dr bob's elementary differential geometry

a slightly different approach
based on elementary undergraduate linear algebra,
multivariable calculus and differential equations

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http://www.homepage.villanova.edu/robert.jantzen/notes/diffgeom/

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Abstract

There are lots of books on differential geometry, including at the introductory level. Why yet another one by an author who doesn't seem to take himself that seriously and occasionally refers to himself in the third person? This one is a bit different than all the rest. dr bob loves this stuff, but how to teach it to students at his own (not elite) university in order to have a little more fun at work than usual? This unique approach may not work for everyone, but it attempts to explain the nuts and bolts of how a few basically simple ideas taken seriously underlie the whole mess of formulas and concepts, without worrying about technicalities like"manifolds," "coordinate coverings" and "differentiability," which only serve to put off students at the first pass through this scenery. It is also presented with an eye towards being able to understand the key concepts needed for the mathematical side of modern physical theories, while still providing the tools that underlie the classical theory of surfaces in space. Examples of curves and surfaces in 2 and 3-dimensional spacetimes have been incorporated as examples, with an Appendix presenting a review of the elementary special relativity (hyperbolic geometry, directly analogous to trigonometry) needed to make sense of them. The continuing theme of symmetry groups and their implications for geometry are have also now been woven into the narrative, which is somewhat uncommon for expositions of differential geometry, but essential to a proper understanding of the implications of the subject for applications to physical theories.

Acknowledgments

The patience of my wife and life partner Ani must be acknowledged in dealing with the obsessional tinkering of the author with the ideas and problems of this text and the supporting Maple worksheets. Fritz Hartmann has to remembered for allowing me to let me make my first overly ambitious attempt at a course on this topic when I had just arrived at Villanova University in my first year ever of teaching in George Orwell's year 1984. Some time later in 1991 I was able to give it the sophomore try with 6 trusting students who probably left the course wondering what they had just done, but the opportunity allowed me to write up the first version of the handwritten notes from which this book eventually sprang to life. Without Hans Kuo, there never would have been a book, since starting to LaTeX such a project from scratch would never have occurred to me, and he gifted me the first draft taken from my scanned handwritten notes I had posted on the internet. These were then seriously developed with the addition of problems and technology graphics and more text in an offering of the course in 2008, after which fours years passed before I had the time to return to the project. Cole Johnston must be credited with pushing me to offer the course again in 2013, which coincided with an awakening of my own interest in surfaces motivated by the opportunity to give a popular talk on differential geometry and relativity in which the surface geometry of corkscrew pasta played a starring role in conveying not only the visual ideas of metric geometry, but tied these mathematical abstractions to the Italian mathematicians who were instrumental in developing the tools for Einstein's general theory of relativity. Remo Ruffini gets credit for drawing me into his obsession with the early work on electromagnetic mass and general relativity done by Enrico Fermi, where the corkscrew pasta surface in a 3-dimensional Minkowski spacetime describes the equator of a classical spherical electron in a circular orbit, inspiring me to incorporate special relativity into the examples and problems of the text. Eduard Bachmakov gave me the missing computer expertise I needed to finally fix my outstanding problem for the coding of my numbered exercises and activate complete hyperlinking of all cross-references in the exported PDF document that tremendously increased the usability of that electronic platform.

Finally without LaTeX such a self-produced book would not have been possible, nor would I have been able to create the graphics illustrations without Maple, nor back up the calculations which make this subject come alive without its computer algebra engine.

Contents

Preface

This book began as a set of handwritten notes from a course given at Villanova University in the spring semester of 1991 that were scanned and posted on the web in 2006 at
http://www34.homepage.villanova.edu/robert.jantzen/notes/dg1991/
and were converted to a LaTeX compuscript and completely revised in 2007–2008 with the help of Hans Kuo of Taiwan through a serendipitous internet collaboration and chance second offering of the course to actual students in the spring semester of 2008, offering the opportunity for serious revision with feedback. Life then intervened and the necessary cleanup operations to put this into a finished form were delayed indefinitely.

Most undergraduate courses on differential geometry are leftovers from the early part of the last century, focusing on curves and surfaces in space, which is not very useful for the most important application of the twentieth century: general relativity and field theory in theoretical physics. Most mathematicians who teach such courses are not well versed in physics, so perhaps this is a natural consequence of the distancing of mathematics from physics, two fields which developed together in creating these ideas from Newton to Einstein and beyond. The idea of these notes is to develop the essential tools of modern differential geometry while bypassing more abstract notions like manifolds, which although important for global questions, are not essential for local differential geometry and therefore need not steal precious time from a first course aimed at undergraduates. On the other hand physicists interested in getting students to the heart of general relativity under time constraints often neglect the mathematical structure that makes tensor analysis more digestible when recast in a more modern light. (One of these shortcuts I think is particularly regrettable is to bypass the understanding of linearity embodied in the concept of the dual space to a vector space by using reciprocal bases to evaluate components along a basis of a vector space. See Appendix E.) Since this is not the primary objective of these notes, we can take a compromise path which tries to give a better view of the overall mathematical structure that will enable interested students to explore applications on their own.

Part 1 (Algebra) develops the vector space structure of $\mathbb{R}^n$ and its dual space of real-valued linear functions, and builds the tools of tensor algebra on that structure, getting the index manipulation part of tensor analysis out of the way first. Part 2 (Calculus) then develops $\mathbb{R}^n$ as a manifold first analyzed in Cartesian coordinates, beginning by redefining the tangent space of multivariable calculus to be the space of directional derivatives at a point, so that all of the tools of Part 1 then can be applied pointwise to the tangent space. Non-Cartesian coordinates and the Euclidean metric are then used as a shortcut to what would be the consideration of more general manifolds with Riemannian metrics in a more ambitious course, followed by the covariant derivative and parallel transport, leading naturally into curvature. The exterior derivative and integration of differential forms is the final topic, showing how conventional vector analysis fits into a more elegant unified framework. Flat Minkowski spacetime geometry is woven into the story together with its symmetry groups, and a few curved space examples from general relativity help drive home the point of truly curved spaces.

The theme of Part 1 is that one needs to distinguish the linearity properties from the inner product ("metric") properties of elementary linear algebra. The inner product geometry governs lengths and angles, and the determinant then enables one to extend the linear measure of length to area and volume in the plane or 3-dimensional space, and to p-dimensional objects in $\mathbb{R}^n$. The determinant also tests linear independence of a set of vectors and hence is key to characterizing subspaces independent of the particular set of vectors we use to describe them while assigning an actual measure to the p-parallelepipeds formed by a particular set, once an inner product sets the length scale for orthogonal directions. By appreciating the details of these basic notions in the setting of $\mathbb{R}^n$, one is ready for the tools needed point by point in the tangent spaces to $\mathbb{R}^n$, once one understands the relationship between each tangent space and the simpler enveloping space. Along the way we discover how basic notions about matrices and vectors and their algebra resurface in so many ways in the tensor algebra needed to do basic differential geometry.

This book is not for everyone. It is verbose, trying to explain in much detail how everything works, with lots of examples interwoven into the discussion. It is aimed at those students who only have the limited foundation of multivariable calculus (see Appendices C, C), linear algebra and differential equations, and tries to avoid abstractions. No inverse function theorem remarks here, for example.

In the spring of 2013, I had a second opportunity to go further with this project by incorporating the mathematics of special relativity into the applications since clearly relativity is a more interesting application than surfaces in space which are the prime target of the usual differential geometry offerings. This in turn led to extending the existing material naturally to include continuous symmetry groups, the missing component of these notes until then. I decided to start the course with a simple multivariable calculus calculation which evaluates the dominant contribution to the geodetic precession effect measured by the GP-B satellite experiment in recent years, and follow with a crash course in hyperbolic geometry (see Appendix A) that is always skipped in our calculus offerings, connecting it up with special relativity (see Appendix B) which would then be woven into the main text in parallel with the more familiar Euclidean geometry associated with the dot product. During the fall of 2012 I tried to think of interesting ways to incorporate relativity into the applications at an elementary level, and having gotten excited about the surface geometry of screw-symmetric surfaces in modeling pasta and circularly orbiting particles, added some more appendices reviewing the basics of special relativity and reviewing curves and surfaces from multivariable calculus. Only at the end of this upgrade will it be clear whether this burst of enthusiasm was successful in exciting the students.

Part 3 is indispensable to students trying to self-study using this book as well as to those rare exceptions who might be in an actual course using it, since it is an ambitious text and includes many options explored in exercises that might appeal to particular interests of the reader that time won't permit discussion of in a course setting. An index of the Maple worksheets which are essential for many of the exercise solutions is first, followed by a complete solution manual electronically linked back to the exercises of the main text in the PDF version of the book for ready access. The Maple worksheets are freely available on dr bob's website while it exists.