DIFFERENTIAL GEOMETRY BASED ON UNDERGRADUATE LINEAR ALGEBRA AND MULTIVARIABLE CALCULUS

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Contents

Preface

This book began as a set of handwritten notes from a course given at Villanova University in the spring semester of 1991 that were scanned and posted on the web in 2006 at

`http://www34.homepage.villanova.edu/robert.jantzen/notes/dg1991/`

and were converted to a LaTeX compuscript and revised in 2007–2008 with the help of Hans Kuo of Taiwan through a serendipitous internet collaboration and chance second offering of the course to actual students in the spring semester of 2008, offering the opportunity for serious revision with feedback.

Most undergraduate courses on differential geometry are leftovers from the early part of the last century, focusing on curves and surfaces in space, which is not very useful for the most important application of the twentieth century: general relativity and field theory in theoretical physics. Most mathematicians who teach such courses are not well versed in physics, so perhaps this is a natural consequence of the distancing of mathematics from physics, two fields which developed together in creating these ideas from Newton to Einstein and beyond. The idea of these notes is to develop the essential tools of modern differential geometry while bypassing more abstract notions like manifolds, which although important for global questions, are not essential for local differential geometry and therefore need not steal precious time from a first course aimed at undergraduates.

Part 1 (Algebra) develops the vector space structure of $\mathbb{R}^n$ and its dual space of real-valued linear functions, and builds the tools of tensor algebra on that structure, getting the index manipulation part of tensor analysis out of the way first. Part 2 (Calculus) then develops $\mathbb{R}^n$ as a manifold first analyzed in Cartesian coordinates, beginning by redefining the tangent space of multivariable calculus to be the space of directional derivatives at a point, so that all of the tools of Part 1 then can be applied pointwise in the space. Non-Cartesian coordinates and the Euclidean metric are then used as a shortcut to what would be the consideration of more general manifolds with Riemannian metrics in a more ambitious course, followed by the covariant derivative and parallel transport, leading naturally into curvature. The exterior derivative and integration of differential forms is the final topic, showing how conventional vector analysis fits into a more elegant unified framework.

The theme of Part 1 is that one needs to distinguish the linearity properties from the inner product ("metric") properties of linear algebra. The inner product geometry governs lengths and angles, and the determinant then enables one to extend the linear measure of length to area and volume in the plane or 3-dimensional space, and to p-dimensional objects in $\mathbb{R}^n$. The determinant also tests linear independence of a set of vectors and hence is key to characterizing subspaces independent of the particular set of vectors we use to describe them while assigning an actual measure to the p-parallelepipeds formed by a particular set, once an inner product sets the length scale for orthogonal directions. By appreciating the details of these basic notions in the setting of $\mathbb{R}^n$, one is ready for the tools needed point by point in the tangent spaces to $\mathbb{R}^n$, once one understands the relationship between each tangent space and the simpler enveloping space.