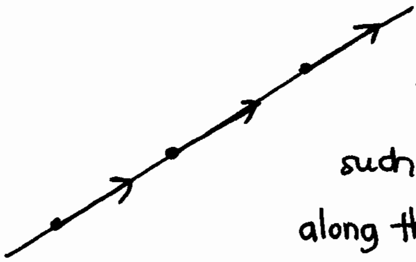
"Straight lines" = autoparallel curves = "geodesics"



How can we characterize the straight lines of $\mathbb{R}^n$? The tangent vector "follows its nose" as it moves along such a line. The tangent vector itself is parallel transported along the line. This condition is

$$\frac{Dc'(\lambda)}{d\lambda} = 0$$

or

$$0 = \frac{Dc^{i\prime}(\lambda)}{d\lambda} = \frac{dc^{i\prime}(\lambda)}{d\lambda} + \Gamma^i{}_{kj} c^{k\prime}(\lambda)\, c^{j\prime}(\lambda)$$

$$= \frac{d^2c^i(\lambda)}{d\lambda} + \Gamma^i{}_{kj}\, c^{k\prime}(\lambda)\, c^{j\prime}(\lambda)$$

$$= \frac{d^2c^i(\lambda)}{d\lambda} + \Gamma^i{}_{kj}\circ c(\lambda)\, \frac{dc^k(\lambda)}{d\lambda}\frac{dc^j(\lambda)}{d\lambda}$$

$$\left(\text{"}\frac{D^2x^i}{d\lambda^2}\text{"}\right) \quad = \text{"}\,\frac{d^2x^i}{d\lambda^2} + \Gamma^i{}_{kj}\frac{dx^k}{d\lambda}\frac{dx^j}{d\lambda}\,\text{"}$$

if one uses $x^i = c^i(\lambda)$ to replace $c^i(\lambda)$

In Cartesian coordinates where $\Gamma^i{}_{kj} = 0$, this reduces to

$$\frac{d^2x^i}{d\lambda^2} = 0 \quad \text{with solution} \quad x^i = a^i\lambda + b$$

or more carefully $\quad c^i(\lambda) = a^i\lambda + b$.

The Cartesian coordinates are linear functions of the parameter, while the Cartesian components of the tangent vector are constants $\quad c^{i\prime}(\lambda) = a^i$.

On the previous page we saw that the tangent vector $c'(\lambda) = e_r$ to the r coordinate lines in their natural parametrization $\lambda = r$ is parallel transported along them. This is clear since these coordinate lines are straight halflines. Since "$\frac{dx^k}{d\lambda}$" $= \delta^k{}_r$ for these curves in spherical coordinates, one can write

$$\frac{D^2x^k}{d\lambda^2} = \frac{d^2x^i}{d\lambda^2} + \Gamma^i{}_{rr} = \Gamma^i{}_{rr} = 0$$

in this "sloppy notation!"

A parametrized curve whose tangent vector is parallel transported along the curve is called a GEODESIC.

__exercise.__ Using the cylindrical coordinate expressions for the components of the covariant derivative given on page 58 or 87, verify that the ρ and z coordinate lines are geodesics (i.e. straight lines).

A "curve" is a set of points with no parametrization, described geometrically or as a solution of a set of equations. A curve is a geodesic if it admits a parametrization which is a geodesic. This generalizes our previous definition of a geodesic. We saw that the previous definition led to straight lines parametrized <u>linearly</u> in Cartesian coordinates, but nonlinear parametrizations can also be used.

Suppose $c(\lambda)$ satisfies $\dfrac{Dc'(\lambda)}{d\lambda} = 0$ and we consider a new parametrization

$$\mathcal{C}(\lambda) = c(f(\lambda))$$

of the curve, where $f(\lambda)$ is a real valued function of λ.

Then $\qquad \mathcal{C}'(\lambda) = f'(\lambda)\, c'(f(\lambda))$ $\qquad$ is the new tangent vector, multiplied by the function $f'(\lambda)$ by the chain rule.

Its covariant derivative along the curve $\mathcal{C}(\lambda)$ is

$$\frac{D\mathcal{C}^{i\,\prime}(\lambda)}{d\lambda} = \underbrace{\frac{d}{d\lambda}\underbrace{\mathcal{C}^{i\,\prime}(\lambda)}_{f'(\lambda)c^{i\,\prime}(f(\lambda))}}_{f''(\lambda)\,c^{i\,\prime}(f(\lambda)) + [f'(\lambda)]^2 c^{i\,\prime\prime}(f(\lambda))} + \Gamma^i{}_{kj}\underbrace{\mathcal{C}^{k\,\prime}(\lambda)\,\mathcal{C}^{j\,\prime}(\lambda)}_{[f'(\lambda)]^2\, c^{k\,\prime}(f(\lambda))\, c^{j\,\prime}(f(\lambda))}$$

$$= [f'(\lambda)]^2 \Big[\underbrace{c^{i\,\prime\prime}(f(\lambda)) + \Gamma^i{}_{kj}c^{k\,\prime}(f(\lambda))c^{j\,\prime}(f(\lambda))}_{\left[\frac{Dc^{i\,\prime}}{d\lambda}\right](f(\lambda)) \;=\; 0}\Big] + f''(\lambda)\,\underbrace{c^{i\,\prime}(f(\lambda))}_{\frac{\mathcal{C}^{i\,\prime}(\lambda)}{f'(\lambda)}}$$

$$= \frac{f''(\lambda)\,\mathcal{C}^{i\,\prime}(\lambda)}{f'(\lambda)}, \quad \text{ie} \quad \boxed{\frac{D\mathcal{C}'(\lambda)}{d\lambda} = \left[\frac{f''(\lambda)}{f'(\lambda)}\right]\mathcal{C}'(\lambda)}$$

Instead of being zero, the covariant derivative of the tangent vector is proportional to the tangent vector. It still "follows its nose" but changes its length as it moves due to the arbitrary parametrization.

A parametrization for which $\frac{Dc'(\lambda)}{d\lambda} = 0$ is called an <u>affine parametrization</u>. of the geodesic. For such a parametrization

$$\frac{D}{d\lambda} \underbrace{\|c'(\lambda)\|^2}_{\substack{\text{length}\\ \text{squared}}} = \frac{D}{d\lambda}\left(g_{ij} c^{i'} c^{j'}\right) = \underbrace{\left(\frac{Dg_{ij}}{d\lambda}\right)}_{0} c^{i'} c^{j'} + g_{ij} \underbrace{\frac{Dc^{i'}}{d\lambda}}_{\overset{\shortparallel}{0}} c^{j'} + g_{ij} c^{i'} \underbrace{\frac{Dc^{j'}}{d\lambda}}_{\overset{\shortparallel}{0}} = 0$$

$$\text{so} \quad \frac{D}{d\lambda} \|c'(\lambda)\| = \frac{D}{d\lambda}\left(\|c'(\lambda)\|^2\right)^{1/2} = \underbrace{\frac{1}{2}\left(\|c'(\lambda)\|^2\right)^{-1/2}}_{\frac{1}{2}\frac{1}{\|c'(\lambda)\|}} \underbrace{\frac{D}{d\lambda}\left(\|c'(\lambda)\|^2\right)}_{=0} = 0$$

i.e., the length of the tangent vector is constant. We already know this since parallel transport preserves length:

$$\frac{D}{d\lambda} \underbrace{\|c'(\lambda)\|}_{\text{function}} = \frac{d}{d\lambda} \|c'(\lambda)\| = 0$$

If we introduce the arclength parametrization as we did in calculus

$$\frac{ds}{d\lambda} = \|c'(\lambda)\|$$

this just says $\quad \frac{d^2s}{d\lambda^2} = \frac{d}{d\lambda} \|c'(\lambda)\| = 0$,

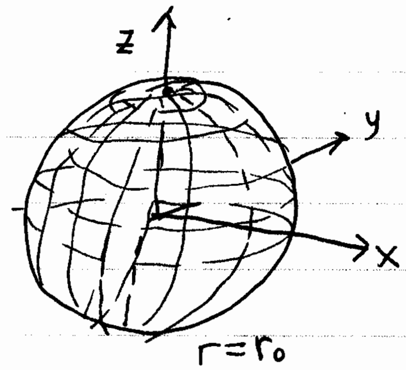
i.e. the arclength and parameter λ are linearly related.

If we use the arclength itself as a parameter we must have

$$\frac{ds}{d\lambda} = \|c'(\lambda)\| = 1,$$

so the tangent vector is a unit vector as we recall from calculus.

The 2-sphere of radius r_0



The Euclidean metric in spherical coordinates is

$$g = dr\otimes dr + r^2 d\theta\otimes d\theta + r^2\sin^2\theta\, d\varphi\otimes d\varphi$$

with coordinate frame $e_r = \frac{\partial}{\partial r},\ e_\theta = \frac{\partial}{\partial\theta},\ e_\varphi = \frac{\partial}{\partial\varphi}$

and dual frame $\omega^r = dr,\ \omega^\theta = d\theta,\ \omega^\varphi = d\varphi$ or

$$g = \omega^{\hat r}\otimes\omega^{\hat r} + \omega^{\hat\theta}\otimes\omega^{\hat\theta} + \omega^{\hat\varphi}\otimes\omega^{\hat\varphi}$$

in the associated orthonormal frame

$$e_{\hat r} = \frac{\partial}{\partial r},\quad e_{\hat\theta} = \frac{1}{r}\frac{\partial}{\partial\theta},\quad e_{\hat\varphi} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\ ;\ \omega^{\hat r} = dr,\ \omega^{\hat\theta} = r\,d\theta,\ \omega^{\hat\varphi} = r\sin\theta\, d\varphi.$$

The coordinate surface $r = r_0$ is a sphere of radius r_0 on which $\{\theta, \varphi\}$ serve as local coordinates with a coordinate singularity at the poles $\theta = 0, \pi$ where φ is undefined. We can live with this problem as long as we are careful.

The 2-sphere is a space in its own right and we can use all the machinery we have developed for $\mathbb{R}^n$ in general coordinate systems to study it. We can also picture the 2-dimensional tangent spaces to the ~~2-~~sphere as subspaces of the full 3-dimensional tangent space of $\mathbb{R}^3$ at each point. The sphere has "intrinsic" or internal geometry of a 2-dimensional nature, plus "extrinsic" or external geometry that has to do with how it sits in the larger space. For example, a cylinder locally has the same flat 2-dimensional geometry as a plane (cut it along a seam & roll it out), but clearly it is bent as a subspace of $\mathbb{R}^3$. To study the intrinsic geometry we simply use the 2-dimensional coordinate system & calculate as though we were studying $\mathbb{R}^2$ in nonCartesian coordinates.

The metric of the sphere tells us the inner products of the frame vectors e_θ, e_φ or $e_{\hat\theta}, e_{\hat\varphi}$:

$$ {}^{(2)}g = g\Big|_{\substack{r=r_0\\ dr=0}} = r_0^2\, d\theta\otimes d\theta + r_0^2 \sin^2\theta\, d\varphi\otimes d\varphi $$
$$ = \underset{\uparrow}{r_0^2}\, \underbrace{\left[d\theta\otimes d\theta + \sin^2\theta\, d\varphi\otimes d\varphi \right]}_{\text{metric for unit sphere } r=1} $$

scaling factor to get metric of sphere of radius $r=r_0$

Now there are no background ~~cover~~ 2-dimensional Cartesian coordinates to use in ~~defining~~ components of the connection. We can only use the general formula

$$ \Gamma^\alpha{}_{\beta\gamma} = \tfrac12 g^{\alpha\delta}\left(g_{\delta\beta,\gamma} - g_{\beta\gamma,\delta} + g_{\gamma\delta,\beta} + C_{\delta\beta\gamma} - \cancel{C_{\beta\gamma\delta}} + C_{\gamma\delta\beta} \right) $$
$$ \alpha,\beta = 1,2 $$

applied to either the coordinate frame or the associated orthonormal frame and using the 2-dimensional metric ${}^{(2)}g$. We have already evaluated all of these components in the 3-dimensional context of spherical coordinates. All we have to do is confine our attention to the components with no radial indices and set $r=r_0$ in their expressions. From page 99 (pages 83 and 75

$$ \Gamma^\theta{}_{\varphi\varphi} = -\cos\theta \overset{\sin\theta}{\cancel{1}},\quad \Gamma^\varphi{}_{\varphi\theta} = \cot\theta $$
only nonzero ones for coordinate frame $\{e_\theta, e_\varphi\}$

$$ \Gamma^{\hat\theta}{}_{\hat\varphi\hat\varphi} = -r_0^{-1}\cot\theta = -\Gamma^{\hat\varphi}{}_{\hat\varphi\hat\theta} $$
ditto for orthonormal frame $\{e_{\hat\theta}, e_{\hat\varphi}\}$.

(copied wrong on p.99)

The latter components correspond to the antisymmetric matrix

$$ \underline{\hat\omega} = (\omega^{\hat\theta}{}_{\hat\varphi}\ \omega^{\hat\varphi}{}_{\hat\theta}) = \begin{pmatrix} 0 & -\cos\theta \\ \cos\theta & 0 \end{pmatrix} d\varphi $$

This just tells us that as we move along the φ coordinate lines, the orthonormal frame vectors begin to rotate with respect to parallel transported vectors on the sphere. Near the north pole $\theta\approx 0$, this rotate is nearly a rotation by the angle φ

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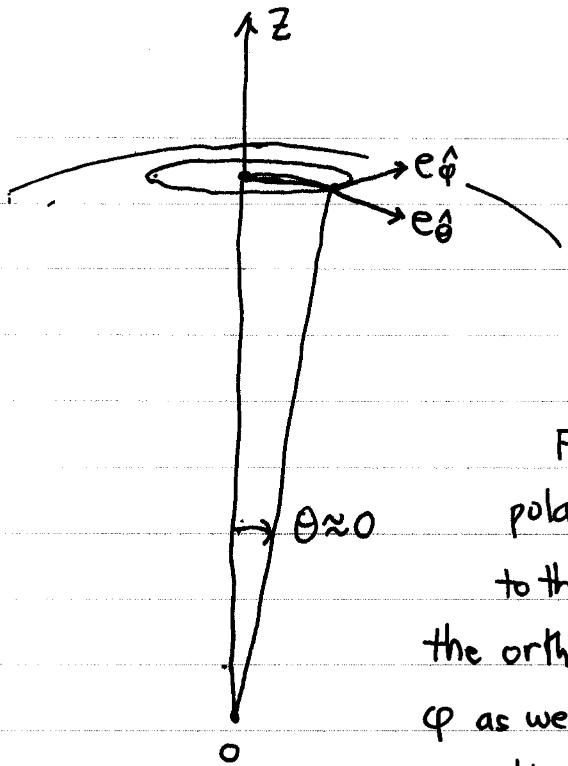
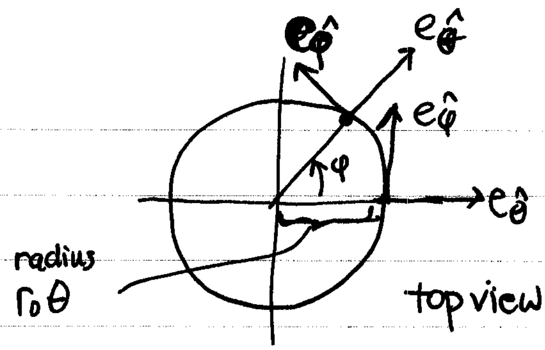
Litho in U.S.A.





For very small θ, this picture looks just like the polar coordinate frame in the plane $z = r_0$ tangent to the sphere at the North pole. You can see that the orthonormal frame rotates by (approximately) the angle φ as we move around the φ coordinate circle, which is exactly what the antisymmetric matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ describes the rate of change of as we begin moving from any particular φ value.

Parallel transported vectors around this circle, however, try to maintain their direction with respect to the enveloping $\mathbb{R}^3$ as best they can while still remaining tangent to the sphere.

In general, the components of the covariant derivative tell us how to parallel transport vectors along the $\{\theta, \varphi\}$ coordinate lines. In particular the θ coordinate circles (great circles through the poles) and the single φ coordinate circle $\theta = \frac{\pi}{2}$ (the equator) are all great circles which we know to be geodesics on the sphere.

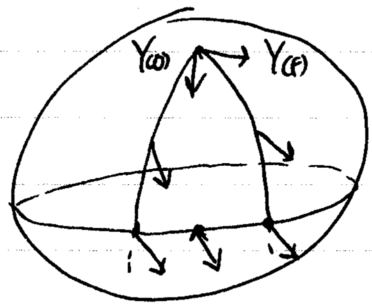
$\nabla_{e_{\hat\theta}} e_{\hat\alpha} = \underbrace{\Gamma^{\hat\delta}{}_{\hat\theta\hat\alpha}}_{=0} e_{\hat\delta} = 0$ tells us that the orthonormal frame is covariant constant along $e_{\hat\theta}$, i.e., is parallel transported along the θ coordinate lines which have $e_{\hat\theta}$ as a unit tangent [so these are geodesics]

$\nabla_{e_{\hat\varphi}} e_{\hat\theta} = +r_0^{-1} \cot\theta \, e_{\hat\varphi}$

$\nabla_{e_{\hat\varphi}} e_{\hat\varphi} = \underbrace{r_0^{-1} \cot\theta}_{=0 \text{ at } \theta = \frac{\pi}{2}} e_{\hat\theta}$

tells us that same thing is true for the orthonormal frame along the equator. [$\frac{De_{\hat\varphi}}{d\lambda} = 0$ along equator $\leftrightarrow$ geodesic]

Thus suppose $Y_{(0)} = \left[Y_{(0)}^{\hat\theta}\, e_{\hat\theta} + Y_{(0)}^{\hat\varphi}\, e_{\hat\varphi}\right]\Big|_{\theta=0}$ is some tangent vector at the north pole. If we parallel transport it down a line of longitude, its orthonormal components remain constants, ie. it maintains its length and its angle with the line of longitude. The same remains true as we move along the equator, and then back up ~~the~~ line of longitude to the North pole again, resulting in the final tangent vector $Y_{(F)}$ which has rotated by the increment in φ between the two lines of longitude, exactly as we described from intuition.



Notice that this tells us that the 2-sphere cannot admit a covariant constant vector field, since if it did, it would coincide with its parallel transport along every such loop and thus the final value would have to equal the original value of the North pole, which we have just shown will not happen in general.

<u>exercise.</u> Suppose we do an analogous discussion with a cylinder $\rho = \rho_0$ in cylindrical coordinates $\{\rho, \varphi, z\}$ on $\mathbb{R}^3$. Then $\{\varphi, z\}$ are local coordinates on this 2-dimensional space.

What are the nonvanishing components of the connection in the coordinate and associated orthonormal frame?

Do covariant constant vector fields exist?

Does a covariant constant ~~for~~ orthonormal frame exist?

Write out the geodesic equations

$$0 = \frac{D^2\varphi}{d\lambda^2} = \dots \,, \quad 0 = \frac{D^2 z}{d\lambda^2} \,.$$

Can you solve these? Can you explain the solutions?

Describing Intrinsic Curvature

$\mathbb{R}^n$ with its Euclidean metric is flat. The 2-sphere is not. How do we describe this mathematically? We need to introduce a quantity that will be called curvature and show how it agrees with our vague intuitive idea of curvature.

We will introduce the "Riemann" curvature tensor in several steps.

① Cartesian coordinates on $\mathbb{R}^n$, where covariant = ordinary differentiation, so $[\nabla_Y Z]^i = Z^i{}_{;j} Y^j = Z^i{}_{,j} Y^j$, hence

$$
\begin{aligned}
(\nabla_X \nabla_Y - \nabla_Y \nabla_X) Z^i &= [\nabla_Y Z]^i{}_{,k} X^k - [\nabla_X Z]^i{}_{,k} Y^k \\
&= [Z^i{}_{,j} Y^j]_{,k} X^k - [Z^i{}_{,j} X^j]_{,k} Y^k \qquad \text{expand using product rule} \\
&= Z^i{}_{,jk} Y^j X^k - Z^i{}_{,jk} X^j Y^k \qquad \text{switch dummy indices on } X, Y \text{ in this term, use distributive law on first 2 terms.} \\
&\quad + \underbrace{Z^i{}_{,j} Y^j{}_{,k} X^k - Z^i{}_{,j} X^j{}_{,k} Y^k}_{\text{common factor on last 2 terms, use distributive law}} \\
&= \underbrace{[Z^i{}_{,jk} - Z^i{}_{,kj}]}_{\text{0 since partial derivatives commute}} X^k Y^j + \underbrace{Z^i{}_{,j} \underbrace{[Y^j{}_{,k} X^k - X^j{}_{,k} Y^k]}_{[X,Y]^j}}_{[\nabla_{[X,Y]} Z]^i}
\end{aligned}
$$

Thus $\underbrace{[\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}]}_{\text{Second order differential operator,}} Z \equiv 0$

In other words this second order differential operator on vector fields is identically zero on $\mathbb{R}^n$. This will be true no matter what coordinates we use to express the operator.

② Coordinate frame calculation for arbitrary coordinates on any space. We just have to include the components of the connection.

$$[\nabla_Y Z]^i = (Z^i{}_{,j} + \Gamma^i{}_{jm} Z^m) Y^j$$

$$[\nabla_X \nabla_Y Z]^i = \left([\nabla_Y Z]^i{}_{,k} + \Gamma^i{}_{km} [\nabla_Y Z]^m \right) X^k$$

$$= \Big(\underbrace{[(Z^i{}_{,j} + \Gamma^i{}_{jm} Z^m) Y^j]}_{\text{expand using product rule}}{}_{,k} + \Gamma^i{}_{km} (Z^m{}_{,j} + \Gamma^m{}_{jp} Z^p) Y^j \Big) X^k$$

$$= \Big\{ [\underset{①}{Z^i{}_{,jk}} + \underset{②}{\Gamma^i{}_{jm} Z^m{}_{,k}} + \underset{③}{\Gamma^i{}_{jm,k} Z^m}] Y^j + \underbrace{(Z^i{}_{,j} + \Gamma^i{}_{jm} Z^m)}_{Z^i{}_{;j} \ ④} Y^j{}_{,k}$$

$$+ \Gamma^i{}_{km} (\underset{⑤}{Z^m{}_{,j}} + \underset{⑥}{\Gamma^m{}_{jp} Z^p}) Y^j \Big\} X^k$$

$$= [\underset{①}{Z^i{}_{,jk}} + \underset{②}{\Gamma^i{}_{jm} Z^m{}_{,k}} + \underset{⑤}{\Gamma^i{}_{km} Z^m{}_{,j}} + \underset{③}{\Gamma^i{}_{jm,k} Z^m} + \underset{⑥}{\Gamma^i{}_{km} \Gamma^m{}_{jp} Z^p}] X^k Y^j$$

$$+ \underset{④}{Z^i{}_{;j} \, Y^j{}_{,k} X^k}$$

symmetric in (jk)

2 terms symmetric in (jk)

switch to n switch to m

Now if we switch X and Y in these formulas

$$[\nabla_X \nabla_Y Z]^i = [\cdots]^i{}_{jk} X^k Y^j + \cdots$$

$$[\nabla_Y \nabla_X Z]^i = \underbrace{[\cdots]^i{}_{jk} Y^k X^j}_{\text{switch dummy indices in these terms}} + \cdots = [\cdots]^i{}_{kj} X^k Y^j + \cdots$$

so $$[\nabla_X \nabla_Y Z]^i - [\nabla_Y \nabla_X Z]^i = \underbrace{([\cdots]^i{}_{jk} - [\cdots]^i{}_{kj})}_{\text{terms symmetric in } (jk) \text{ cancel out}} X^k Y^j + \cdots$$

$$[\nabla_X \nabla_Y - \nabla_Y \nabla_Z] Z^i = [\underset{③}{\Gamma^i{}_{jm,k}} - \underset{③}{\Gamma^i{}_{km,j}} + \underset{⑥}{\Gamma^i{}_{kn} \Gamma^n{}_{jm}} - \underset{⑥}{\Gamma^i{}_{jn} \Gamma^n{}_{km}}] X^k Y^j Z^m$$

$$+ \underbrace{Z^i{}_{;j} [Y^j{}_{,k} X^k - X^j{}_{,k} Y^k]}_{[\nabla_{[X,Y]} Z]^i}$$

so the result is

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}) Z^i = R^i{}_{mkj} X^k Y^j Z^m$$

where $R^i{}_{mkj} = \Gamma^i{}_{jm,k} - \Gamma^i{}_{km,j} + \Gamma^i{}_{kn}\Gamma^n{}_{jm} - \Gamma^i{}_{jn}\Gamma^n{}_{km}$.

This operator is actually linear in X, Y, and Z, i.e., defines the components of a tensor field

$$R = R^i{}_{mkj}\, e_i \otimes \omega^m \otimes \omega^k \otimes \omega^j$$

which is explicitly antisymmetric in its last pair of indices.
It is called the Riemann curvature tensor and the previous calculation shows that in $\mathbb{R}^n$ with the Euclidean metric, this curvature tensor is identically zero.

③ Calculation in an arbitrary frame.

The components of this tensor in an arbitrary frame are

$R^i{}_{mkj} = R(\omega^i, e_m, e_k, e_j)$ or

$$[\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} - \nabla_{[e_i,e_j]}] e_k = R^\ell{}_{kij}\, e_\ell$$

but $\nabla_{e_j} e_k = \Gamma^\ell{}_{jk} e_\ell$

$$\nabla_{e_i}\nabla_{e_j} e_k = \nabla_{e_i}(\Gamma^\ell{}_{jk} e_\ell) = (\nabla_{e_i}\underbrace{\Gamma^\ell{}_{jk}}_{\text{just functions}}) e_\ell + \underbrace{\Gamma^\ell{}_{jk} \underbrace{\nabla_{e_i} e_\ell}_{\Gamma^m{}_{i\ell} e_m}}_{\text{switch } \ell \text{ and } m}$$

$$= (\Gamma^\ell{}_{jk,i} + \Gamma^\ell{}_{im}\Gamma^m{}_{jk}) e_\ell$$

and $\nabla_{[e_i,e_j]} e_k = \nabla_{C^m_{ij} e_m} e_k = C^m_{ij} \nabla_{e_m} e_k$ by linearity

$$= C^m_{ij}\Gamma^\ell{}_{mk}\, e_\ell$$

so

$$[\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} - \nabla_{[e_i,e_j]}] e_k = \underbrace{(\Gamma^\ell{}_{jk,i} - \Gamma^\ell{}_{ik,j} - C^m_{ij}\Gamma^\ell{}_{mk} + \Gamma^\ell{}_{im}\Gamma^m{}_{jk} - \Gamma^\ell{}_{jm}\Gamma^m{}_{ik})}_{R^\ell{}_{kij}} e_\ell$$

The only difference is the extra structure function term which appears compared to the coordinate frame calculation.

We can lower the first index on the curvature tensor to obtain a $\binom{0}{4}$-tensor with components $R_{mnij} = g_{m\ell} R^{\ell}{}_{nij}$.

One can show that this tensor is antisymmetric in its first pair of indices as well as its second as well as under the interchange of the two pairs:

$$R_{nmij} = -R_{mnij}$$
$$R_{mnji} = -R_{mnij}$$
$$R_{mnij} = R_{ijmn}.$$

For a 2-dimensional space, it has therefore at most one independent nonzero component R_{1212}.

In an orthogonal frame this means that $R^{1}{}_{212}$ is the single independent component.

<u>exercise</u> For the 2-dimensional spaces

1) cylindrical coordinates $\{\rho, \varphi\}$ on the plane $z=0$ $[{}^{(2)}g = d\rho\otimes d\rho + \rho^2 d\varphi\otimes d\varphi]$

2) spherical coordinates on the sphere $r=r_0$

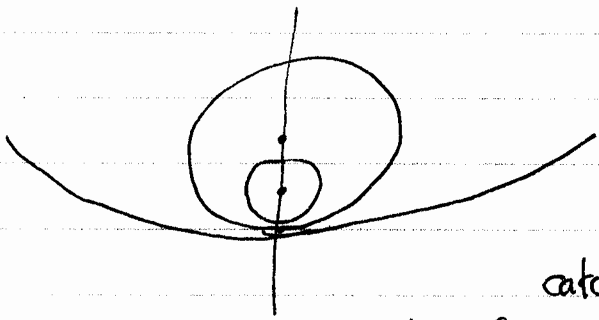
a) calculate the single independent component $R^{1}{}_{212}$ of the curvature tensor in the coordinate frame

b) repeat in the associated orthonormal frame, to get $R^{\hat 1}{}_{\hat 2\hat 1\hat 2}$.

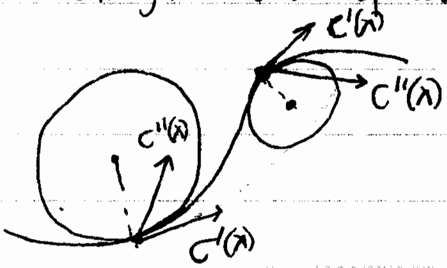
<u>exercise</u>

Verify that the curvature tensor of Euclidean $\mathbb{R}^3$ vanishes in cylindrical coordinates. Note that at most $9 = 3\times 3$ independent components $\{R^{2}{}_{3ij},\ R^{3}{}_{1ij},\ R^{1}{}_{2ij}\}$ $ij = \{23, 31, 12\}$ need to be checked.

Interpretation of Curvature

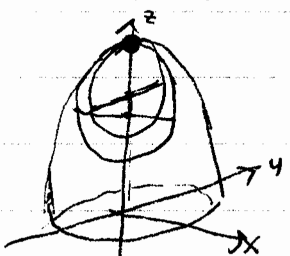


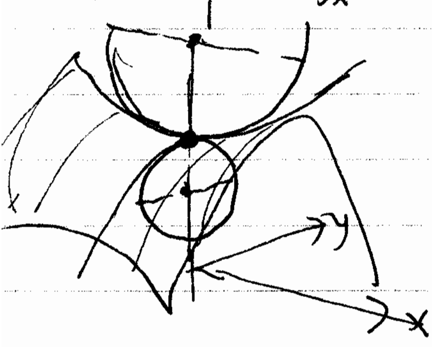
Curvature is a notion inversely related to radius of curvature. A circle is more curved if it has a smaller radius, less curved or "flatter" if it has a larger radius. In fact in multivariable calculus one learns that the curvature κ of a circle is the reciprocal of its radius r_0 : $\kappa = 1/r_0$.
A straight line corresponds to the limit $r_0 \to \infty$ and has zero curvature.



For an arbitrary curve at each point we can determine a circle of best fit to the curve in the plane of the tangent vector $c'(\lambda)$ and its derivative (osculating plane and osculating circle) with radius $1/\kappa$ in terms of the curvature at the given point. [All of this in $\mathbb{R}^3$ or $\mathbb{R}^2$]

To handle curvature for surfaces in $\mathbb{R}^3$, it turns out we can always find two circles of best fit at right angles to each other which best describe how the surface curves at each point. These circles may lie on the same side or on opposite sides of the tangent plane as with an ellipsoid (upper figure) or a saddle surface (lower figure). The curvature is taken to be



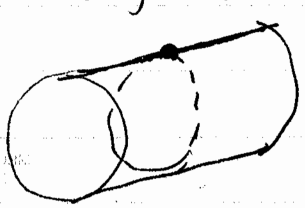


$$K = \pm \left(\frac{1}{r_1}\right)\left(\frac{1}{r_2}\right)$$

radii of two circles of best fit.

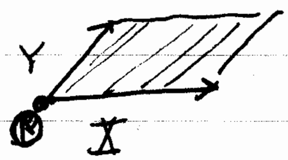
$$\pm : \begin{cases} + & \text{if same side} \\ - & \text{if opposite sides} \end{cases}$$

Increasing either radius flattens out the surface at the point.



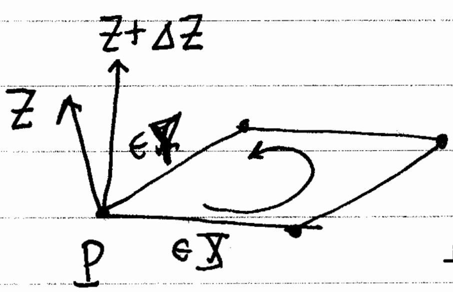
A cylinder is an example where one of the two radii has become infinite, so one of these two circles of best fit reduces to a straight line, leading to zero curvature.

A sphere has both radii equal to its radius at each point, leading to curvature $K = 1/r_0^2$. Notice that curvature has dimensions of inverse length in the 1-dimensional case and of inverse area in the 2-dimensional case. The 1-dimensional curvature concept generalizes to a notion of curvature in highdimensional spaces.



Pick a 2-plane in the tangent space at a given point. This can be done by specifying two linearly independent tangent vectors X and Y. The 2-vector $X \wedge Y$ contains both orientation information as well as length information. It's length using the p-vector innerproduct which avoids overcounting gives the area of the parallelogram formed by the two vectors.

Now multiply X and Y by a small enough positive number ϵ that we can identify them with directed curve segments in the space itself, i.e., the part of the tangent space they occupy is so small that we can use it as a good approximation to part of the space itself. The parallelogram in the tangent space may be interpreted as a closed curve in the space itself beginning and ended at the point P of our discussion.



Take any ~~tangent~~ vector Z at P and parallel transport it around the loop. Its length must remain constant so at most it can rotate relative to its original value. The difference is a small rotation which we have seen is described by an antisymmetric matrix in an orthonormal frame. This difference is approximately

$$\Delta Z^i \approx R^i{}_{jmn} (\epsilon X)^m (\epsilon Y)^n Z^j$$

This approximation gets better as $\epsilon \to 0$. Note that its value is proportional to ϵ^2, or more precisely, to the area of the parallelogram formed by X and Y.

The antisymmetry in the last pair of indices means that only the components of $X \wedge Y$, not $X \otimes Y$, contribute to this value. The antisymmetry of the first pair of indices, when both lowered, means that in an orthonormal frame the mixed indices are also antisymmetric and hence the matrix $R^i{}_{jmn} (\epsilon X)^m (\epsilon Y)^n$ represents a small rotation, which when contracted with Z^j, rotates it by this small amount, to produce the increment in Z.

If we fix the indices (mn) on $R^i{}_{jmn}$, we are looking at the subspace of the tangent space spanned by the frame vectors e_m and e_n or $\partial/\partial x^m$ and $\partial/\partial x^n$ in a coordinate frame. The remaining 2 indices describe the small rotation (in the sense of the increment of a vector under the rotation) associated with that 2-plane of directions. We can basically think of the curvature tensor as a linear-transformation valued function on 2-planes in the tangent space. As long as we have a metric with Euclidean signature as we have assumed in these discussions, the linear transformations describe the rate of change of rotations.

In 2-dimensions, the choice of 2-plane is fixed (the whole 2-dim tangent space) and a single number characterizes the rate of change of a rotation in that 2-plane, explaining why the curvature tensor has only 1 independent component $R^1{}_{212}$.