

dr bob's elementary differential geometry

a slightly different approach
based on elementary undergraduate linear algebra,
multivariable calculus and differential equations

by bob jantzen

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Abstract

There are lots of books on differential geometry, including at the introductory level. Why yet another one by an author who doesn't seem to take himself that seriously and occasionally refers to himself in the third person? This one is a bit different than all the rest. Dr Bob loves this stuff, but how to teach it to students at his own (not elite) university in order to have a little more fun at work than usual? This unique approach may not work for everyone, but it attempts to explain the nuts and bolts of how a few basically simple ideas taken seriously underlie the whole mess of formulas and concepts, without worrying about technicalities like "manifolds," "coordinate coverings" and "differentiability," which only serve to put off students at the first pass through this scenery. It is also presented with an eye towards being able to understand the key concepts needed for the mathematical side of modern physical theories, while still providing the tools that underlie the classical theory of surfaces in space. Examples of curves and surfaces in 2 and 3-dimensional spacetimes have been incorporated as examples, with an Appendix presenting a review of the elementary special relativity (hyperbolic geometry, directly analogous to trigonometry) needed to make sense of them. The continuing theme of symmetry groups and their implications for geometry are also now woven into the narrative, which is somewhat uncommon for expositions of differential geometry, but essential to a proper understanding of the implications of the subject for applications to physical theories.

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first overly ambitious attempt at a course on this topic when I had just arrived at Villanova
University in my first year ever of teaching in George Orwell's year 1984. Some time later in
1991 I was able to give it the sophomore try with 6 trusting students who probably left the
course wondering what they had just done, but the opportunity allowed me to write up the first
version of the handwritten notes from which this book eventually sprang to life. Without Hans
Kuo, there never would have been a book, since starting to LaTeX such a project from scratch
would never have occurred to me, and he gifted me the first draft taken from my scanned
handwritten notes I had posted on the internet. These were then seriously developed with the
addition of problems and technology graphics and more text in an offering of the course in 2008,
after which four years passed before I had the time to return to the project. Cole Johnston
must be credited with pushing me to offer the course again in 2013, which coincided with an
awakening of my own interest in surfaces motivated by the opportunity to give a popular talk
on differential geometry and relativity in which the surface geometry of corkscrew pasta played
a starring role in conveying not only the visual ideas of metric geometry, but tied these math-
ematical abstractions to the Italian mathematicians who were instrumental in developing the
tools for Einstein's general theory of relativity. Remo Ruffini gets credit for drawing me into his
obsession with the early work on electromagnetic mass and general relativity done by Enrico
Fermi, where the corkscrew pasta surface in a 3-dimensional Minkowski spacetime describes
the equator of a classical spherical electron in a circular orbit, inspiring me to incorporate
special relativity into the examples and problems of the text. Eduard Bachmakov gave me the
missing computer expertise I needed to finally fix my outstanding problem for the coding of my
numbered exercises and activate complete hyperlinking of all cross-references in the exported
PDF document that tremendously increased the usability of that electronic platform.

Finally without LaTeX such a self-produced book would not have been possible, nor would I
have been able to create the graphics illustrations without Maple, nor back up the calculations
which make this subject come alive without its computer algebra engine.

Contents

Abstract	2
Acknowledgments	3
Table of Contents	4
Preface	11
I ALGEBRA	13
0 Introduction: motivating index algebra	15
Geometry?	20
1 Foundations of tensor algebra	25
1.1 Index conventions	26
1.2 A vector space V	27
Elementary linear algebra: solving systems of linear equations	33
Elementary linear algebra: the eigenvalue problem and linear transformations	37
1.3 The dual space V^*	40
1.4 Linear transformations of a vector space V into itself (and tensors)	56
More than 2 indices: general tensors	64
1.5 Linear transformations of V into itself and a change of basis	72
Matrix form of the “transformation law” for $\binom{1}{1}$ -tensors	77
Matrices of symmetric $\binom{0}{2}$ -tensors	78
A 3-index example	79
1.6 Linear transformations between V and V^*	86
Invertible maps between V and V^*	87
Inner products	87
Index shifting with an inner product	94
Index shifting conventions	97
Partial evaluation of a tensor and index shifting	100
Contraction of tensors	101
Geometric interpretation of index shifting	101
Cute fact (an aside for your reading pleasure): geometric interpretation of index lowering on vectors	116

1.7	Matrix groups	122
	Transformation groups	130
	Non-Abelian groups	132
	Representations of Lie groups	141
2	Symmetry properties of tensors	147
2.1	Measure motivation and determinants	148
2.2	Tensor symmetry properties	153
2.3	Epsilons and deltas	158
2.4	Antisymmetric tensors	171
2.5	Symmetric tensors and multivariable Taylor series	174
	Symmetric tensors and multipole moments in physics	176
	Moments of inertia?	178
3	Time out	183
3.1	Whoa! Review of what we've done so far	184
4	Antisymmetric tensors, subspaces and measure	197
4.1	Determinants gone wild	198
4.2	The wedge product	204
4.3	Subspace orientation and a new star duality	211
	Rescaled inner product for antisymmetric tensors	218
	The unit n -form on an oriented vector space with inner product	220
	The metric dual	222
	Inner product, duality and wedge product relations	226
	Determining subspaces	235
4.4	Wedge and star duality on R^n in practice	240
4.5	Matrix generators of the generalized orthogonal matrix groups	248
II	CALCULUS	262
5	From multivariable calculus to the foundation of differential geometry	263
5.1	The tangent space in multivariable calculus	264
5.2	More motivation for the re-interpretation of the tangent space	274
5.3	Flow lines of vector fields	279
5.4	Frames and dual frames and Lie brackets	291
5.5	Non-Cartesian coordinates on \mathbb{R}^n (polar coordinates in \mathbb{R}^2)	299
5.6	Cylindrical and spherical coordinates on R^3	309
5.7	Cylindrical coordinate frames	313
5.8	Spherical coordinate frames	321
5.9	Lie brackets and noncoordinate frames	331

6	Covariant derivatives	339
6.1	Covariant derivatives on R^n with Euclidean metric	340
6.2	Notation for covariant derivatives	343
6.3	Covariant differentiation and the general linear group	352
6.4	Covariant constant tensor fields	360
6.5	The clever way of evaluating the components of the covariant derivative	363
	Key to Riemannian geometry and surface geometry in R^3	364
6.6	Noncoordinate frames	367
6.7	Geometric interpretation of the Lie bracket	372
	Lie brackets and transformation groups	373
6.8	Isometry groups and Killing vector fields	379
6.9	Noncoordinate frames and $SO(3, \mathbb{R})$	395
7	More on covariant derivatives	397
7.1	Gradient, curl and divergence	398
7.2	Second covariant derivatives and the Laplacian	403
7.3	Spherical coordinate orthonormal frame	411
7.4	Rotations and derivatives	417
8	Parallel transport and geodesics	423
8.1	Covariant differentiation along a curve and parallel transport	425
8.2	Parallel transport within coordinate surfaces in space	436
8.3	Geodesics	442
	Conserved momentum, symmetries and Killing vector fields	446
8.4	Surfaces of revolution	448
8.5	Parametrized curves as “motion of point particles” and the geodesic motion approach	460
8.6	The Euclidean plane and the Kepler problem	467
	Kepler’s problem	471
8.7	2-spheres, pseudospheres and other conics of revolution	478
	spheres	479
	Minkowski geometry	487
	unit spacelike pseudosphere	488
	unit timelike pseudosphere	489
8.8	The torus	494
8.9	Geodesics as extremal curves: a peek at the calculus of variations	504
	The boundary value problem for geodesics	514
8.10	The rigid body example and $SO(3, \mathbb{R})$	515
8.11	The screw-symmetric helical tube	517
8.12	The Schwarzschild equatorial plane geometry	526
	Circular geodesic equations of motion	526

9	Intrinsic curvature	529
9.1	Calculating the curvature tensor and its symmetries	530
9.2	Interpretation of curvature	541
9.3	The limiting loop parallel transport calculation of curvature	546
	The limiting loop parallel transport frame curvature calculation	549
	The symmetry of the covariant derivative	550
9.4	Positive curvature focusing of geodesics and negative curvature defocusing	553
10	Extrinsic curvature	561
10.1	The extrinsic curvature tensor	562
10.2	Spheres, cylinders and cones: some useful concrete examples	569
	Cones: a useful cautionary example	571
10.3	Extrinsic curvature as a quadratic approximation	573
10.4	Total curvature: intrinsic plus extrinsic curvature	577
10.5	Tube/tubular surfaces	587
10.6	Surface geodesics studied from the outside	592
11	Differential forms: integration and differentiation	599
11.1	Changing the variable in a single variable integral	600
11.2	Changing variables in multivariable integrals	602
11.3	Parametrized p -surfaces and pushing forward the coordinate grid and tangent vectors	606
11.4	Pulling back functions, covariant tensors and differential forms	609
11.5	Changing the parametrization	614
11.6	Integration and a metric	618
	Curves in a 3-dimensional flat space	618
	Surfaces in a 3-dimensional flat space	618
	Integrating over a p -surface in an n -dimensional space	621
11.7	The exterior derivative d	623
11.8	The exterior derivative and a metric	636
	star, sharp, flat, d and ∇	636
	grad, div and curl	639
	the codifferential δ	643
	Commutative diagrams?	647
11.9	Induced orientation on a boundary	649
	First coordinate adapted to boundary	649
	Induced orientation for any coordinate ordering	652
11.10	Stokes' theorem	655
	The case $p = 2$ in \mathbb{R}^3	655
	The case $p = 3$ in \mathbb{R}^3	657
	Idea of the proof of Stokes' Theorem	658
11.11	Worked examples of Stokes' theorem and Gauss's law for \mathbb{R}^3	661
	The ordinary Stokes' theorem in \mathbb{R}^3	661

A Gauss's law problem	665
11.12 Examples in \mathbb{R}^4 and \mathbb{M}^4	668
3-spheres, 3-cylinders and 2-cylinders in \mathbb{R}^4	668
3-cylinders in \mathbb{R}^4	673
2-cylinders in \mathbb{R}^4	673
pseudospheres in \mathbb{M}^4	673
cylinders in \mathbb{M}^4	673
constant inertial time hypersurface	673
12 Wrapping things up	675
12.1 Final remarks	676
1991	676
2013	676
12.2 MATH 5600 Spring 1991 Differential Geometry: Take Home Final	677
Appendices	685
A From trigonometry to hyperbolic functions and hyperbolic geometry	687
B Hyperbolic geometry and special relativity	699
C Curves in 3-space	713
The Euclidean helix	713
Circles to pseudo-circles: hyperbolas	721
The Lorentz helix	725
D Surfaces in 3-space	733
E Visualizing vector space duality in the vector space R^3	741
III Supplementary materials	748
Maple worksheets	749
Solutions to Exercises	751
Chapter 0	753
Chapter 1	754
Chapter 2	762
Chapter 3	765
Chapter 4	766
Chapter 5	770
Chapter 6	776
Chapter 7	782
Chapter 8	784

Chapter 9	785
Chapter 10	786
Chapter 11	787
Chapter 12: final exam worked	788
The last page is:	800

Preface

This book began as a set of handwritten notes from a course given at Villanova University in the spring semester of 1991 that were scanned and posted on the web in 2006 at <http://www34.homepage.villanova.edu/robert.jantzen/notes/dg1991/> and were converted to a L^AT_EX compuscript and completely revised in 2007–2008 with the help of Hans Kuo of Taiwan through a serendipitous internet collaboration and chance second offering of the course to actual students in the spring semester of 2008, offering the opportunity for serious revision with feedback. Life then intervened and the necessary cleanup operations to put this into a finished form were delayed indefinitely.

Most undergraduate courses on differential geometry are leftovers from the early part of the last century, focusing on curves and surfaces in space, which is not very useful for the most important application of the twentieth century: general relativity and field theory in theoretical physics. Most mathematicians who teach such courses are not well versed in physics, so perhaps this is a natural consequence of the distancing of mathematics from physics, two fields which developed together in creating these ideas from Newton to Einstein and beyond. The idea of these notes is to develop the essential tools of modern differential geometry while bypassing more abstract notions like manifolds, which although important for global questions, are not essential for local differential geometry and therefore need not steal precious time from a first course aimed at undergraduates. On the other hand physicists interested in getting students to the heart of general relativity under time constraints often neglect the mathematical structure that makes tensor analysis more digestible when recast in a more modern light. (One of these shortcuts I think is particularly regrettable is to bypass the understanding of linearity embodied in the concept of the dual space to a vector space by using reciprocal bases to evaluate components along a basis of a vector space. See Appendix E.) Since this is not the primary objective of these notes, we can take a compromise path which tries to give a better view of the overall mathematical structure that will enable interested students to explore applications on their own.

Part 1 (Algebra) develops the vector space structure of \mathbb{R}^n and its dual space of real-valued linear functions, and builds the tools of tensor algebra on that structure, getting the index manipulation part of tensor analysis out of the way first. Part 2 (Calculus) then develops \mathbb{R}^n as a manifold first analyzed in Cartesian coordinates, beginning by redefining the tangent space of multivariable calculus to be the space of directional derivatives at a point, so that all of the tools of Part 1 then can be applied pointwise to the tangent space. Non-Cartesian coordinates and the Euclidean metric are then used as a shortcut to what would be the consideration of more general manifolds with Riemannian metrics in a more ambitious course, followed by the covariant derivative and parallel transport, leading naturally into curvature. The exterior derivative and integration of differential forms is the final topic, showing how conventional vector analysis fits into a more elegant unified framework. Flat Minkowski spacetime geometry is woven into the story together with its symmetry groups, and a few curved space examples from general relativity help drive home the point of truly curved spaces.

The theme of Part 1 is that one needs to distinguish the linearity properties from the inner product (“metric”) properties of elementary linear algebra. The inner product geometry

governs lengths and angles, and the determinant then enables one to extend the linear measure of length to area and volume in the plane or 3-dimensional space, and to p -dimensional objects in \mathbb{R}^n . The determinant also tests linear independence of a set of vectors and hence is key to characterizing subspaces independent of the particular set of vectors we use to describe them while assigning an actual measure to the p -parallelepipeds formed by a particular set, once an inner product sets the length scale for orthogonal directions. By appreciating the details of these basic notions in the setting of \mathbb{R}^n , one is ready for the tools needed point by point in the tangent spaces to \mathbb{R}^n , once one understands the relationship between each tangent space and the simpler enveloping space. Along the way we discover how basic notions about matrices and vectors and their algebra resurface in so many ways in the tensor algebra needed to do basic differential geometry.

This book is not for everyone. It is verbose, trying to explain in much detail how everything works, with lots of examples interwoven into the discussion. It is aimed at those students who only have the limited foundation of multivariable calculus (see Appendices C, C), linear algebra and differential equations, and tries to avoid abstractions. No inverse function theorem remarks here, for example.

In the spring of 2013, I had a second opportunity to go further with this project by incorporating the mathematics of special relativity into the applications since clearly relativity is a more interesting application than surfaces in space which are the prime target of the usual differential geometry offerings. This in turn led to extending the existing material naturally to include continuous symmetry groups, the missing component of these notes until then. I decided to start the course with a simple multivariable calculus calculation which evaluates the dominant contribution to the geodetic precession effect measured by the GP-B satellite experiment in recent years, and follow with a crash course in hyperbolic geometry (see Appendix A) that is always skipped in our calculus offerings, connecting it up with special relativity (see Appendix B) which would then be woven into the main text in parallel with the more familiar Euclidean geometry associated with the dot product. During the fall of 2012 I tried to think of interesting ways to incorporate relativity into the applications at an elementary level, and having gotten excited about the surface geometry of screw-symmetric surfaces in modeling pasta and circularly orbiting particles, added some more appendices reviewing the basics of special relativity and reviewing curves and surfaces from multivariable calculus. Only at the end of this upgrade will it be clear whether this burst of enthusiasm was successful in exciting the students.

Part 3 is indispensable to students trying to self-study using this book as well as to those rare exceptions who might be in an actual course using it, since it is an ambitious text and includes many options explored in exercises that might appeal to particular interests of the reader that time won't permit discussion of in a course setting. An index of the Maple worksheets which are essential for many of the exercise solutions is first, followed by a complete solution manual electronically linked back to the exercises of the main text in the PDF version of the book for ready access. The Maple worksheets are freely available on dr bob's website while it exists.