Group Action on a Space

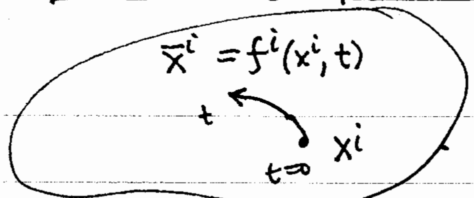
by bob jantzen [2001]

17 pages of notes for an independent study by Christopher Pilman, in addition to the previous sets of differential geometry notes [1984, 1991], with several pages on the Lie derivative copied from Introduction to Cosmological Models 2.

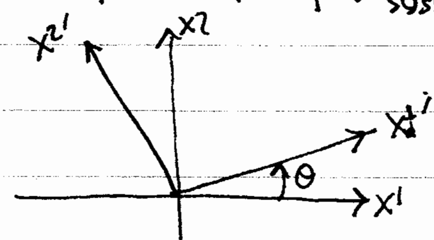
The rotations in the plane are used to motivate a general 1-parameter group of transformations, using the exponential map, then specializing back to the rotations showing the relation to the matrix generators and matrix transformation. Comoving coordinates are found for this example. Then dragging and the Lie derivative are introduced. Then the relationship between the rotation generators and linear and angular momentum is discussed. Next r-parameter groups and rotations of space, then boosts of 2-d Minkowski spacetime, and the generators of the 4-d Lorentz group. Finally the Lie derivative and isometry actions are touched upon, with a final exercise.

- PDF 700K

$\boxed{\text{Action of a group on a space}}$: from finite to "infinitesimal" back to finite transformations (1-D case)



active transformation: give coords of new point as function of old pt (same coord system)

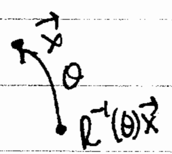


new coord axes: $x^{1\prime}, x^{2\prime}$

new coords of point = old coords of point from which it comes

"dragged along coordinates"

$$x^{i\prime}(\vec{x}) = x^i(R^{-1}(\theta)\, x) \longrightarrow \begin{bmatrix} x^{1\prime} \\ x^{2\prime} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta\, x_1 + \sin\theta\, x_2 \\ -\sin\theta\, x_1 + \cos\theta\, x_2 \end{bmatrix}$$

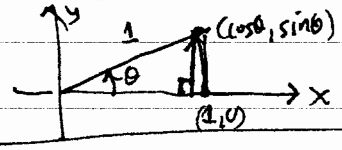


__active rotations of plane__ by an angle θ (counterclockwise)

$$\begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta\, x_1 - \sin\theta\, x_2 \\ \sin\theta\, x_1 + \cos\theta\, x_2 \end{bmatrix}$$

$$\vec{\bar{x}} = R(\theta)\vec{x}$$

__EX.__ $(x^1, x^2) = (1, 0) \longmapsto (\bar{x}^1, \bar{x}^2) = (\cos\theta, \sin\theta)$ in first quad if $0 \le \theta < \pi/2$



1-D abelian group: $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$ (closure)

inverses: $R(\theta)^{-1} = R(-\theta)$

identity: $R(0) = I_2$

automatically associative "multiplication"

__passive coordinate transformation__: just giving new coordinates of same point with respect to the new coord system.

As we vary the single parameter, we trace out an "orbit" (circle); fixing (x^1, x^2) we can think of $\vec{\bar{x}} = R(\theta)\vec{x}$ as a parametrized curve so we can evaluate its tangent at $\theta = 0$ corresponding to the identity to see how points start to move as we increase θ from 0:

$$\bar{x}^i = f^i(x, t)$$

$$\frac{d\bar{x}^i}{dt}\Big|_{t=0} = \frac{\partial f^i}{\partial t}\Big|_{t=0} = \xi^i(x) \quad \text{"generating" vector field}$$

$$\xi = \xi^i(x)\partial_i$$

orbits are tangent to ξ as we start out from $t=0$.

$$\bar{x}^1 = \cos\theta x^1 - \sin\theta x^2 \qquad \frac{\partial \bar{x}^1}{\partial\theta}\Big|_{\theta=0} = -x^2$$

$$\bar{x}^2 = \sin\theta x^1 + \cos\theta x^2 \qquad \frac{\partial \bar{x}^2}{\partial\theta}\Big|_{\theta=0} = x^1$$

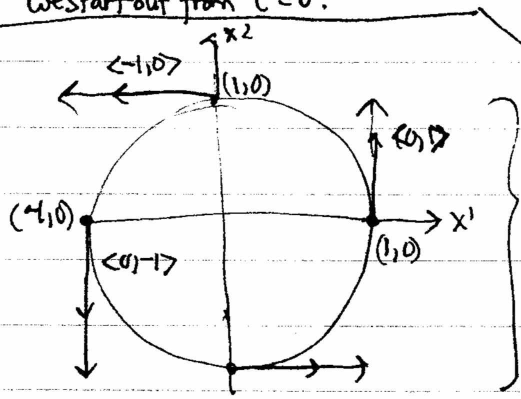
$$(\xi^1, \xi^2) = (-x^2, x^1) \qquad \xi = x^1\partial_2 - x^2\partial_1 = (S_3)^i{}_j x^j \partial_i \qquad S_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\bar{x}^i = R(\theta)^i{}_j x^j$$

$$\frac{d\bar{x}^i}{d\theta}\Big|_{\theta=0} = \underbrace{\frac{dR(\theta)^i{}_j}{d\theta}\Big|_{\theta=0}}_{\equiv S_3(\theta)^i{}_j} x^j$$

$R(\theta)$ curve of matrices. S_3 is its "tangent" at $\theta = 0$.

In vector spaces like $\mathbb{R}^n$, we can identify tangent vectors with vectors as usual. (2x2 matrices form a vector space)



plot of ξ on unit circle.

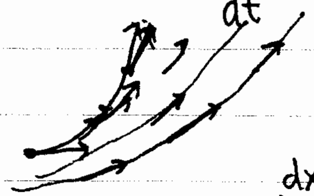
Action of a group on a space (2)

Given any vector field $\xi = \xi^i(x)\partial_i$, we can construct its 1-D group of transformations or "flow along its integral curves".

An integral curve is defined by

$$\frac{dx^i(t)}{dt} = \xi^i(x(t))$$

tangent to curve at $X(t)$ is value of ξ there.
There is one integral curve through each point of space.



solve this system of DEs by iteration and Taylor series:

$$\frac{dx^i(t)}{dt} = \xi^i(x(t)) \rightarrow \left.\frac{dx^i(t)}{dt}\right|_{t=0} = \xi^i(x) = \xi(x)x^i \quad (\text{note } x^i(0) = x^i)$$

$$\frac{d^2x^i(t)}{dt^2} = \frac{\partial \xi^i(x(t))}{\partial x^j}\frac{dx^j(t)}{dt} \rightarrow \left.\frac{d^2x^i(t)}{dt^2}\right|_{t=0} = \underbrace{\frac{dx^j(0)}{dt}}_{\xi^j(x(0))}\frac{\partial \xi^i(x(0))}{\partial x^j}\Bigg) = \underbrace{\xi^j(x(0))\frac{\partial}{\partial x^j}}_{\xi}\underbrace{\xi^i(x(0))}_{\xi x^i}$$

$$= \xi^2 x^i$$

$$\xi x^i = \xi^j \partial_j x^i = \xi^i \quad (\partial_j x^i = \delta^i_j)$$

...

$$x^i(t) = x^i(0) + \left.\frac{dx^i(t)}{dt}\right|_{t=0} t + \left.\frac{d^2x^i}{dt^2}\right|_{t=0}\frac{t^2}{2!} + \cdots$$

$$= x^i(0) + t\xi(x(0))x^i + \xi^2(x(0))x^i\frac{t^2}{2} + \cdots$$

$$= \left(1 + t\xi + \frac{t^2}{2}\xi^2 + \cdots\right)x^i = \boxed{e^{t\xi}x^i = x^i(t)}$$ exponential form of finite transformation.

By definition $\xi x^i = \xi^i$ so if $\xi^i = (S_3)^i{}_j x^j$ for a rotation.

$$= (S_3)^i{}_j x^j$$

$$\xi^2 x^i = (S_3{}^i{}_j(\xi x^j)) = (S_3{}^i{}_j)(S_3)^j{}_k x^k = (S_3{}^2)^i{}_k x^k$$

...

so $\boxed{x^i = (e^{tS_3})^i{}_j x^j}$ (can just replace ξ by S_3, diff. by matrix mult)

<u>carrying out matrix exponential</u>

$$(S_3)^2 = \begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix} = \begin{pmatrix}-1&0\\0&-1\end{pmatrix} = -I_2$$

$$(S_3)^3 = (S_3)^2 S_3 = -I_2 S_3 = -S_3$$

$$(S_3{}^4) = -S_3{}^2 = I_2$$

...

$$e^{tS_3} = I_2 + tS_3 + \frac{t^2}{2}S_3^2 + \frac{t^3}{3!}S_3^3 + \frac{t^4}{4!}S_3^4 + \cdots = I_2\left(1 - \frac{t^2}{2} + \frac{t^4}{4} - \cdots\right) + S_3\left(t - \frac{t^3}{3} + \frac{t^5}{5}\cdots\right)$$

$$= \cos t\, I_2 + \sin t\, S_3$$

$$= \begin{pmatrix}\cos t&0\\0&\cos t\end{pmatrix} + \begin{pmatrix}0&-\sin t\\\sin t&0\end{pmatrix} = \begin{pmatrix}\cos t&-\sin t\\\sin t&\cos t\end{pmatrix} = R(t),$$ ie t is the polar angle

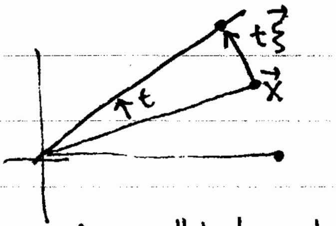
so we are back to the finite transformations of the group.

$$\underbrace{R(\theta)}_{\text{elements of group}} = e^{\theta S_3}$$ → multiples of $\begin{pmatrix}0&-1\\1&0\end{pmatrix} \sim \begin{pmatrix}0&-a\\a&0\end{pmatrix}$ → antisymmetric 2x2 matrices = vector space called "Lie algebra" of matrix group.

↑ exponential map from Lie algebra into group.

Action of a group on a space (3)

"infinitesimal transformations" $\leftrightarrow$ "small values of parameter t like in calculus when we work with differentials dx, dy. $\cos t \to 1$, $\sin t \to t$:



$$\left.\begin{aligned} \bar{x}^1 &= x^1 - t x^2 \\ \bar{x}^2 &= t x^1 + x^2 \end{aligned}\right\} \quad |t| \ll 1$$

$$\bar{x}^1 - x^1 \approx \text{"}\delta x^1\text{"} = -t x^2 \qquad \delta \vec{x}^i = t\,\xi^i$$
$$\bar{x}^2 - x^2 = \text{"}\delta x^2\text{"} = t x^1$$

for small t begin to move along tangent vector ξ at x. "infinitesimal rotation"

But in fact the generating vector field of this infinitesimal transformation also describes finite transformations thru its integral curve flow

<u>flow</u>: $\dfrac{dx^i(t)}{dt} = \xi^i(x(t))$, $x^i(0) = x^i$ $\leftarrow$ start at x^i, move along integral curve through x^i by parameter amount t. This moves all points of space simultaneously — for each t we get a transformation & this is in fact a group.

<u>comoving coordinates.</u>

$$\left.\begin{aligned} \xi y^1 &= 1 \\ \xi y^2 &= 0 \end{aligned}\right\}$$ system of PDE's for 2 new coords y^1, y^2 (2 decoupled equations) (can solve independently)

$\xi = x^2 \partial_1 - x^1 \partial_2$: $x^2 \dfrac{\partial y^1}{\partial x^1} - x^1 \dfrac{\partial y^1}{\partial x^2} = 1$, $x^2 \dfrac{\partial y^2}{\partial x^1} - x^1 \dfrac{\partial y^2}{\partial x^2} = 0$

$\xi y^i = \delta^i_1 =$ components of ξ w.r.t. coords y^i

ξ will have only a unit component along y^1 and zero other components.

$$e^{t\xi} y^i = y^i + t \underbrace{\xi y^i}_{\delta^i_1} + \underbrace{\frac{t^2}{2} \xi^2 y^i + \dots}_{0} = y^i + t\,\delta^i_1. \qquad \begin{aligned} y^1 &\to y^1 + t \\ y^2 &\to y^2 \end{aligned}$$

so in comoving coords the flow of ξ is just translation in the first coordinate.

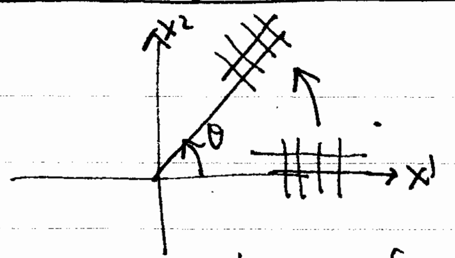
for rotations, solns are $y^1 = \arctan\left(\dfrac{x^2}{x_1}\right) = \theta$ (can still add a constant)

$y^2 = \sqrt{x_1^2 + x_2^2} = r$ (or any function of r)

[easy to check by backsubstitution]

These are just polar coordinates in the plane so we could express rotations in the form $\left.\begin{aligned} \bar{r} &= r \\ \bar{\theta} &= \theta + t \end{aligned}\right\}$ rotation by angle t

Action of a group on a space (4)

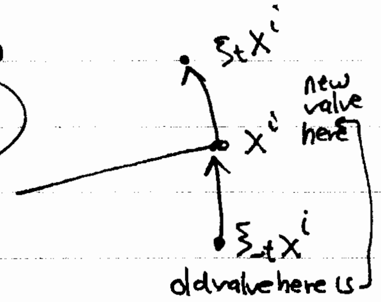


dragging along the old coordinate grid by the transformation makes the new coordinate grid whose values at a given point are the same as the coordinates at the inverse transformation point. $x^{i'}(x) = x^i(R(\theta)^{-1})$

we can drag along any functions on the space in the same way

$$[R(\theta)\, F](x) \equiv F(R(\theta)^{-1}x) \quad \text{"rotation operator"}$$

or $\quad (\xi_t F)(x) = F(\xi_{-t}(x))$



This just drags the graph of the function along with the points.

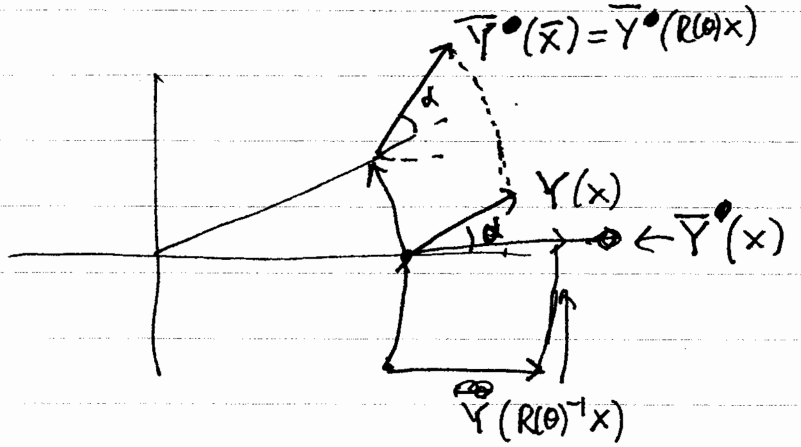
But by the same Taylor series argument that we used for the coords (active transformation) we can repeat here.

$$(\xi_t F)(x) = e^{-t\xi} F(x).$$

$$\frac{d}{dt}(\xi_t F)(x)\Big|_{t=0} = -\xi F(x)$$

← shows how function begins to change as we begin moving with the group. Its change comes from the nearby point value of F at about $-t\xi x^i$ from which x^i comes.

We can ask: what happens if we drag along vector fields? They are tangents to curves. Drag along the curve and take the new tangent.



value at $\vec{\bar{x}} = R(\theta)\vec{x}$ comes from $\vec{x}$

or

value at $\vec{x}$ comes from $R(\theta)^{-1}\vec{x}$

The key idea is the vectors are like "infinitesimal" displacement vectors between points in the coordinate grid and if we drag along the coordinate grid, the vector goes with it, namely will have the same components in the dragged along coordinates as it did at the original point before dragging along.

new comps of new field at x $\quad \bar{Y}^{i\bullet\prime}(x) = Y^i(R(\theta)^{-1}x) \quad$ old components of old field at $R(\theta)^{-1}x$

Action of a group on a space (5)

$$\bar{Y} = \bar{Y}^{i'}\partial_{i'} = \bar{Y}^{i}\partial_{i}$$

$$\bar{Y}^{i} = \bar{Y}x^{i} = \bar{Y}^{j'}\partial_{j'}x^{i} = \frac{\partial x^{i}}{\partial x^{j'}}\underbrace{\bar{Y}^{j'}}_{Y(R(-\theta)x)}$$

vector transforms under coord. transformation.

$$\underbrace{Y(R(-\theta)x)}_{e^{-\theta\xi}Y(x)}$$

But $x'^{i} = R(-\theta)^{i}{}_{j}x^{j}$ passive coord tranf.

$x^{i} = R(\theta)^{i}{}_{j}x^{j'}$ inverse

$\frac{\partial x^{i}}{\partial x^{j'}} = R(\theta)^{i}{}_{j}$

$$\underbrace{\bar{Y}^{i}(x)}_{\substack{\text{components of new}\\ \text{"dragged along" field in}\\ \text{original coords}}} = \underbrace{R(\theta)^{i}{}_{j}}_{\substack{\text{active rotation}\\ \text{of components}}}\, Y^{j}\underbrace{(R(-\theta)x)}_{\text{evaluated at inverse pt.}} = e^{\theta\xi}(Y)^{i}$$

↓ inverse taylor series.

$$= \left(e^{\theta S_3}\right)^{i}{}_{j}\; e^{-\theta\xi}\,Y^{j}$$

spin ang mom operator

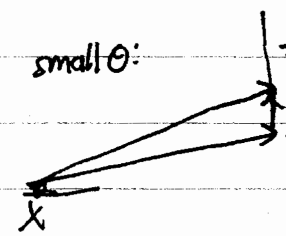
orbital ang mom operator

total ang mom J_3 operator.

$$\frac{d}{d\theta}\bar{Y}^{i}(x)\Big|_{\theta=0} = S_3{}^{i}{}_{j}Y^{j} - \underbrace{\xi}_{L_3}Y^{i} = [(L_3+S_3)Y]^{i}$$

$$= -\underbrace{(\delta^{i}{}_{j}\xi - S_3{}^{i}{}_{j})Y^{j}}.$$

small θ:



change in $Y \approx \theta\frac{d}{d\theta}\bar{Y}^{i}(x)$

$$\equiv (\mathcal{L}_{\xi}Y)^{i} = \xi^{j}\partial_{j}Y^{i} - Y^{j}\underbrace{\partial_{j}\xi^{i}}$$

$$\equiv [\xi, Y]^{i}$$

$\partial_{j}(S_3)^{i}{}_{k}x^{k})$

$S_3{}^{i}{}_{k}\delta^{k}{}_{j}$

$S_3{}^{i}{}_{j}$ ✓

Lie derivative of Y along ξ

$\equiv$ Lie bracket (commutator) of ξ and Y

---

compute: $[\xi, Y] = [\xi^{i}\partial_{i}, Y^{j}\partial_{j}]$

$$= \xi^{i}\partial_{i}(Y^{j}\partial_{j}) - Y^{j}\partial_{j}(\xi^{i}\partial_{i})$$

$$= \xi^{i}(\partial_{i}Y^{j})\partial_{j} + \xi^{i}Y^{j}\partial_{i}\partial_{j}$$

$$- Y^{j}(\partial_{j}\xi^{i})\partial_{i} - Y^{j}\xi^{i}\partial_{j}\partial_{i} \to 0$$

but $\partial_{i}\partial_{j} = \partial_{j}\partial_{i}$ partial derivatives commute.

$$[\xi, Y]^{j} = \xi^{i}\partial_{i}Y^{j} - Y^{i}\partial_{i}\xi^{j}$$

$$= \xi^{i}(\partial_{i}Y^{j})\partial_{j} - Y^{i}(\partial_{i}\xi^{j})\partial_{j}$$

switch dummy indices

$$= (\xi^{i}\partial_{i}Y^{j} - Y^{i}\partial_{i}\xi^{j})\,\partial_{j}$$

Action of a group on a space (6)

By the same Taylor series argument as above:

$$\bar{Y}^i(x) = \bar{Y}^i(x)\Big|_{\theta=0} + \theta \frac{d\bar{Y}^i(x)}{d\theta}\Big|_{\theta=0} + \frac{1}{2}\theta^2 \frac{d\bar{Y}^i(x)}{d\theta}\Big|_{\theta=0}$$

$$= Y^i(x) + \theta(-\mathcal{L}_\xi Y^i) + \frac{1}{2}\theta^2(-\mathcal{L}_\xi(-\mathcal{L}_\xi Y^i)) + \cdots$$

$$= e^{-\theta \mathcal{L}_\xi} Y^i(x)$$

$$= [e^{-\theta\underbrace{(\mathcal{L}_3 + L_3)}_{J_3}} Y(x)]^i$$

exponentiating the Lie derivative captures the finite transformation of a vector field induced by "dragging along".

For rotations the tangent to the 1-parameter family of rotations acting on a vector field leads to a sum of 2 operators — an orbital ang mom operator which acts on the component functions to move their evaluation point and a spin angular momentum which rotates the component functions.

Why "angular momentum"?

we can repeat this for the translations of the plane

$\bar{x}^i = x^i + a^i$ or $\bar{x}^1 = x^1 + a^1$, $\bar{x}^2 = x^2 + a^2$ } 2 parameter abelian group (a^1, a^2)

we can consider each parameter separately.

$$\xi_1^i = \frac{\partial \bar{x}^i}{\partial a^1}\Big|_{a^1=0} = \delta^i_1 \qquad \xi_1 = \xi_1^i \partial_i = \delta^i_1 \partial_i = \partial_1$$

$$\xi_2^i = \frac{\partial \bar{x}^i}{\partial a^2}\Big|_{a^2=0} = \delta^i_2 \qquad \xi_2 = \xi_2^i \partial_i = \delta^i_2 \partial_i = \partial_2$$

note $[\xi_1, \xi_2] = [\partial_1, \partial_2] = 0$ ∴ commute

"translation operator" $F(\vec{x} + a) = e^{-a^1\partial_1 - a^2\partial_2} F(\vec{x})$ ← dragging along

dragging along cartesian coords does not change grid so no need to change components here, only re-evaluate them at inverse point.

$\xi_1 = \partial_1 = p_1$
$\xi_2 = \partial_2 = p_2$ } "linear momentum operators"

$$\xi = x^1\partial_2 - x^2\partial_1 = x^1 p_2 - x^2 p_1 = (\vec{x} \times \vec{p})^3$$

"3rd component of angular momentum"

Action of a group on a space (7)

rotations & translations of plane form 3-parameter group (θ, a^1, a^2)

$$\bar{x}^i(x;\theta,a) = R(\theta)^i{}_j x^j + a^i$$

3 generating vector fields $\{ \xi = x^1\partial_2 - x^2\partial_1,\ \partial_1,\ \partial_2 \}$

commutation relations or "Lie algebra" structure:

$$[\xi, \partial_i] = -(S_3)^j{}_i \partial_j \ , \qquad [\partial_1, \partial_2] = 0 \qquad (\text{check!})$$

important $\boxed{\text{FACT}}$: if $\bar{x}^i = f^i(x^1 \ldots x^n; a^1 \ldots a^r)$ is an r-parameter group of transformations of the space with coordinates $(x^1 \ldots x^n)$, then letting $\bar{x}^i = f^i(x,a)$ abbreviate this, the composition of 2 transformations must again be a transformation of the group:

$$f(f(x, a_{(1)}), a_{(2)}) = f(x, a_{(3)}) \qquad \text{where } a_{(3)} = \varphi(a_{(1)}, a_{(2)})$$

is the group "multiplication function".

Defining its generating vector fields by

$$\xi_a{}^j = \frac{\partial f^j(x,a)}{\partial a^a}\bigg|_{a^a=0} \qquad \xi_a = \xi^j{}_a \partial_j \qquad a = 1, \ldots r$$

Then closure of the group composition translates into closure of the Lie algebra: $[\xi_a, \xi_b] = C^c{}_{ab}\xi_c$ where $C^c{}_{ab}$ are constants, ie the Lie brackets of the generators must also belong to the vector space of constant linear combinations of those generators. This vector space with the Lie bracket is the (vector field) Lie algebra of the transformation group.

Example: Rotations of $\mathbb{R}^3$

Let $\epsilon_{ijk} = \begin{cases} 1 \text{ if } ijk \text{ is } 123, 231, 312 \\ -1 \text{ if } ijk \text{ is } 132, 213, 321 \end{cases}$ (totally antisymmetric)

Let $(S_a)^i{}_j = -\epsilon_{aij}$ and $L_a = S_a{}^i{}_j x^j \partial_i$

Compute $[L_a, L_b] = -\epsilon_{abc} L_c$ (check!) (sum over c)

Then $e^{\theta^a L_a} x^i = \underbrace{(e^{\theta^a S_a})^i{}_j}_{R(\theta)} x^j = \bar{x}^i$

Let $\delta_{ij} = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i \neq j. \end{cases}$ Kronecker delta with both indices down.
= components of Euclidean inner product in an orthonormal basis of 3-vectors

so $r^2 = \delta_{ij}x^i x^j = (x^1)^2 + (x^2)^2 + (x^3)^2$.

Action of a group on a space (8)

invariance of r^2 under rotations means:

$$r^2 = \delta_{ij} x^i x^j = \delta_{ij} \bar{x}^i \bar{x}^j = \delta_{ij} (R^i{}_m x^m)(R^j{}_n x^n)$$

$$= \underbrace{\delta_{mn}}_{} x^m x^n \quad \left\{\begin{array}{l}\text{change of}\\ \text{dummy}\\ \text{indices}\end{array}\right. = \underbrace{(\delta_{ij} R^i{}_m R^j{}_n)}_{} x^m x^n$$

same:

$$\delta_{mn} = \delta_{ij} R^i{}_m R^j{}_n$$

or $\quad R^i{}_m \delta_{ij} R^j{}_n = \delta_{mn}$

not adjacent, must exchange row column to be adjacent for matrix product.

adjacent: matrix product

3x3 identity matrix: I_3

transpose

$$R^T I_3 R = I_3$$

or $\quad R^T R = I_3$

$\therefore \quad R^T = R^{-1}$

transpose is inverse for an "orthogonal matrix" representing a rotation.

FACT if $R = e^{tA} = I + tA + \frac{t^2}{2} A^2 + \ldots$

then. $\frac{dR}{dt} = Ae^{tA} = e^{tA} A$ (check from exp series!)

so $\left.\frac{dR}{dt}\right|_{t=0} = A$

and $\left.\frac{d}{dt}\right|_{t=0} \left[(e^{tA})^T (e^{tA}) = I_3 \right]$

$$(e^{0A} A)^T e^{0A} + (e^{0A})^T (e^{0A} A) = 0$$

$A^T + A = 0 \qquad A^T = -A \qquad$ so A is an antisymmetric matrix.

The Lie algebra of the orthogonal group (rotation matrices) consists of antisymmetric matrices since their exponentials are automatically orthogonal.

In components: $\quad \left.\frac{d}{dt}\right|_{t=0} \left[R^i{}_m \delta_{ij} R^j{}_n = \delta_{mn} \right]$

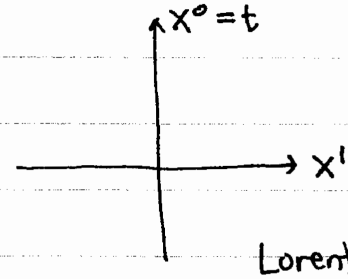
$$\underbrace{A^i{}_m \delta_{ij}}_{A_{jm}} \delta^j{}_n + \delta^i{}_m \underbrace{\delta_{ij} A^j{}_n}_{A_{in}} = 0$$

$A_{jm} \leftarrow$ definition of index lowering $\rightarrow A_{in}$

so $\quad A_{jm} \delta^j{}_n + \delta^i{}_m A_{in} = 0$

or $\quad A_{nm} + A_{mn} = 0 \qquad \boxed{A_{nm} = -A_{mn}}$ $\therefore$ antisymmetry of totally covariant form of mixed component $A^i{}_j$ representing linear transformation.

Action of a group on a space (9)



2-D Minkowski spacetime of special relativity

- Let $(\eta_{\alpha\beta}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ ie: $\eta_{00} = -1$, $\eta_{11} = 1$, $\eta_{01} = \eta_{10} = 0$

spacetime interval: $s^2 = -(x^0)^2 + (x^1)^2 = \eta_{\alpha\beta} x^\alpha x^\beta$

Lorentz transformation leaves it invariant:

$$\bar{x}^\alpha = L^\alpha{}_\beta x^\beta$$

equals original value

$$\eta_{\alpha\beta} \bar{x}^\alpha \bar{x}^\beta = \eta_{\alpha\beta} (L^\alpha{}_\gamma x^\gamma)(L^\beta{}_\delta x^\delta) = (\eta_{\alpha\beta} L^\alpha{}_\gamma L^\beta{}_\delta) x^\gamma x^\delta = \eta_{\gamma\delta} x^\gamma x^\delta$$

$\cancel{\eta_{\gamma\delta} \bar{x}^\gamma \bar{x}^\delta}$

$\therefore$ same

$$\eta_{\alpha\beta} L^\alpha{}_\gamma L^\beta{}_\delta = \eta_{\gamma\delta}$$

If $L = e^{tB}$, $\left.\frac{dL}{dt}\right|_{t=0} = B$, so $\quad \leftarrow \left. L^\alpha{}_\beta \right|_{t=0} = \delta^\alpha{}_\beta$, identity ($e^0 = I$)

$$\eta_{\alpha\beta} B^\alpha{}_\gamma \delta^\beta{}_\delta + \eta_{\alpha\beta} \delta^\alpha{}_\gamma B^\beta{}_\delta = 0$$

$\cancel{\eta_{\alpha\delta} B^\alpha{}_\gamma + \eta_{\alpha\delta} \delta^\alpha{}_\gamma}$

$$\eta_{\alpha\delta} B^\alpha{}_\gamma + \eta_{\gamma\beta} B^\beta{}_\delta = 0$$

$\cancel{B_{\gamma\delta} + B_{\delta\gamma}}$ $\quad B_{\delta\gamma} + B_{\gamma\delta} = 0 \qquad B_{\delta\gamma} = -B_{\gamma\delta}$ antisymmetry of index lowered matrix

$$B^\alpha{}_\beta = \eta^{\alpha\gamma} B_{\gamma\beta}$$

antisymmetric

extra minus for 0 component leads to symmetric matrix. condition if has a zero index.

where $\eta^{\alpha\beta} = \eta_{\alpha\beta}$ is the same matrix but with indices up for raising indices:

$$\eta^{\alpha\beta} \eta_{\beta\gamma} = \delta^\alpha{}_\gamma$$

$$B^0{}_1 = \eta^{00} B_{01} = -B_{01} = +B_{10} = \eta_{11} B^1{}_0 = B^1{}_0$$

But $B^0{}_0 = B^1{}_1 = 0$ still, so if B is an offdiagonal symmetric matrix, its exponential is a Lorentz transformation preserving spacetime interval

$$e^{\alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\alpha^2}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 + \cdots$$

but $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ etc.

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left(1 + \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} + \cdots\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cosh\alpha$$

$$+ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\alpha + \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} + \cdots\right) \quad + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sinh\alpha$$

$$= \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}$$

"family of boosts along x^1 axis", boost parameter α

Action of a group on a space (10)

So $\begin{pmatrix}\bar{X}^0\\ \bar{X}^1\end{pmatrix} = \underbrace{\begin{pmatrix}\cosh\alpha & \sinh\alpha\\ \sinh\alpha & \cosh\alpha\end{pmatrix}}\begin{pmatrix}X^0\\ X^1\end{pmatrix} = \begin{pmatrix}\cosh\alpha\, X^0 + \sinh\alpha\, X^1\\ \sinh\alpha\, X^0 + \cosh\alpha\, X^1\end{pmatrix}$

determinant: $\cosh^2\alpha - \sinh^2\alpha = 1$

2x2 unit determinant matrices belong to group $SL(2,\mathbb{R})$
"special linear group" in 2-D, special since unit determinant.

Let $\cosh\alpha = \gamma$ (gamma factor)
$\tanh\alpha = \frac{\sinh\alpha}{\cosh\alpha} = v$ speed
so $\sinh\alpha = v\cosh\alpha = \gamma v$

$\frac{\cosh^2\alpha - \sinh^2\alpha}{\cosh^2\alpha} = \frac{1}{\cosh^2\alpha}$

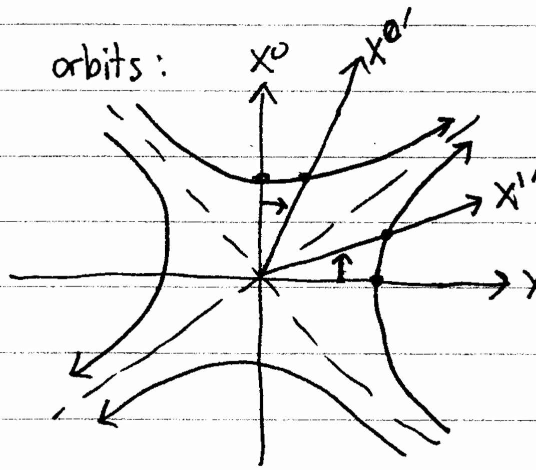
$1 - \underbrace{\tanh^2\alpha}_{v^2} = \frac{1}{\gamma^2}$

$\gamma^2 = \frac{1}{1-v^2}$

$\boxed{\gamma = \frac{1}{\sqrt{1-v^2}}}$ Lorentz gamma factor

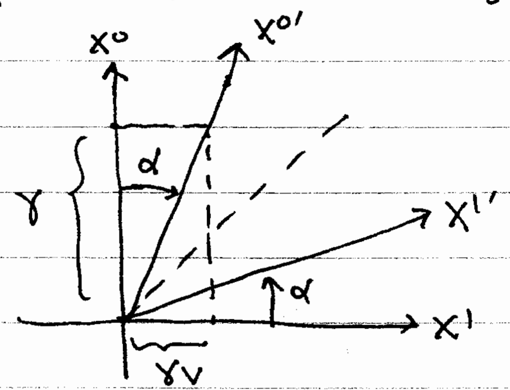
$\begin{pmatrix}\bar{X}^0\\ \bar{X}^1\end{pmatrix} = \begin{pmatrix}\gamma(X^0 + vX^1)\\ \gamma(vX^0 + X^1)\end{pmatrix}$

orbits:



hyperbolas in the plane of constant spacetime interval $(-X^0)^2 + (X^1)^2 = S^2$
distance from the origin in hyperbolic geometry.

new coord axes dragged along $(X^{0\prime}, X^{1\prime})$.



α is not the Euclidean angle but the boost parameter.
$\gamma = \tanh\alpha$
("hyperbolic angle α")

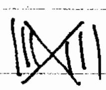
comoving coords
now "pseudo-polar" coords:
if $|X^0| > |X^1|$ (inside "light cone")

$X^0 = \tau\cosh\chi$
$X^1 = \tau\sinh\chi$

$S^2 = (-X^0)^2 + (X^1)^2 = -\tau^2$ = square of proper time interval.

or outside light cone: $|X^0| < |X^1|$

$X^0 = \ell\sinh\chi$
$X^1 = \ell\cosh\chi$



$S^2 = (-X^0)^2 + (X^1)^2 = \ell^2$ = square of proper length interval

along $X^{0\prime}$ the reciprocal slope is
$\frac{dX^1}{dX^0} = \frac{\gamma v}{\gamma} = v$ = speed of worldline of particle moving along $X^{0\prime}$ axis with $X^{1\prime}$ fixed, ie. at rest in new coordinates.

okay, stop. This should wet your appetite, no?

Action of a group on a space (II)

Exercises: (1) Compute the generator ξ for the boosts with parameter α.

(2) compute $[\xi, \partial_i]$, for the only nonzero Lie bracket commutator of the Lie algebra of the 2-D Poincare group of boosts and translations.

~~4D~~ (3) Let $\xi(A) = A^i{}_j x^j \partial_i$

Show $[\xi(A), \xi(B)] = \pm \xi([A,B])$ where $[A,B]^i{}_j = A^i{}_k B^k{}_j - B^i{}_k A^k{}_j$

$\uparrow$ get sign right.

is the matrix commutator $[A,B] = AB - BA$.

This gives an isomorphism between a matrix Lie algebra and a vector field Lie algebra of generators

(4) 4D Lorentz group.

$$(\eta_{\alpha\beta}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (\eta^{\alpha\beta}) \quad \text{ie:} \quad \begin{matrix} \eta_{00} = -1 = \eta^{00} \\ \eta_{ii} = 1 = \eta^{ii} \end{matrix} \qquad \alpha, \beta = 0,1,2,3$$

$$L = e^{tB} \qquad \eta_{\alpha\beta} L^\alpha{}_\gamma L^\beta{}_\delta = \eta_{\gamma\delta} \rightarrow B_{\alpha\beta} = -B_{\beta\alpha}$$

so $B^\alpha{}_\beta = \eta^{\alpha\gamma} B_{\gamma\beta}$.

$\rightarrow$ Show that $B^0{}_i = +B^i{}_0$, $B^i{}_j = -B^j{}_i$ if $i \neq j = 1,2,3$

$$\text{so} \quad B = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & E_1 & E_2 & E_3 \\ 1 & E_1 & 0 & B_3 & -B_2 \\ 2 & E_2 & -B_3 & 0 & B_1 \\ 3 & E_3 & & -B_1 & 0 \end{array}$$

has this form.

Let
$$K_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad K_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad K_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
boost generators

$$S_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad S_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad S_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
rotation generators.

$\rightarrow$ Compute the commutators:

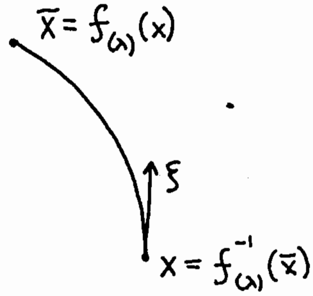
$[S_2, S_3]$ $[K_2, K_3]$ $[S_1, K_1]$

$[S_3, S_1]$ $[K_3, K_1]$ $[S_1, K_2]$

$[S_1, S_2]$ $[K_1, K_2]$ $[S_1, K_3]$

LIE DERIVATIVE

$x^\mu \to \bar{x}^\mu = f^\mu_{(\lambda)}(x)$ 1-parameter family of point transformations ("diffeomorphisms")



$f^\mu_{(0)}(x) = x^\mu$ identity transformation

GENERATING VECTOR FIELD :

$$\xi^\mu(x) \equiv \frac{df^\mu_{(0)}(x)}{d\lambda} \quad , \qquad \xi = \xi^\mu \frac{\partial}{\partial x^\mu}$$

(vector fields identified with differential operators)

power series expansion :

$$\bar{x}^\mu = f^\mu_{(0)}(x) + \lambda \frac{df^\mu_{(0)}}{d\lambda}(x) + \tfrac{1}{2}\lambda^2 \frac{d^2 f^\mu_{(0)}}{d\lambda^2}(x) + \cdots \quad = x^\mu + \lambda \xi^\mu(x) + \cdots$$

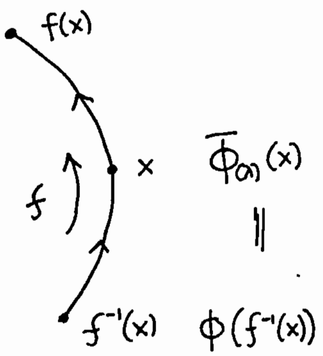
$$\approx x^\mu + \lambda \xi^\mu(x) \quad \text{for } \lambda \ll 1$$

inverse transformation satisfies $f^{-1\,\mu}_{(\lambda)}(f_{(\lambda)}(x)) = x^\mu$, $f^{-1\,\mu}_{(0)}(x) = x^\mu$,
so by chain rule, $\left.\frac{d}{d\lambda}\right|_{\lambda=0}$ of the first equation gives :

$$\frac{df^{-1\,\mu}_{(0)}}{d\lambda}(\underbrace{f_{(0)}(x)}_{x}) + \left.\frac{d}{d\lambda}\right|_{\lambda=0} \underbrace{f^{-1}_{(0)}(f_{(\lambda)}(x))}_{f^\mu_{(\lambda)}(x)} = 0 \to \frac{df^{-1\,\mu}_{(0)}(x)}{d\lambda} - \frac{df^\mu_{(0)}(x)}{d\lambda} = -\,\xi^\mu(x)$$

so a similar power series expansion yields

$$f^{-1\,\mu}_{(\lambda)}(x) = x^\mu - \lambda \xi^\mu(x) + \cdots \quad \approx x^\mu - \lambda \xi^\mu(x) \qquad \lambda \ll 1 .$$



If $\phi(x)$ is a (scalar) function, let $\bar{\Phi}_{(\lambda)}(x)$ be the function transformed by the point transformation

$$\bar{\Phi}_{(\lambda)}(\bar{x}) = \phi(x) \qquad \text{or} \qquad \bar{\Phi}_{(\lambda)}(x) = \phi(f^{-1}_{(\lambda)}(x))$$

new value at new point — old value at old point

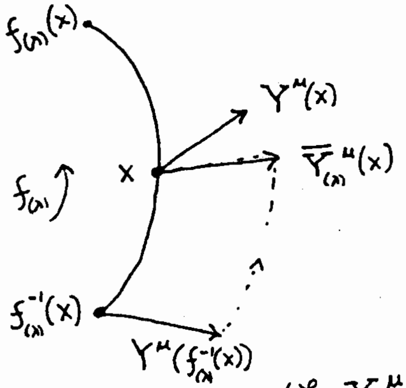
This definition moves the function in the direction of the point transformation.

The rate of change of $\bar{\Phi}_{(\lambda)}$ with respect to λ at $\lambda = 0$ tells how ϕ begins to change under the point transformation and defines the negative of the Lie derivative of ϕ with respect to the generating vector field

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} \bar{\Phi}_{(\lambda)}(x) = \frac{\partial \phi}{\partial x^\mu}(f^{-1}_{(0)}(x)) \frac{df^{-1\,\mu}_{(0)}}{d\lambda}(x) = -\,\xi^\mu(x) \frac{\partial \phi}{\partial x^\mu}(x) = -\,\xi(x)\,\phi$$

$$\pounds_\xi \phi \equiv -\left.\frac{d}{d\lambda}\right|_{\lambda=0} \bar{\Phi}_{(\lambda)} = \xi \phi = \xi^\mu \frac{\partial}{\partial x^\mu} \phi = \phi_{,\mu} \xi^\mu$$

The Lie derivative of a scalar by a vector field ξ is just the directional derivative of the scalar along that vector field



A vector field is transformed by the point transformation as follows:

$$\underbrace{\bar{Y}^\mu_{(\lambda)}(x)}_{\text{value at } x} = \underbrace{\overbrace{\frac{\partial f^\mu_{(\lambda)}}{\partial x^\nu}(f^{-1}_{(\lambda)}(x))}^{\text{jacobian}}}_{\text{this maps tanspace at } f^{-1}(x) \text{ to tan space at } x} \underbrace{Y^\nu(f^{-1}_{(\lambda)}(x))}_{\text{value at point mapped onto } x \text{ by } f_{(\lambda)}}$$

see note on next page

Now calculate its Lie derivative exactly as for the scalar:

$$(\mathcal{L}_\xi Y^\mu)(x) = -\frac{d}{d\lambda}\Big|_{\lambda=0} \bar{Y}^\mu_{(\lambda)}(x)$$

$$= -\Big[\underbrace{\frac{\partial f^\mu_{(0)}}{\partial x^\nu}(\underbrace{f^{-1}_{(0)}(x)}_{x})}_{\frac{\partial x^\mu}{\partial x^\nu}=\delta^\mu{}_\nu}\Big] \underbrace{\frac{\partial Y^\nu}{\partial x^\rho}(f^{-1}_{(0)}(x))\,\frac{d f^{-1\rho}_{(0)}}{d\lambda}(x)}_{\underbrace{-\xi^\rho(x)\,\frac{\partial Y^\nu}{\partial x^\rho}(x)}_{\text{just like in scalar case}}} - \frac{\partial}{\partial x^\nu}\Big[\underbrace{\frac{d f^\mu_{(0)}}{d\lambda}}_{\xi^\mu}\Big](f^{-1}_{(0)}(x))\; Y^\nu(\underbrace{f^{-1}_{(0)}(x)}_{x})$$

$$= \Big[\underbrace{\xi^\rho \frac{\partial Y^\nu}{\partial x^\rho}}_{\text{directional derivative of components}} - \underbrace{\frac{\partial \xi^\mu}{\partial x^\nu} Y^\nu}_{\text{extra from changing components}}\Big](x)$$

compared to vector transformation law

$$\boxed{\mathcal{L}_\xi \phi = \phi_{,\rho}\,\xi^\rho}$$

$$\boxed{\mathcal{L}_\xi Y^\mu = Y^\mu{}_{,\rho}\,\xi^\rho - \xi^\mu{}_{,\rho}\,Y^\rho}$$

$$\boxed{\mathcal{L}_\xi Z_\mu = Z_{\mu,\rho}\,\xi^\rho + Z_\rho\,\xi^\rho{}_{,\mu}}$$

$$\boxed{\mathcal{L}_\xi g_{\mu\nu} = \underbrace{g_{\mu\nu,\rho}}_{\Gamma_{\nu\mu\rho}+\Gamma_{\mu\nu\rho}}\xi^\rho + g_{\rho\nu}\,\xi^\rho{}_{,\mu} + g_{\mu\rho}\,\xi^\rho{}_{,\nu}}$$

If we do the same thing for a covariant vector field

$$\bar{Z}^{(\lambda)}_\mu = \underbrace{\overbrace{\frac{\partial f^{-1\,\nu}_{(\lambda)}}{\partial x^\mu}}^{\text{inverse jacobian}}}(x)\, Z_\nu(f^{-1}_{(\lambda)}(x))$$

the λ-derivative of this term leads instead to $-\xi^\nu{}_{,\mu} Z_\nu$ so we get

For the metric we get one of these terms for each covariant index, but always the first term is just the directional derivative of the components

The expression for the $\mathcal{L}_\xi g_{\mu\nu}$ can be rewritten using

to yield:

$$\mathcal{L}_\xi g_{\mu\nu} = g_{\rho\nu}\,\xi^\rho{}_{;\mu} + g_{\mu\rho}\,\xi^\rho{}_{;\nu} = (g_{\rho\nu}\xi^\rho)_{;\mu} + (g_{\mu\rho}\xi^\rho)_{;\nu}$$

$$= \xi_{\nu;\mu} + \xi_{\mu;\nu} \qquad (\text{since } g_{\rho\nu;\sigma}=0)$$

We'll get to this later

* Note that $\mathcal{L}_\xi Y = [\xi, Y] = -\mathcal{L}_Y \xi$, where $[\,,]$ is the commutator of the vector fields as differential operators.

top of page 11 Lie derivative explanation in terms of dragged along coords

active transformation

$x^\mu \to \bar{x}^\mu = f^\mu_{(\lambda)}(x)$ λ = parameter of 1-parameter family of transformations.

passive transformation:

$$\underbrace{x^{\mu'}(x)}_{\text{new coords at } x} = \underbrace{x^\mu(f^{-1}_{(\lambda)}(x))}_{\text{old coords at inverse pt } f^{-1}_{(\lambda)}(x)} = x^\mu(f_{(-\lambda)}(x)) = f^\mu_{(-\lambda)}(x)$$

μth component of functions $f^\mu_{(-\lambda)}(x)$

inverse parameter

so $\dfrac{\partial x^{\mu'}}{\partial x^\nu}(x) = \dfrac{\partial f^\mu_{(-\lambda)}(x)}{\partial x^\nu}$ Jacobian.

and inverse $\dfrac{\partial x^\nu}{\partial x^{\mu'}}(x) = \dfrac{\partial f^\nu_{(\lambda)}(x)}{\partial x^\mu}$ inverse corresponds to $\lambda \to -\lambda$

evaluated at $f^\mu_{(-\lambda)}(x)$:

$$\frac{\partial x^\nu}{\partial x^{\mu'}}(f^\mu_{(-\lambda)}(x)) = \frac{\partial f^\nu_{(\lambda)}(f^\mu_{(-\lambda)}(x))}{\partial x^\mu}$$

so changing coords at old point of a vector there

$$Y^\mu(f_{(-\lambda)}(x)) = \frac{\partial x^\nu}{\partial x^{\mu'}}(f^\mu_{(-\lambda)}(x))\, Y^{\mu'}(f^\mu_{(-\lambda)}(x))$$

but the transformed components of the field at the inverse point define the components at x of a new field, the dragged along field at x.

$$\bar{Y}^\mu_{(\lambda)}(x) = Y^\mu(f_{(-\lambda)}(x)) = \ldots$$

which is the equation which starts page 11.

page 11 insert

In the March notes, I neglected to talk explicitly about the invariance of a field under a transformation.
In the case of a 1-parameter family of point transformations, invariance of a tensor field $T^{\mu\ldots}{}_{\nu\ldots}$ means that

$$\bar{T}_{(\lambda)}{}^{\mu\ldots}{}_{\nu\ldots} = T^{\mu\ldots}{}_{\nu\ldots}$$

the transformed field equals the original field for all λ, hence taking the λ-derivative at $\lambda=0$, one has vanishing Lie derivative:

$$\pounds_\xi T^{\mu\ldots}{}_{\nu\ldots} = 0.$$

For a metric, invariance means:

later: covariant derivative.

$$\bar{g}_{(\lambda)\,\mu\nu} = g_{\mu\nu}$$

$$\pounds_\xi g_{\mu\nu} = 0. \qquad \text{or} \qquad 0 = \xi_{(\mu;\nu)}.$$

ξ is called a Killing vector field, and the equation Killing's equation. Its solutions are the generators of the full group of motions of the metric. We evaluated explicity the Killing vectors for the flat spaces of arbitrary signature, and consequently, for the imbedded pseudospheres, homogeneous and isotropic spaces with maximum symmetry.

later

On a group G, the generators of right translations $\{e_a\}$ and the generators of left translations $\{\tilde{e}_a\}$ commute since the left translations commute with the right translations:

$[e_a,\tilde{e}_b]=0$: $\rightarrow$ $\pounds_{\tilde{e}_b} e_a = 0$ means $\{e_a\}$ are left invariant

$\rightarrow$ $\pounds_{e_a} \tilde{e}_b = 0$ means $\{\tilde{e}_a\}$ are right invariant

CONSTANTS OF THE MOTION FOR GEODESICS

later when we do covariant derivatives.

A very useful property of Killing vector fields is that each independent KVF yields a conserved momentum for a geodesic.

Suppose $x^\mu = x^\mu(\tau)$ is a timelike geodesic parametrized by the proper time τ. The unit four-velocity $u^\mu = dx^\mu(\tau)/d\tau$ satisfies $\frac{Du^\mu}{d\tau} \equiv u^\mu{}_{;\nu} u^\nu = 0$, where $\frac{D}{dt}$ = "$;_\nu u^\nu$" is the covariant derivative along the tangent.

If ξ^μ is a KVF, then the momentum like quantity $p = \xi_\mu u^\mu$, a sort of component of the velocity along the symmetry direction, is conserved:

only symmetric part contributes

$$\frac{D}{d\tau}(\xi_\mu u^\mu) = (\xi_\mu u^\mu)_{;\nu} u^\nu = \xi_\mu \underbrace{(u^\mu{}_{;\nu} u^\nu)}_{\substack{=0 \\ \text{(geodesic)}}} + \underbrace{\xi_{(\mu;\nu)}}_{\substack{=0 \\ \text{(Killing eq)}}} u^\mu u^\nu = 0$$

If ξ^μ is timelike, then $-p$ can be interpreted as an energy, and if instead spacelike, as some kind of momentum (linear or angular).

Action of a group on a space

short exercise.

Suppose we have a linear group of transformations

$$X^\alpha \to \bar{X}^\alpha = A^\alpha{}_\beta X^\beta$$

If this leaves a metric invariant then

$$\bar{g}_{\alpha\beta} = g_{\gamma\delta} A^{-1\gamma}{}_\alpha A^{-1\delta}{}_\beta = g_{\alpha\beta}.$$

If the matrix generators are $A = e^B$

$$\xi(B) = B^\alpha{}_\beta X^\beta \partial_\alpha$$

then $\xi(B)^\alpha = B^\alpha{}_\beta X^\beta$

$$\xi(B)^\alpha{}_{,\beta} = B^\alpha{}_\beta$$

Now evaluate $\mathcal{L}_{\xi(B)} g_{\mu\nu}$ using the formula on page 11 and use index lowering notation $B_{\alpha\beta} = g_{\alpha\gamma} B^\gamma{}_\beta$ for the case of a constant metric $g_{\mu\nu,\alpha} = 0$ corresponding to a global inner product on the space like the Euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$ of $\mathbb{R}^n$ or the Lorentz metric $\eta_{\alpha\beta}$ of 4-D spacetime.

What condition does the Killing equation

$$\mathcal{L}_{\xi(B)} g_{\mu\nu} = 0$$

place on the covariant components of B?