad \omega^\alpha(X) = X^\beta \omega^\alpha(e_\beta) = \delta^\alpha{}_\beta X^\beta = X^\alpha.$$

Then $\sigma(X) = \sigma(X^\alpha e_\alpha) = X^\alpha \sigma(e_\alpha) = \sigma_\alpha X^\alpha = \sigma_\alpha \omega^\alpha(X)$ for all $X \in V$,

i.e. $\sigma = \sigma_\alpha \omega^\alpha$,

so every linear form can be expressed in terms of these n (easily seen to be linearly independent) linear forms, which therefore are a basis of V^*, called the basis dual to $\{e_\alpha\}$.

If $e_{\alpha'} = e_\beta A^\beta{}_{\alpha'}$, $e_\alpha = e_{\beta'} A^{\beta'}{}_\alpha$ is a change of basis then

$\omega^{\beta'}(e_\alpha) = \omega^{\beta'}(e_{\gamma'} A^{\gamma'}{}_\alpha) = \delta^{\beta'}{}_{\gamma'} A^{\gamma'}{}_\alpha = A^{\beta'}{}_\alpha$

so $\omega^{\beta'} = A^{\beta'}{}_\alpha \omega^\alpha$ or $\omega^\alpha = A^\alpha{}_{\beta'} \omega^{\beta'}$.

$$\boxed{\begin{aligned} e_{\alpha'} &= e_\beta A^{-1\beta}{}_\alpha \\ \omega^{\alpha'} &= A^\alpha{}_\beta \omega^\beta \end{aligned}}$$

The dual basis transforms by the inverse of the transformation of the basis itself.

The dual basis determines a vector space isomorphism from V^* onto R^n

$\sigma \longrightarrow (\sigma_\alpha) \in R^n$ (row vector).

The evaluation of a linear form on a vector can then be represented as:

$$\sigma(X) = \sigma_\alpha X^\alpha = (\sigma_1, \dots, \sigma_n) \begin{pmatrix} X^1 \\ \vdots \\ X^n \end{pmatrix}.$$

We could go on to consider the space of linear forms on V^*, denoted by $(V^*)^*$, but this is unnecessary since we can identify linear forms on V^* with evaluation of elements of V^* on vectors: $\sigma \to \sigma(X) \in R$ is a linear form on V^*, ie. $(V^*)^* \cong V$. The elements of V^* are often called covectors.

MULTILINEAR FUNCTIONS

Denote the k-fold product $V \times \cdots \times V$ by V^k.

A function $T: V^k \to R$ given by $(v_1,\dots,v_k) \to T(v_1,\dots,v_k) \in R$ is called <u>multilinear</u> if it is linear in each argument

$$T(v_1,\dots, av_i+bu_i,\dots,v_k) = a\,T(v_1,\dots,v_i,\dots,v_k) + b\,T(v_1,\dots,u_i,\dots,v_k).$$

T is called a k-tensor on V or more precisely a <u>$\binom{0}{k}$ tensor over V</u>.

Let $\mathcal{T}^{0,k}(V)$ be the space of k-tensors on V, clearly a real vector space. Its elements are also called <u>covariant tensors</u> over V.

TENSOR PRODUCT

If $S \in \mathcal{T}^{0,k}(V)$ and $T \in \mathcal{T}^{0,\ell}(V)$, define the <u>tensorproduct</u> $S \otimes T \in \mathcal{T}^{0,k+\ell}(V)$ by

$$S\otimes T\,(v_1,\dots,v_k, v_{k+1},\dots,v_{k+\ell}) = S(v_1,\dots,v_k)\,T(v_{k+1},\dots,v_{k+\ell}).$$

This is clearly associative $[\,(S\otimes T)\otimes U = S\otimes(T\otimes U)$ so write simply $S\otimes T\otimes U\,]$, while tensor products of linear combinations result in the same linear combinations of the tensor products.

[This converts the ∞-dimensional space $\overset{\infty}{\underset{k=0}{\oplus}} \mathcal{T}^{0,k}(V) = \mathcal{T}^{0,0}(V) \oplus \mathcal{T}^{0,1}(V) \oplus \cdots$ into an algebra, called the algebra of forms over V, where $\mathcal{T}^{0,0}(V) = R$.]

BASIS FOR $\mathcal{T}^{0,k}(V)$

Given a basis $\{e_\alpha\}$ of V, introduce the set of n^k k-tensors $\{\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_k} \mid 1 \le \alpha_1,\dots,\alpha_k \le n\}$ which satisfy

$$\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_k}(e_{\beta_1},\dots,e_{\beta_k}) = \delta^{\alpha_1}{}_{\beta_1}\cdots\delta^{\alpha_k}{}_{\beta_k}$$

so

$$T(X_1,\dots,X_k) = T(X_1^{\alpha_1}e_{\alpha_1},\dots,X_k^{\alpha_k}e_{\alpha_k}) = \underbrace{X^{\alpha_1}_1\cdots X^{\alpha_k}_k}_{\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_k}(X_1,\dots,X_k)}\ \underbrace{T(e_{\alpha_1},\dots,e_{\alpha_k})}_{\equiv T_{\alpha_1\dots\alpha_k}}$$

$$= T_{\alpha_1\dots\alpha_k}\,\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_k}(X_1,\dots,X_k).$$

Since $(X_1,\dots,X_k)$ are arbitrary:

$$\boxed{T = T_{\alpha_1\dots\alpha_k}\,\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_k} \qquad T_{\alpha_1\dots\alpha_k} = T(e_{\alpha_1},\dots,e_{\alpha_k})}$$

$T_{\alpha_1\dots\alpha_k}$ are called the components of T with respect to the basis $\{e_\alpha\}$.

In fact $\{\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_k}\}$ is a basis for $\mathcal{T}^{0,k}(V)$ since

$$a_{\alpha_1\dots\alpha_k}\,\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_k} = 0 \;\to\; 0 = a_{\alpha_1\dots\alpha_k}\,\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_k}(e_{\beta_1},\dots,e_{\beta_k}) = a_{\alpha_1\dots\alpha_k}$$

implies that the set is linearly independent.

We may write in an obvious notation $\mathcal{T}^{0,k}(V) = \otimes^k V^* \quad (= \underbrace{V^* \otimes \cdots \otimes V^*}_{k\text{-times}})$.

$\mathcal{T}^{k,0}(V) = \otimes^k V = \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}}$

We could repeat everything for covariant tensors over V^* identified with tensor products of vectors which act on V^* by evaluation $\quad X(\sigma) \equiv \sigma(X)$.
These are contravariant tensors over V, or $\binom{k}{0}$ tensors over V.

$\{e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}\}$ is a basis for $\mathcal{T}^{k,0}(V)$ and

$$S = S^{\alpha_1 \ldots \alpha_k} e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k} \quad , \quad S^{\alpha_1 \ldots \alpha_k} = S(\omega^{\alpha_1}, \ldots, \omega^{\alpha_k})$$

MIXED TENSORS

We can introduce the real vector space $\mathcal{T}^{j,k}(V)$ of $\binom{j}{k}$-tensors over V, namely multilinear functions from $(V^*)^j \times V^k$ into R:

$$\underbrace{(\sigma^1, \ldots, \sigma^j}_{j \text{ covector arguments}}, \underbrace{v_1, \ldots, v_k}_{k \text{ vector arguments}}) \longrightarrow T(\sigma^1, \ldots, \sigma^j, v_1, \ldots, v_k) \in R.$$

$\{e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_j} \otimes \omega^{\beta_1} \otimes \cdots \otimes \omega^{\beta_k}\}$ is a basis of this n^{j+k}-dimensional space, where the tensor product may be extended to $\binom{j}{k}$-tensors in a natural way,

for example $\quad X \otimes \sigma\,(\Upsilon, Y) = \Upsilon(X)\,\sigma(Y)$.

Then $\quad T = T^{\alpha_1 \ldots \alpha_j}{}_{\beta_1 \ldots \beta_k}\, e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_j} \otimes \omega^{\beta_1} \otimes \cdots \otimes \omega^{\beta_k}$

$$T^{\alpha_1 \ldots \alpha_j}{}_{\beta_1 \ldots \beta_k} = T(\omega^{\alpha_1}, \ldots, \omega^{\alpha_j}, e_{\alpha_1}, \ldots, e_{\alpha_k})$$

We can write $\quad \mathcal{T}^{j,k}(V) = (\otimes^j V) \otimes (\otimes^k V^*)$.

Its elements are also called tensors of valence (j,k), k times covariant (lower indices of components) and j times contravariant (upper indices of components).

TENSOR TRANSFORMATION LAW

If $\quad e_{\alpha'} = e_\beta A^{-1\beta}{}_\alpha$, $\omega^{\alpha'} = A^\alpha{}_\beta \omega^\beta$ then

$$T^{\alpha_1' \ldots \alpha_j'}{}_{\beta_1' \ldots \beta_k'} = T(\omega^{\alpha_1'}, \ldots, \omega^{\alpha_j'}, e_{\beta_1'}, \ldots, e_{\beta_k'}) = T(A^{\alpha_1}{}_{\gamma_1} \omega^{\gamma_1}, \ldots)$$

$$= A^{\alpha_1}{}_{\gamma_1} \cdots A^{\alpha_j}{}_{\gamma_j}\, A^{-1\delta_1}{}_{\beta_1} \cdots A^{-1\delta_k}{}_{\beta_k}\, T^{\gamma_1 \ldots \gamma_j}{}_{\delta_1 \ldots \delta_k}.$$

$\binom{1}{1}$-tensors and linear transformations

The $\binom{1}{1}$-tensor $B = B^{\alpha}{}_{\beta}\, e_{\alpha} \otimes \omega^{\beta} \in \mathcal{T}^{1,1}(V)$ can naturally be identified with the linear transformation of V into itself:

$$X = X^{\alpha} e_{\alpha} \longrightarrow BX = B^{\alpha}{}_{\beta} X^{\beta} e_{\alpha} \qquad \text{where} \begin{cases} B e_{\alpha} = B^{\beta}{}_{\alpha} e_{\beta} \\ B^{\beta}{}_{\alpha} = \omega^{\beta}(B e_{\alpha}) \end{cases}$$

This may be written $X \to B \llcorner X$, the <u>contraction</u> of $B \in \mathcal{T}^{1,1}(V)$ by X, namely the evaluation of the last vector argument of B on X

$$B \llcorner X = B^{\alpha}{}_{\beta}\, e_{\alpha} \otimes \omega^{\beta}(\ , X) = B^{\alpha}{}_{\beta}\, e_{\alpha} X^{\beta}.$$

A contraction corresponds to summing off a covariant and a contravariant index against each other. We can also introduce the left contraction $X \lrcorner T$ which evaluates the first vector argument of T on X.

Note that each basis determines an isomorphism from the space $gl(V)$ of linear transformations on V onto the space $gl(n,R)$ of $n \times n$ real matrices

$$B \in gl(V) \rightarrow \omega^{\alpha}(B e_{\beta})\, e^{\beta}{}_{\alpha} = (B^{\alpha}{}_{\beta}) \in gl(n,R).$$

Under a change of basis $e_{\alpha'} = e_{\beta} A^{\beta}{}_{\alpha'} \equiv e_{\beta} A^{-1\beta}{}_{\alpha}$,

$$B^{\gamma'}{}_{\delta'} = A^{\gamma'}{}_{\alpha} B^{\alpha}{}_{\beta} A^{\beta}{}_{\delta'} = A^{\gamma}{}_{\alpha} B^{\alpha}{}_{\gamma} A^{-1\gamma}{}_{\delta}$$

Thus for a given basis $\{e_{\alpha}\}$, the corresponding points of R^n undergo matrix multiplication by the matrix $(B^{\alpha}{}_{\beta})$:

$$\underbrace{(X^{\alpha}) \rightarrow (B^{\alpha}{}_{\beta})(X^{\beta})}_{\text{column vectors}}$$

while this matrix undergoes a <u>similarity transformation</u> under a change of basis

$$(B^{\alpha}{}_{\beta}) \longrightarrow A\,(B^{\alpha}{}_{\beta})\,A^{-1}.$$

INNER PRODUCTS

An <u>inner product</u> on V: $\quad X, Y \longrightarrow X \cdot Y \equiv g_{\alpha\beta} X^{\alpha} Y^{\beta}$

$$g_{\alpha\beta} \equiv e_{\alpha} \cdot e_{\beta} = g_{\beta\alpha}$$

is defined by a symmetric $\binom{0}{2}$-tensor g over V with components $g_{\alpha\beta} = g(e_{\alpha}, e_{\beta})$ with respect to the basis $\{e_{\alpha}\}$, i.e.

$$g = g_{\alpha\beta}\, \omega^{\alpha} \otimes \omega^{\beta}, \quad X \cdot Y = g(X, Y).$$

g is called the <u>covariant metric tensor</u>. g is <u>nonsingular</u> if $(g_{\alpha\beta})$ is a nonsingular matrix; i.e. $\quad g \equiv |\det(g_{\alpha\beta})| \neq 0.$

For a nonsingular metric g, one can introduce the contravariant metric tensor whose symmetric matrix of components is the inverse of the matrix of components of g:

$$g^{-1} = g^{\alpha\beta} e_\alpha \otimes e_\beta \quad , \quad g^{\alpha\beta} = g^{-1}(\omega^\alpha, \omega^\beta) \quad , \quad (g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}.$$

The contravariant metric tensor determines an inner product on the dual space

$$\sigma \cdot \rho = g^{-1}(\sigma, \rho) = g^{\alpha\beta} \sigma_\alpha \rho_\beta .$$

RAISING AND LOWERING INDICES

A nonsingular metric tensor g determines an isomorphism between V and V^*:

$$X = X^\alpha e_\alpha \in V \longrightarrow g \llcorner X = g_{\alpha\beta} X^\beta \omega^\alpha \equiv X_\alpha \omega^\alpha \in V^*$$

or in components: $\quad X^\alpha \longrightarrow X_\alpha = g_{\alpha\beta} X^\beta.$

This is called lowering the index α.

The inverse map uses the contravariant metric tensor and is called raising the index α:

$$\sigma = \sigma_\alpha \omega^\alpha \in V^* \longrightarrow \sigma^\alpha e_\alpha \equiv g^{\alpha\beta} \sigma_\beta e_\alpha$$

$$\sigma_\alpha \longrightarrow \sigma^\alpha = g^{\alpha\beta} \sigma_\beta .$$

These maps allow us to raise and lower indices indiscriminantly. Any index on a $\binom{p}{q}$-tensor can be raised and lowered at will, so any tensor can be brought to fully covariant or contravariant form. In component form:

$$T^{\alpha_1 \ldots \alpha_p}{}_{\beta_1 \ldots \beta_p} \longrightarrow \begin{cases} T^{\alpha_1 \ldots \alpha_p \, \beta_1 \ldots \beta_q} = T^{\alpha_1 \ldots \alpha_p}{}_{\gamma_1 \ldots \gamma_q} g^{\gamma_1 \beta_1} \cdots g^{\gamma_q \beta_q} \\ T_{\alpha_1 \ldots \alpha_p \, \beta_1 \ldots \beta_q} = g_{\alpha_1 \gamma_1} \cdots g_{\alpha_p \gamma_q} T^{\gamma_1 \ldots \gamma_p}{}_{\beta_1 \ldots \beta_q} \end{cases}$$

INNER PRODUCT ON $\mathcal{T}^{p,q}(V)$

If $T = T^{\alpha_1 \ldots \alpha_p}{}_{\beta_1 \ldots \beta_q} e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_p} \otimes \omega^{\beta_1} \otimes \cdots \otimes \omega^{\beta_q}$ and $S = \ldots \in \mathcal{T}^{p,q}(V)$ define their inner product

$$\begin{aligned} T \cdot S &= T^{\alpha_1 \ldots \alpha_p}{}_{\beta_1 \ldots \beta_q} S_{\alpha_1 \ldots \alpha_p}{}^{\beta_1 \ldots \beta_q} \\ &= T^{\alpha_1 \ldots \alpha_p}{}_{\beta_1 \ldots \beta_q} g_{\alpha_1 \gamma_1} \cdots g_{\alpha_p \gamma_p} g^{\beta_1 \delta_1} \cdots g^{\beta_q \delta_q} S^{\gamma_1 \ldots \gamma_p}{}_{\delta_1 \ldots \delta_q} . \end{aligned}$$

EVALUATION OF $\binom{p}{q}$-tensors on $\binom{q}{p}$-tensors

If $T \in \mathcal{T}^{(p,q)}(V)$ and $S \in \mathcal{T}^{(q,p)}(V)$ are tensors of opposite valence, a natural pairing of indices occurs which allows contraction of corresponding indices against each other

$$\langle T, S \rangle = T^{\alpha_1 \ldots \alpha_p}{}_{\beta_1 \ldots \beta_q} S^{\beta_1 \ldots \beta_q}{}_{\alpha_1 \ldots \alpha_p}$$

COMPONENT VERSION OF TENSOR PRODUCT AND CONTRACTION

In case it is not clear, if

$$S = S^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \cdots \beta_q}\, e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_p} \otimes \omega^{\beta_1} \otimes \cdots \otimes \omega^{\beta_q} \qquad \text{and}$$

$$T = T^{\alpha_1 \cdots \alpha_r}{}_{\beta_1 \cdots \beta_s}\, e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_r} \otimes \omega^{\beta_1} \otimes \cdots \otimes \omega^{\beta_s}$$

are a $\binom{p}{q}$-tensor and an $\binom{r}{s}$-tensor respectively, then their tensor product is the following $\binom{p+r}{q+s}$-tensor

$$S \otimes T = \underbrace{S^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \cdots \beta_q}\, T^{\alpha_{p+1} \cdots \alpha_{p+r}}{}_{\beta_{q+1} \cdots \beta_{q+s}}}_{(S\otimes T)^{\alpha_1 \cdots \alpha_{p+r}}{}_{\beta_1 \cdots \beta_{q+s}}}\, e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_{p+r}} \otimes \omega^{\beta_1} \otimes \cdots \otimes \omega^{\beta_{q+s}}.$$

Summing over a pair of indices, one contravariant and one covariant, is called contraction of those indices.

For example $\quad T^{\alpha\beta}{}_{\gamma\delta} \rightarrow T^{\alpha\beta}{}_{\alpha\delta} \quad$ or

$$T = T^{\alpha\beta}{}_{\gamma\delta}\, e_\alpha \otimes e_\beta \otimes \omega^\gamma \otimes \omega^\delta \rightarrow T^{\alpha\beta}{}_{\alpha\delta}\, e_\beta \otimes \omega^\delta.$$

Contraction of a covariant tensor by a vector is accomplished by first taking their tensor product and then contracting a pair of indices of the result:

For example

$$g_{\alpha\beta}, X^\gamma \longrightarrow T^\gamma{}_{\alpha\beta} = X^\gamma g_{\alpha\beta} \longrightarrow T^\gamma{}_{\alpha\gamma} = X^\gamma g_{\alpha\gamma} = X_\alpha.$$

Similar statements hold for the contraction of contravariant tensor by a covector, etc. (contractions of mixed tensors...).

CANONICAL FORM , ORTHONORMAL BASES

A basis $\{e_\alpha\}$ can always be chosen so that the matrix of components of a metric tensor assumes the following canonical form

$$(g_{\alpha\beta}) = \mathrm{diag}\,(\underbrace{1,\dots,1}_{p}, \underbrace{-1,\dots,-1}_{q}, \underbrace{0,\dots,0}_{n-(p+q)}) = (e_\alpha\cdot e_\beta)$$

When $p+q=n$, as will be assumed here, then g is nondegenerate and $\{e_\alpha\}$ is called an <u>orthonormal basis</u>.

$s = p-q$ is called the <u>signature</u> of g : $p = \frac{n+s}{2}$, $q = \frac{n-s}{2}$.

The change $g \to -g$ interchanges p and q and changes the sign of the signature. If $s=n$, the metric is <u>positive definite</u> and V is called a Euclidean space. If $s=-n$, it is convenient to change the negative definite metric into a positive definite one by changing its sign. If g is not definite, it is a matter of choice which sign is taken for the signature, i.e. only $|s|$ is of physical significance.

<u>EX</u>. Relable the natural basis of R^4 as $\{e_\alpha\} = \{e_0, e_1, e_2, e_3\}$

The Lorentz metric may have either signature:

$s=2\,(-+++)$: $(g_{\alpha\beta}) = \mathrm{diag}(-1,1,1,1) \equiv (\eta_{\alpha\beta})$

$s=-2\,(+---)$: $(g_{\alpha\beta}) = \mathrm{diag}(1,-1,-1,-1)$.

<u>EX</u>. R^n with the inner product which makes the natural basis orthogonal (and which is positive definite) is called Euclidean space E^n:

$$\delta = \delta_{\alpha\beta}\,\omega^\alpha \otimes \omega^\beta \quad , \quad e_\alpha\cdot e_\beta = \delta_{\alpha\beta}\,.$$

<u>EX</u>. Introduce the spaces $M^{p,q}$ with $p+q=n$ as R^n with the inner product whose components in the natural basis are

$$({}^{(p,q)}\eta_{\alpha\beta}) = \mathrm{diag}\,(\underbrace{1,\dots,1}_{p}, \underbrace{-1,\dots,-1}_{q})$$

$M^{n,0}$ and $M^{0,n}$ are equivalent to E^n.

$M^{n-1,0}$ and $M^{0,n-1}$ are n-dimensional Minkowski spaces M^n.

Note that $\det({}^{(p,q)}\eta_{\alpha\beta}) = (-1)^q = (-1)^{\frac{n-s}{2}}$.

Thus for a general metric of the same signature:

$$\det(g_{\alpha\beta}) = (-1)^{\frac{n-s}{2}}\, g \qquad g \equiv |\det(g_{\alpha\beta})|\,.$$

PSEUDO-ORTHOGONAL GROUPS

Suppose V has an inner product of signature s. Its components in an orthonormal basis are:
$$e_\alpha \cdot e_\beta = {}^{(p,q)}\eta_{\alpha\beta} \qquad \begin{pmatrix} s = p-q \\ n = p+q \end{pmatrix}$$

The pseudo-orthogonal group $O(p,q) \subset GL(n,\mathbb{R})$ is the matrix group ($GL(n,\mathbb{R})$ = group of nonsingular real $n \times n$ matrices) which maps orthonormal bases onto orthonormal bases:
$$e_\alpha \rightarrow e_{\alpha'} = e_\beta B^\beta{}_\alpha$$
$${}^{(p,q)}\eta_{\alpha\beta} = e_{\alpha'} \cdot e_{\beta'} = B^\gamma_\alpha \, {}^{(p,q)}\eta_{\gamma\delta} \, B^\delta{}_\beta$$

or in matrix form: $\quad \boldsymbol{\eta} = B^T \boldsymbol{\eta} B \quad$ $\left(\begin{array}{l} \rightarrow \det \eta = \det \eta (\det B)^2 \\ \text{or } \det B = \pm 1 \end{array}\right)$

where "T" indicates the transpose.

The subgroup of unimodular matrices ($\det B = 1$) is called the special pseudo-orthogonal group $SO(p,q)$.

For $q=0$ we write $O(n,0) = O(n,\mathbb{R})$ and $SO(n,0) = SO(n,\mathbb{R})$, the orthogonal groups in n-dimensions.

SYMMETRIZATION AND ANTISYMMETRIZATION

Suppose $T = T_{\alpha_1 \cdots \alpha_p}\, \omega^{\alpha_1} \otimes \cdots \otimes \omega^{\alpha_p} \in \mathcal{T}^{0,p}(V)$.

Introduce new tensors $ALT(T)$ and $SYM(T) \in \mathcal{T}^{0,p}(V)$ by

$$SYM(T)_{\alpha_1 \cdots \alpha_p} = \frac{1}{p!} \sum_{\pi} T_{\alpha_{\pi_1} \cdots \alpha_{\pi_p}} \equiv T_{(\alpha_1 \cdots \alpha_p)}$$

$$ALT(T)_{\alpha_1 \cdots \alpha_p} = \frac{1}{p!} \sum_{\pi} (-1)^{\pi}\, T_{\alpha_{\pi_1} \cdots \alpha_{\pi_p}} \equiv T_{[\alpha_1 \cdots \alpha_p]}$$

where $(\pi_1, \ldots, \pi_p) = \pi(1, \ldots, p)$ is a permutation of $(1, \ldots, p)$ and $(-1)^{\pi} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$ if π is an $\begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$ permutation.

$SYM(T)$ and $ALT(T)$ are the totally symmetric and totally antisymmetric parts of T. ALT stands for "alternating".

The same discussion holds for $\mathcal{T}^{p,0}(V)$.

Let
$$\Lambda^p(V) = ALT(\mathcal{T}^{p,0}(V)) = ALT(\otimes^p V)$$
$$\Lambda^p(V^*) = ALT(\mathcal{T}^{0,p}(V)) = ALT(\otimes^p V^*)$$
be the linear subspaces of totally antisymmetric $\binom{p}{0}$-tensors and $\binom{0}{p}$-tensors.

GENERALIZED KRONECKER DELTAS, LEVI-CIVITA EPSILON

Define
$$\delta^{\beta_1 \cdots \beta_p}_{\alpha_1 \cdots \alpha_p} = p!\, \delta^{\beta_1}_{[\alpha_1} \cdots \delta^{\beta_p}_{\alpha_p]} = p!\, \delta^{[\beta_1}_{\alpha_1} \cdots \delta^{\beta_p]}_{\alpha_p} = \begin{cases} \begin{pmatrix} 1 \\ -1 \end{pmatrix} & \text{if } (\beta_1, \ldots, \beta_p) \text{ is an } \begin{pmatrix} \text{even} \\ \text{odd} \end{pmatrix} \text{ permutation of } (\alpha_1, \ldots, \alpha_p) \text{ and they are all distinct integers} \\ 0 & \text{in all other cases} \end{cases}$$

$$\epsilon^{\beta_1 \cdots \beta_n} = \delta^{\beta_1 \cdots \beta_n}_{1 \cdots n} = \epsilon^{[\beta_1 \cdots \beta_n]} \qquad \epsilon^{1 \cdots n} = 1$$

$$\epsilon_{\alpha_1 \cdots \alpha_n} = \delta^{1 \cdots n}_{\alpha_1 \cdots \alpha_n} = \epsilon_{[\alpha_1 \cdots \alpha_n]} \qquad \epsilon_{1 \cdots n} = 1$$

Some useful formulas are then

$$\delta^{\beta_1 \cdots \beta_n}_{\alpha_1 \cdots \alpha_n} = \epsilon^{\beta_1 \cdots \beta_n}\, \epsilon_{\alpha_1 \cdots \alpha_n}$$

$$\delta^{\beta_1 \cdots \beta_p \gamma_{p+1} \cdots \gamma_q}_{\alpha_1 \cdots \alpha_p \gamma_{p+1} \cdots \gamma_q} = \frac{(n-p)!}{(n-q)!}\, \delta^{\beta_1 \cdots \beta_p}_{\alpha_1 \cdots \alpha_p}$$

$$\epsilon^{\beta_1 \cdots \beta_p \gamma_1 \cdots \gamma_q}\, \epsilon_{\alpha_1 \cdots \alpha_p \gamma_1 \cdots \gamma_q} = (n-p)!\, \delta^{\beta_1 \cdots \beta_p}_{\alpha_1 \cdots \alpha_p}$$

These symbols may be used to antisymmetrize indices

$$T_{[\alpha_1 \cdots \alpha_p]} = \frac{1}{p!}\, \delta^{\beta_1 \cdots \beta_p}_{\alpha_1 \cdots \alpha_p}\, T_{\beta_1 \cdots \beta_p}$$

and to express determinants

$$\det(A^\alpha{}_\beta) = \epsilon^{\alpha_1\cdots\alpha_n} A^1{}_{\alpha_1}\cdots A^n{}_{\alpha_n} = \epsilon_{\beta_1\cdots\beta_n} A^{\beta_1}{}_1\cdots A^{\beta_n}{}_n$$

$$= \frac{1}{n!}\,\delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n} A^{\beta_1}{}_{\alpha_1}\cdots A^{\beta_n}{}_{\alpha_n}$$

$$\epsilon^{\alpha_1\cdots\alpha_n}\det(A^\alpha{}_\beta) = \delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n} A^{\beta_1}{}_1\cdots A^{\beta_n}{}_n$$

$$\det(g_{\alpha\beta}) = \epsilon^{\alpha_1\cdots\alpha_n} g_{\alpha_1 1}\cdots g_{\alpha_n n} = \frac{1}{n!}\,\epsilon^{\alpha_1\cdots\alpha_n} g_{\alpha_1\beta_1}\cdots g_{\alpha_n\beta_n}\epsilon^{\beta_1\cdots\beta_n}$$

Note: raising and lowering conventions do not apply to $\epsilon^{\alpha_1\cdots\alpha_n}$ and $\epsilon_{\alpha_1\cdots\alpha_n}$.

BASES FOR THE SPACES OF ANTISYMMETRIC TENSORS

Antisymmetric tensors turn out to be very important so it is useful to have a basis of the spaces of totally antisymmetric covariant contravariant tensors.

Define $e_{\alpha_1\cdots\alpha_p} = p!\, e_{[\alpha_1}\otimes\cdots\otimes e_{\alpha_p]}$

$\{e_{\alpha_1\cdots\alpha_p}\}_{\alpha_1<\cdots<\alpha_p}$ is a basis of $\Lambda^p(V)$

Elements of $\Lambda^p(V)$ are called <u>p-vectors</u> or in general <u>multivectors</u>.

$\omega^{\alpha_1\cdots\alpha_p} = p!\,\omega^{[\alpha_1}\otimes\cdots\otimes\omega^{\alpha_p]}$

$\{\omega^{\alpha_1\cdots\alpha_p}\}_{\alpha_1<\cdots<\alpha_p}$ is a basis of $\Lambda^p(V^*)$

Elements of $\Lambda^p(V)$ are called <u>p-forms</u> or in general <u>multiforms</u>.

If $T\in\Lambda^p(V)$ then

$$T = T^{\alpha_1\cdots\alpha_p} e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_p} = T^{[\alpha_1\cdots\alpha_p]} e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_p} = T^{\alpha_1\cdots\alpha_p} e_{[\alpha_1}\otimes\cdots\otimes e_{\alpha_p]}$$

$$= \frac{1}{p!}\,T^{\alpha_1\cdots\alpha_p} e_{\alpha_1\cdots\alpha_p} = T^{\alpha_1\cdots\alpha_p} e_{|\alpha_1\cdots\alpha_p|}$$

indicates the sum over $\alpha_1<\cdots<\alpha_p$ only

Similarly if $S\in\Lambda^p(V^*)$

$$S = S_{\alpha_1\cdots\alpha_p}\omega^{\alpha_1}\otimes\cdots\otimes\omega^{p} = \cdots = \frac{1}{p!} S_{\alpha_1\cdots\alpha_p}\omega^{\alpha_1\cdots\alpha_p} = S_{\alpha_1\cdots\alpha_p}\omega^{|\alpha_1\cdots\alpha_p|}$$

The final expressions for T and S indicate why $e_{\alpha_1\cdots\alpha_p}$ and $\omega^{\alpha_1\cdots\alpha_p}$ have been defined with the factorial factor.

Note that $\frac{1}{p!}\,\delta^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_p}$ is the identity operator on $\Lambda^p(V)$ and $\Lambda^p(V^*)$.

For example if $T\in\Lambda^p(V)$: $\quad T^{\alpha_1\cdots\alpha_p} = \frac{1}{p!}\,\delta^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_p} T^{\beta_1\cdots\beta_p}$.

EXTERIOR ALGEBRA

Consider the two (vector) spaces of linear combinations of multivectors or multiforms for any value $0 \leq p \leq n$ of the "rank" p [identify 0-vectors and 0-forms with real numbers]. These spaces can be designated by

$$\Lambda(V) = \bigoplus_{p=0}^{n} \Lambda^p(V) = R \oplus V \oplus \Lambda^2(V) \oplus \cdots \oplus \Lambda^n(V)$$

$$\Lambda(V^*) = \bigoplus_{p=0}^{n} \Lambda^p(V^*) = R \oplus V^* \oplus \Lambda^2(V^*) \oplus \cdots \oplus \Lambda^n(V^*).$$

Since $\Lambda^p(V)$ and $\Lambda^p(V^*)$ have dimension $\binom{n}{p}$, so these spaces have dimension $\sum_{p=0}^{n} \binom{n}{p} = (1+1)^n = 2^n$.

An algebra is a vector space with a product operation defined for pairs of vectors satisfying the distributive law and such that multiplying the product by a real number is equivalent to multiplying either of the factors by this number.

A product operator can be introduced on the space of multiforms or multivectors by simply antisymmetrizing the tensor product

If S is a p-form, T a q-form, define the $(p+q)$-form

$$S \wedge T = \frac{(p+q)!}{p!\,q!} \, ALT(S \otimes T)$$

$$\begin{aligned}(S \wedge T)_{\alpha_1 \cdots \alpha_p} &= \frac{(p+q)!}{p!\,q!} S_{[\alpha_1 \cdots \alpha_p} T_{\alpha_{p+1} \cdots \alpha_{p+q}]} \\ &= \frac{1}{p!\,q!} \delta^{\beta_1 \cdots \beta_{p+q}}_{\alpha_1 \cdots \alpha_{p+q}} S_{\beta_1 \cdots \beta_p} T_{\beta_{p+1} \cdots \beta_{p+q}} \\ &= \delta^{\beta_1 \cdots \beta_{p+q}}_{\alpha_1 \cdots \alpha_{p+q}} S_{|\beta_1 \cdots \beta_p|} T_{|\beta_{p+1} \cdots \beta_{p+q}|}\end{aligned}$$

The final line shows why the factorial factor is included in the definition [Some people omit this factor but then it pops up elsewhere.] The product $\wedge$ is called the exterior or wedge product. This product is associative, namely if U is an r-form:

$$(S \wedge T) \wedge U = S \wedge (T \wedge U) = \binom{\text{parentheses}}{\text{unnecessary}} \; S \wedge T \wedge U$$

This is true since if one antisymmetrizes a subset of indices first and then the whole set, the same result is obtained; we only have to check the factorial factor:

$$((S\wedge T)\wedge U)_{\alpha_1\cdots\alpha_{p+q+r}} = \frac{1}{(p+q)!\,r!}\,\delta^{\beta_1\cdots\beta_{p+q+r}}_{\alpha_1\cdots\alpha_{p+q+r}}\left(\frac{(p+q)!}{p!\,q!}S_{\beta_1\cdots\beta_p}T_{\beta_{p+1}\cdots\beta_{p+q}}\right)U_{\beta_{p+q+1}\cdots\beta_{p+q+r}}$$

$$= \frac{1}{p!\,q!\,r!}\,\delta^{\beta_1\cdots\beta_{p+q+r}}_{\alpha_1\cdots\alpha_{p+q+r}}\,S_{\beta_1\cdots\beta_p}T_{\beta_{p+1}\cdots\beta_{p+q}}U_{\beta_{p+q+1}\cdots\beta_{p+q+r}}$$

$$= \frac{(p+q+r)!}{p!\,q!\,r!}\,S_{[\beta_1\cdots\beta_p}T_{\beta_{p+1}\cdots\beta_{p+q}}U_{\beta_{p+q+1}\cdots\beta_{p+q+r}]}$$

$$\text{or}\qquad S\wedge T\wedge U = \frac{(p+q+r)!}{p!\,q!\,r!}\,\mathrm{ALT}(S\otimes T\otimes U)\ .$$

(Clearly the same result would be obtained for $S\wedge(T\wedge U)$.) From this formula it is obvious what the coefficient would be for an arbitrary number of factors.

For the exterior product of p 1-forms:

$$\sigma^1\wedge\cdots\wedge\sigma^p = p!\,\mathrm{ALT}(\sigma^1\otimes\cdots\otimes\sigma^p) = p!\,\sigma^{[1}\otimes\cdots\otimes\sigma^{p]}.$$

In particular $\quad \omega^{\alpha_1}\wedge\cdots\wedge\omega^{\alpha_p} = p!\,\omega^{[\alpha_1}\otimes\cdots\otimes\omega^{\alpha_p]} = \omega^{\alpha_1\cdots\alpha_p}$.

The entire discussion may be repeated for multivectors, yielding

$$e_{\alpha_1}\wedge\cdots\wedge e_{\alpha_p} = p!\,e_{[\alpha_1}\otimes\cdots\otimes e_{\alpha_p]} = e_{\alpha_1\cdots\alpha_p}\ .$$

The exterior product converts the vector space of all multivectors or multiforms into an algebra called the <u>exterior algebra</u> (of V and V^* respectively).

Since it takes pq interchanges of adjacent pairs of indices to move p ordered indices past q ordered indices, each interchange changing the sign, one has the result

$$S\wedge T = (-1)^{pq}\,T\wedge S \qquad p \text{ and } q \text{ the ranks of } S \text{ and } T$$

Note the relations

$$e_{\alpha_1\cdots\alpha_p}\wedge e_{\alpha_{p+1}\cdots\alpha_{p+q}} = e_{\alpha_1\cdots\alpha_{p+q}}\,.$$
$$\omega^{\alpha_1\cdots\alpha_p}\wedge\omega^{\alpha_{p+1}\cdots\alpha_{p+q}} = \omega^{\alpha_1\cdots\alpha_{p+q}}\ ,$$

and

$$e_{\alpha_1\cdots\alpha_n} = \epsilon_{\alpha_1\cdots\alpha_n}\,e_{1\cdots n}$$
$$\omega^{\alpha_1\cdots\alpha_n} = \epsilon^{\alpha_1\cdots\alpha_n}\,\omega^{1\cdots n}\ .$$

RENORMALIZED INNER PRODUCT

A nondegenerate metric tensor g on V induces an inner product on each space $\mathcal{T}^{j,k}(V)$ denoted above by "$\cdot$". For multivectors and multiforms, where $\{e_{\alpha_1\cdots\alpha_p}\}_{\alpha_1<\cdots<\alpha_p}$ and $\{\omega^{\alpha_1\cdots\alpha_p}\}_{\alpha_1<\cdots<\alpha_p}$ are bases, it is convenient to renormalize this inner product by a factorial factor which always appears

$$S\cdot T = S^{\alpha_1\cdots\alpha_p}T_{\alpha_1\cdots\alpha_p} = p!\, S^{\alpha_1\cdots\alpha_p}T_{|\alpha_1\cdots\alpha_p|}$$

Let $\langle S,T\rangle$ be the new inner product defined without the $p!$ factor

$$\langle S,T\rangle = \frac{1}{p!} S^{\alpha_1\cdots\alpha_p}T_{\alpha_1\cdots\alpha_p} = S^{\alpha_1\cdots\alpha_p}T_{|\alpha_1\cdots\alpha_p|} .$$

The inner products of the basis vectors are, for example,

$$\langle e_{\alpha_1\cdots\alpha_p}, e_{\beta_1\cdots\beta_p}\rangle = \frac{1}{p!}\,\delta^{\gamma_1\cdots\gamma_p}_{\alpha_1\cdots\alpha_p}\, g_{\gamma_1\delta_1}\cdots g_{\gamma_p\delta_p}\,\delta^{\delta_1\cdots\delta_p}_{\beta_1\cdots\beta_p} .$$

When $\{e_\alpha\}$ is an orthonormal basis, then $\{e_{|\alpha_1\cdots\alpha_p|}\}$ and $\{\omega^{|\alpha_1\cdots\alpha_p|}\}$ are orthonormal with respect to $\langle\, ,\rangle$.

Consider the bases $\{e_{1\cdots n}\}$ and $\{\omega^{1\cdots n}\}$ of the 1-dimensional spaces $\Lambda^n(V)$ and $\Lambda^n(V^*)$. Then we obtain

$$\langle e_{1\cdots n}, e_{1\cdots n}\rangle = \det(g_{\alpha\beta}) = (-1)^{\frac{n-s}{2}}\, g \qquad\qquad g\equiv|\det(g_{\alpha\beta})|$$
$$\langle \omega^{1\cdots n}, \omega^{1,\cdots,n}\rangle = \det(g^{\alpha\beta}) = (-1)^{\frac{n-s}{2}}\, g^{-1} .$$

If we define

$$\eta = g^{1/2}\omega^{1\cdots n} = \frac{1}{n!}\eta_{\alpha_1\cdots\alpha_n}\omega^{\alpha_1\cdots\alpha_n} , \qquad \eta_{\alpha_1\cdots\alpha_n} = g^{1/2}\epsilon_{\alpha_1\cdots\alpha_n}$$

and then raise its indices:

$$\eta^{\alpha_1\cdots\alpha_n} = g^{\alpha_1\beta_1}\cdots g^{\alpha_n\beta_n}\eta_{\beta_1\cdots\beta_n} = g^{1/2}\epsilon^{\alpha_1\cdots\alpha_n}\det(g^{\alpha\beta}) = (-1)^{\frac{n-s}{2}} g^{-1/2}\epsilon^{\alpha_1\cdots\alpha_n}$$
$$\eta^{\#} = \frac{1}{n!}\eta^{\alpha_1\cdots\alpha_n} e_{\alpha_1\cdots\alpha_n} = (-1)^{\frac{n-s}{2}} g^{-1/2} e_{1\cdots n} .$$

The indices of η may be raised and lowered at will.

η and $\eta^{\#}$ are orthonormal bases of $\Lambda^n(V)$ and $\Lambda^n(V^*)$ which reduce to $\omega^{1\cdots n}$ and $(-1)^{\frac{n-s}{2}} e_{1\cdots n}$ when $\{e_\alpha\}$ is orthonormal:

$$\langle\eta,\eta\rangle = (-1)^{\frac{n-s}{2}} = \langle\eta^{\#},\eta^{\#}\rangle .$$

The only other orthonormal elements of these spaces are $-\eta$ and $-\eta^{\#}$.

ORIENTATION

Suppose $e_\alpha = e_{\beta'} A^{\beta'}{}_\alpha$ is a change of basis, then

$$g_{\alpha\beta} = A^{\gamma'}_\alpha A^{\delta'}_\beta \, g_{\gamma'\delta'} \quad , \quad g = |\det A^{\alpha'}{}_\beta| \, g'$$

$$\eta = g^{1/2} \omega^{1\ldots n} = |\det(A^{\alpha'}{}_\beta)| \, g'^{1/2} \underbrace{A^1{}_{\alpha_1'} \cdots A^n{}_{\alpha_n'} \underbrace{\omega^{\alpha_1'\ldots\alpha_n'}}_{\epsilon^{\alpha_1\ldots\alpha_n}\,\omega^{1'\ldots n'}}}_{\det(A^\alpha{}_{\beta'})}$$

$$= \frac{|\det(A^{\alpha'}{}_\beta)|}{\det(A^{\alpha'}_\beta)} \, g'^{1/2} \omega^{1'\ldots n'} = \operatorname{sgn}(\det A^{\alpha'}{}_\beta) \, (g'^{1/2} \omega^{1'\ldots n'}).$$

In other words if we define η by the formula $g^{1/2}\omega^{1\ldots n}$ in a particular basis, its formula will have the opposite sign in any other basis obtained from the first by a transformation with negative determinant. In order for η to be well defined, one must pick one of the two classes of bases for V (each class consisting of bases related by transformations of positive determinant) and define η in that class by the above formula.

This choice is called an <u>orientation</u> for V. The choice can be made by specifying a basis $\{e_\alpha\}$ of V which will be called positively oriented; any other positively oriented basis $\{e_{\alpha'}\}$ will satisfy

$$e_{1'\ldots n'} = k \, e_{1\ldots n} \quad , \quad k > 0 ,$$

while negatively oriented bases will have $k < 0$. When V has a nondegenerate metric, one need only specify one of the two orthonormal elements of $\Lambda^n(V)$ (or $\Lambda^n(V^*)$).

THE STAR OPERATOR

Introduce the following special (n-p)-forms

$$\eta_{\alpha_1\cdots\alpha_p} = \frac{1}{(n-p)!}\,\eta_{\alpha_1\cdots\alpha_p\alpha_{p+1}\cdots\alpha_n}\,\omega^{\alpha_{p+1}\cdots\alpha_n} = \eta_{\alpha_1\cdots\alpha_n}\,\omega^{|\alpha_{p+1}\cdots\alpha_n|}.$$

Raise and lower the indices at will.

Define a linear star operation on the bases of the spaces $\Lambda^p(V^*)$ by

$$\begin{cases} *\,\omega^{\alpha_1\cdots\alpha_p} = \eta^{\alpha_1\cdots\alpha_p} \\ *\,1 = \eta \qquad \text{(the case } p=0) \end{cases}$$

so for a p-form S, one obtains an (n-p)-form

$$*S = \frac{1}{p!}S_{\alpha_1\cdots\alpha_p}\,{*\omega^{\alpha_1\cdots\alpha_p}} = \frac{1}{p!}S_{\alpha_1\cdots\alpha_p}\,\eta^{\alpha_1\cdots\alpha_p} = S_{\alpha_1\cdots\alpha_p}\,\eta^{|\alpha_1\cdots\alpha_p|}.$$

$*S$ is called the dual of S.

To evaluate the operation $**$ one can apply it to the basis vectors:

$$**\,\omega^{\alpha_1\cdots\alpha_p} = *\eta^{\alpha_1\cdots\alpha_p} = \frac{1}{(n-p)!}\,\eta^{\alpha_1\cdots\alpha_p}{}_{\alpha_{p+1}\cdots\alpha_n}\,{*\omega^{\alpha_{p+1}\cdots\alpha_n}}$$

$$= \frac{1}{(n-p)!}\,\eta^{\alpha_1\cdots\alpha_p\alpha_{p+1}\cdots\alpha_n}\,\underbrace{\eta_{\alpha_{p+1}\cdots\alpha_n}}_{\frac{1}{p!}\underbrace{\eta_{\alpha_{p+1}\cdots\alpha_n\beta_1\cdots\beta_p}}_{(-1)^{p(n-p)}\eta_{\beta_1\cdots\beta_p\alpha_{p+1}\cdots\alpha_n}}\omega^{\beta_1\cdots\beta_p}}$$

$$= \frac{(-1)^{p(n-p)}}{(n-p)!\,p!}(-1)^{\frac{n-s}{2}}\underbrace{\underbrace{\epsilon^{\alpha_1\cdots\alpha_p\alpha_{p+1}\cdots\alpha_n}\,\epsilon_{\beta_1\cdots\beta_p\alpha_{p+1}\cdots\alpha_n}}_{(n-p)!\,\delta^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_p}}\,\omega^{\beta_1\cdots\beta_p}}_{p!\,\omega^{\alpha_1\cdots\alpha_p}} = (-1)^{p(n-p)+\frac{n-s}{2}}\,\omega^{\alpha_1\cdots\alpha_p}$$

So $\boxed{** = (-1)^{p(n-p)}(-1)^{\frac{n-s}{2}}}$ on $\Lambda^p(V^*)$.

Consider the following exterior product:

$$\omega^{\alpha_1\cdots\alpha_p}\wedge\eta_{\beta_1\cdots\beta_p} = \frac{1}{(n-p)!}\underbrace{\underbrace{\eta_{\beta_1\cdots\beta_p\beta_{p+1}\cdots\beta_n}}_{g^{1/2}\epsilon_{\beta_1\cdots\beta_p\beta_{p+1}\cdots\beta_n}}\underbrace{\omega^{\alpha_1\cdots\alpha_p\beta_{p+1}\cdots\beta_n}}_{\epsilon^{\alpha_1\cdots\alpha_p\beta_{p+1}\cdots\beta_n}\,\omega^{1\cdots n}}}_{(n-p)!\,\delta^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_p}} = \delta^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_p}\,\eta$$

This is useful for the following calculation.

Suppose S and T are p-forms, then $S\wedge {}^*T$ is an n-form:

$$S\wedge{}^*T = \frac{1}{p!} S_{\alpha_1\ldots\alpha_p}\omega^{\alpha_1\ldots\alpha_p}\wedge \frac{1}{p!} T^{\beta_1\ldots\beta_p}\eta_{\beta_1\ldots\beta_p}$$

$$= \frac{1}{(p!)^2} S_{\alpha_1\ldots\alpha_p} T^{\beta_1\ldots\beta_p}\delta^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_p}\eta = \frac{1}{p!} S_{\alpha_1\ldots\alpha_p}T^{\alpha_1\ldots\alpha_p}\eta = S_{\alpha_1\ldots\alpha_p}T^{|\alpha_1\ldots\alpha_p|}\eta$$

$$\boxed{S\wedge{}^*T = \langle S,T\rangle\,\eta}$$

In fact the dual can be defined by this formula, as is done in certain presentations. The star operator is defined for multivectors in exactly the same way, beginning with $\eta^\#$ instead of η.

VECTOR SUBSPACES

A linear subspace or vector subspace of V is a substance which contains every linear combination of vectors belonging to it, so that it is a vector space in its own right.

Every vector $X\in V$ determines a 1-dimensional subspace:

$$\operatorname{span}\{X\} = \{aX \mid a\in R\}.$$

Every covector σ determines an $(n-1)$-dimensional subspace of V, namely all vectors which are mapped to zero by σ:

$$\ker\sigma = \{X\in V \mid \sigma(X)=0\}.$$

In the same way certain p-vectors and p-forms, respectively, determine p-dimensional and $(n-p)$-dimensional subspaces of V.

Suppose $W\subset V$ is a p-dimensional vector subspace of V with basis $\{e_i\}_{i=1,\ldots,p}$. One can always complete this basis to a basis $\{e_\alpha\}=\{e_i,e_a\}_{\substack{i=1\ldots p\\ a=p+1\ldots n}}$ of V with dual basis $\{\omega^\alpha\}=\{\omega^i,\omega^a\}$. This basis is said to be adapted to the subspace W.

Since $\omega^a(e_i)=0$, the p-form $\omega^{p+1}\wedge\ldots\wedge\omega^n$ vanishes if any argument (it doesn't matter which one) is evaluated on an element of $W=\operatorname{span}\{e_i\}$

$$W = \ker(\omega^{p+1}\wedge\ldots\wedge\omega^n) = \{X\in V \mid (\omega^{p+1}\wedge\ldots\wedge\omega^n)\llcorner X = 0\}$$

evaluation of last argument on X

On the other hand the p-vector $e_1\wedge\ldots\wedge e_p$ spans a 1-dimensional subspace of $\Lambda^p(V)$ which results from any set of linearly independent vectors which span W.

For example, if $e_{i'} = A^j{}_{i'} e_j$ is another basis of $\bar{W}$, completed to an adapted basis $\{e_{\alpha'}\}$ of V, then

$$e_{1'} \wedge \dots \wedge e_{p'} = A^{j_1}{}_{1'} \cdots A^{j_p}{}_{p'}\, e_{j_1} \wedge \dots \wedge e_{j_p} = \underbrace{A^{j_1}{}_{1'} \cdots A^{j_p}{}_{p'}\, \delta^{1\dots p}_{j_1 \dots j_p}}_{\det(A^j{}_{i'})} e_1 \wedge \dots \wedge e_p ,$$

while $\{\omega^{p+1'}, \dots, \omega^{n'}\}$ must still annihilate $\bar{W} = \mathrm{span}\{e_{i'}\}$, so $\omega^{a'} = B^{a'}{}_b \omega^b$ and

$$\omega^{p+1'} \wedge \dots \wedge \omega^{n'} = \dots = \det(B^{a'}{}_b)\ \omega^{p+1} \wedge \dots \wedge \omega^n .$$

Thus each p-dimensional subspace $\bar{W}$ of V determines a 1-dimensional subspace of $\Lambda^p(V)$ and a 1-dimensional subspace of $\Lambda^{n-p}(V^*)$ and vice versa. In an adapted basis of V, these spaces are spanned by $e_{1\dots p}$ and $\omega^{p+1\dots n}$ respectively.

When V has a metric we can use the $*$ operator and index manipulation to relate these two antisymmetric tensors

$$*\omega^{p+1\dots n} = \eta^{p+1\dots n} = \eta^{p+1\dots n}{}_{\alpha_1 \dots \alpha_p}\, \omega^{|\alpha_1 \dots \alpha_p|}$$

The contravariant form of this tensor obtained by raising all its indices is

$$(*\omega^{p+1\dots n})^\# = \eta^{p+1\dots n\, \alpha_1 \dots \alpha_p} e_{|\alpha_1 \dots \alpha_p|} = \eta^{p+1 \dots n\, 1 \dots p} e_{1\dots p}$$

$$= (-1)^{p(n-p)} \eta^{1\dots n} e_{1\dots p}$$

$$= (-1)^{p(n-p) + \frac{n-s}{2}} g^{-1/2} \quad e_{1\dots p} .$$

Thus these two tensors are related to each other up to an unimportant constant by first taking the dual of one and then changing the position (up or down) of all its indices.