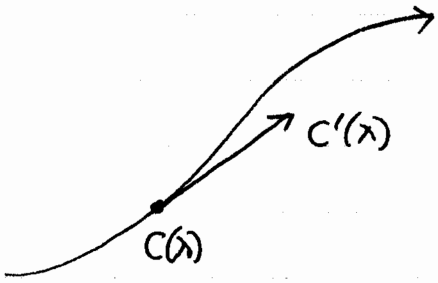
Covariant differentiation along a curve and Parallel translation



Suppose we have a parametrized curve $c(\lambda)$ in $\mathbb{R}^n$ with tangent vector $c'(\lambda)$:

standard cartesian coords ↓

$$\begin{cases} c(\lambda) = (c^1(\lambda), \dots, c^n(\lambda)) \;, & c^i(\lambda) = x^i \circ c(\lambda) \\ c'(\lambda) = c^{i\prime}(\lambda) \left.\dfrac{\partial}{\partial x^i}\right|_{c(\lambda)} & \end{cases}$$

In a general coordinate system $\{\bar{x}^i\}$ define analogously

$$\bar{c}^i(\lambda) = \bar{x}^i \circ c(\lambda) \qquad \text{(evaluate coord functions on curve)}$$

so that
$$c'(\lambda) = \bar{c}^{i\prime}(\lambda) \left.\frac{\partial}{\partial \bar{x}^i}\right|_{c(\lambda)}$$

The equivalence of these two expressions for $c'(\lambda)$ is the chain rule.

Example. On $\mathbb{R}^3$:

Cartesian	cylindrical	spherical
$(x^1,x^2,x^3) = (x,y,z)$	$(\bar{x}^1,\bar{x}^2,\bar{x}^3) = (\rho,\varphi,z)$	$(\bar{x}^1,\bar{x}^2,\bar{x}^3) = (r,\theta,\varphi)$
$x = 2\sin\theta_0 \cos\lambda \equiv c^1(\lambda)$	$\rho = 2\sin\theta_0 \equiv \bar{c}^1(\lambda) \equiv c^\rho(\lambda)$	$r = 2 \equiv \bar{c}^1(\lambda) = c^r(\lambda)$
$y = 2\sin\theta_0 \sin\lambda \equiv c^2(\lambda)$	$\varphi = \lambda \equiv \bar{c}^2(\lambda) \equiv c^\varphi(\lambda)$	$\theta = \theta_0 \equiv \bar{c}^2(\lambda) = c^\theta(\lambda)$
$z = 2\cos\theta_0 \equiv c^3(\lambda)$	$z = 2\cos\theta_0 \equiv \bar{c}^3(\lambda) \equiv c^z(\lambda)$	$\varphi = \lambda \equiv \bar{c}^3(\lambda) = c^\varphi(\lambda)$
$dx/d\lambda = -2\sin\theta_0 \sin\lambda = c^{1\prime}(\lambda)$	$d\rho/d\lambda = 0 = \bar{c}^{1\prime}(\lambda) = c^{\rho\prime}(\lambda)$	$dr/d\lambda = 0 = \bar{c}^{1\prime}(\lambda) = c^{r\prime}(\lambda)$
$dy/d\lambda = 2\sin\theta_0 \cos\lambda = c^{2\prime}(\lambda)$	$d\varphi/d\lambda = 1 = \bar{c}^{2\prime}(\lambda) = c^{\varphi\prime}(\lambda)$	$d\theta/d\lambda = 0 = \bar{c}^{2\prime}(\lambda) = c^{\theta\prime}(\lambda)$
$dz/d\lambda = 0 = c^{3\prime}(\lambda)$	$dz/d\lambda = 0 = \bar{c}^{3\prime}(\lambda) = c^{z\prime}(\lambda)$	$d\varphi/d\lambda = \underline{1} = \bar{c}^{3\prime}(\lambda) = c^{\varphi\prime}(\lambda)$

$$c'(\lambda) = 2\sin\theta_0 \left(-\sin\lambda \left.\frac{\partial}{\partial x}\right|_{c(\lambda)} + \cos\lambda \left.\frac{\partial}{\partial y}\right|_{c(\lambda)}\right) = \left.\frac{\partial}{\partial \varphi}\right|_{c(\lambda)} = \left.\frac{\partial}{\partial \varphi}\right|_{c(\lambda)}$$

For $\lambda: 0 \to 2\pi$, this is a ~~circle~~ one revolution of a circle which is a coordinate line of φ in both cylindrical and spherical coordinates.

If f is any function on $\mathbb{R}^n$, then the chain rule states that its derivative by the tangent vector $c'(\lambda)$ at $c(\lambda)$ is just the derivative of f along the parametrized curve at that point

$$\nabla_{c'(\lambda)} f = c'(\lambda) f = \bar{c}^{i\prime}(\lambda) \frac{\partial f}{\partial \bar{x}^i}\bigg|_{c(\lambda)} = "\frac{d\bar{x}^i(\lambda)}{d\lambda}" \frac{\partial f}{\partial \bar{x}^i}\bigg|_{c(\lambda)}$$

$$= \frac{d}{d\lambda}\left[f \circ c(\lambda)\right]$$

Since the covariant derivative of a function by a tangent vector is the ordinary derivative of the function by the tangent vector, this relates that covariant derivative to the derivative of the function along the curve.

We can extend this operation to tensor fields to measure their change along the curve with respect to covariant constant tensor fields.

Preliminary example: $f = xy = \frac{1}{2}\rho^2 \sin 2\varphi = \frac{1}{2} r^2 \sin^2\theta \sin 2\phi$

$$\nabla_{c'(\lambda)} f = 2\sin\theta_0\left(-\sin\lambda\ y + \cos\lambda\ x\right)\Big|_{c(\lambda)} = 2\sin\theta_0\left(-\sin\lambda\,(2\sin\theta_0 \sin\lambda) + \cos\lambda\,(2\sin\theta_0\cos\lambda)\right)$$

$$= 4\sin^2\theta_0\left[\cos^2\lambda - \sin^2\lambda\right] = 4\sin^2\theta_0 \cos 2\lambda$$

$$\text{or} \quad = \frac{\partial}{\partial\varphi}\left(\frac{1}{2}\rho^2 \sin 2\phi\right)\Big|_{c(\lambda)} = \frac{1}{2}(2\sin\theta_0)^2\, 2\cos 2\phi\,\Big|_{c(\lambda)} = 4\sin^2\theta_0 \cos 2\lambda$$

$$\text{or} \quad = \frac{\partial}{\partial\varphi}\left(\frac{1}{2} r^2 \sin^2\theta \sin 2\phi\right)\Big|_{c(\lambda)} = \cdots \qquad = \quad \text{same}$$

<u>exercise.</u> Evaluate $\nabla_{c'(\lambda)} f$ for $f = x^2 - y^2$.

Now suppose Y is a vector field. Then in ~~general~~ coordinates

$$\overline{[\nabla_{C'(\lambda)} Y]}^i = \bar Y^i{}_{;j}\, \bar C^{j\prime}(\lambda) = (\bar Y^i{}_{,j} + \bar\Gamma^i{}_{jk}\bar Y^k)\,\bar C^{j\prime}(\lambda)$$

$$= \bar Y^i{}_{,j}\,\bar C^{j\prime}(\lambda) + \bar\Gamma^i{}_{jk}\bar Y^k \bar C^{j\prime}(\lambda)$$

$$= \frac{d}{d\lambda}[\bar Y^i\circ c(\lambda)] + \underbrace{\bar\Gamma^i{}_{jk}\circ c(\lambda)\; C^{j\prime}(\lambda)}\; Y^k\circ c(\lambda)$$

$$\equiv \frac{D[\bar Y^i\circ c(\lambda)]}{d\lambda} \equiv \overline{\left[\frac{DY\circ c(\lambda)}{d\lambda}\right]}^i$$

$\bar\omega^i{}_k(C'(\lambda))$

$\parallel$

matrix giving change of frame along curve.

where in the next to last line the fact that all component functions are evaluated at $c(\lambda)$ is made explicit (suppressed earlier to avoid ugly expressions).

All we need to know about Y to evaluate its <u>covariant derivative along the curve</u> are its component functions along the curve. Their values at all points of $\mathbb{R}^n$ off the curve are irrelevant.

<u>continuing example</u> Let $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. In Cartesian coordinates $D/d\lambda$ just reduces to $d/d\lambda$ of the components of X along $C(\lambda)$, i.e.,

$$\frac{DX\circ c(\lambda)}{d\lambda} = \frac{d}{d\lambda}[y\circ c(\lambda)]\frac{\partial}{\partial x}\Big|_{c(\lambda)} + \frac{d}{d\lambda}[x\circ c(\lambda)]\frac{\partial}{\partial y}\Big|_{c(\lambda)} = 2\sin\theta_0\left[\cos\lambda\frac{\partial}{\partial x}\Big|_{c(\lambda)} - \sin\lambda\frac{\partial}{\partial y}\Big|_{c(\lambda)}\right]$$

In cylindrical coordinates $\bar c^{i\prime}(\lambda) = \delta^i{}_\varphi$ so $\bar\omega^i{}_k(C'(\lambda)) = \bar\Gamma^i{}_{\varphi k}$ and

$$X^\rho = \rho\sin 2\varphi\,,\quad X^\varphi = \cos 2\varphi \to X^\rho\circ c(\lambda) = 2\sin\theta_0\sin 2\lambda,\quad X^\varphi\circ c(\lambda) = \cos 2\lambda$$

$$\frac{DX^\rho}{d\lambda} = \frac{dX^\rho}{d\lambda} + \underbrace{\Gamma^\rho{}_{\varphi\varphi}}_{-\rho}X^\varphi = 4\sin\theta_0\cos 2\lambda - 2\sin\theta_0\cos 2\lambda = 2\sin\theta_0\cos 2\lambda$$

given on page 58 middle

$$\frac{DX^\varphi}{d\lambda} = \frac{dX^\varphi}{d\lambda} + \underbrace{\Gamma^\varphi{}_{\varphi\rho}}_{\rho^{-1}}X^\rho = -2\sin 2\lambda + \frac{1}{2\sin\theta_0}(2\sin\theta_0\sin 2\lambda) = -\sin 2\lambda \quad \left(\text{and } \frac{DX^z}{d\lambda} = 0\right)$$

$$\text{so}\quad \frac{DX\circ c(t)}{d\lambda} = 2\sin\theta_0\cos 2\lambda\,\frac{\partial}{\partial\rho}\Big|_{c(\lambda)} - \sin 2\lambda\,\frac{\partial}{\partial\varphi}\Big|_{c(\lambda)}.$$

~~exercise. Using bottom of p. 33, verify that this is the transform of~~ the Carlesian expression.

One can do this for any tensor field T

$$\overline{\left[\frac{D\, T\circ c(\lambda)}{d\lambda}\right]}^{i\ldots}{}_{j\ldots} = \frac{d}{d\lambda} \overline{T}^{i}{}_{j\ldots}\circ c(\lambda) + \overline{\Gamma}^{i}{}_{k\ell}\circ c(\lambda)\, \overline{c}^{k\prime}(\lambda)\, \overline{T}^{\ell\ldots}{}_{j\ldots} + \cdots$$
$$- \overline{\Gamma}^{\ell}{}_{kj}\circ c(\lambda)\, \overline{c}^{k\prime}(\lambda)\, \overline{T}^{i\ldots}{}_{\ell\ldots} - \cdots$$

which is just $\overline{[\nabla_{c'(\lambda)} T]}^{i\ldots}{}_{j\ldots}$ ~~with~~ its first term re-expressed using the chain rule.

The metric is covariant constant so $\overline{\left[\frac{D\, g\circ c(\lambda)}{d\lambda}\right]}_{ij} = \overline{[\nabla_{c'(t)} g]}_{ij} = 0$, i.e., the covariant derivative of the metric along any curve is zero.

If Y is any covariant constant vector field, $\nabla Y = 0$, then it too will have vanishing covariant derivative along any curve

$$0 = \overline{\left[\frac{D\, Y\circ c(\lambda)}{d\lambda}\right]}^{i} = \frac{d\, \overline{Y}^{i}\circ c(\lambda)}{d\lambda} + \overline{\Gamma}^{i}{}_{jk}\circ c(\lambda)\, \overline{c}^{j\prime}(\lambda)\, \overline{Y}^{k}\circ c(\lambda)$$

Notice that everything in this first order linear ordinary differential equation for the functions $\overline{Y}^{i}\circ c(\lambda)$ is a function of λ alone. Suppose we just specify arbitrary particular values

$$\overline{Y}^{i}(c(0)) \equiv Y^{i}_{(0)}$$

at $\lambda = 0$. These are initial conditions for the differential equation, which then has a unique solution $\overline{Y}^{i}(\lambda)$. Then we have succeeded in defining a vector $\overline{Y}^{i}(\lambda) \frac{\partial}{\partial \overline{x}^{i}}\Big|_{c(\lambda)}$ along the curve which doesnt change its components with respect to a Cartesian frame as we move along the curve.

This process describes the "parallel transport" of the initial tangent vector along the curve.

continuing example From the previous example the equations $\frac{DY^a(\lambda)}{d\lambda}=0$ are

$$\frac{dY^\rho}{d\lambda} - 2\sin\theta_0 Y^\varphi = 0 \quad \rightarrow \quad \frac{dY^\rho}{d\lambda} = (2\sin\theta_0 Y^\varphi) \quad \leftrightarrow \quad \frac{du^1}{d\lambda} = u^2$$

$$\frac{dY^\varphi}{d\lambda} + \frac{1}{2\sin\theta_0} Y^\rho = 0 \qquad \frac{d}{d\lambda}(2\sin\theta_0 Y^\varphi) = -Y^\rho \qquad \frac{du^2}{d\lambda} = -u^1$$

$$\frac{dY^z}{d\lambda} = 0 \qquad \frac{dY^z}{d\lambda} = 0 \qquad \frac{du^3}{d\lambda} = 0$$

This is a constant coefficient linear differential equation which is studied in every first course on ordinary differential equations.

In fact compare $\frac{d}{d\lambda}\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}$

with page 81 rewritten for the rotation of the coordinates x and y:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \frac{d}{d\theta}\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

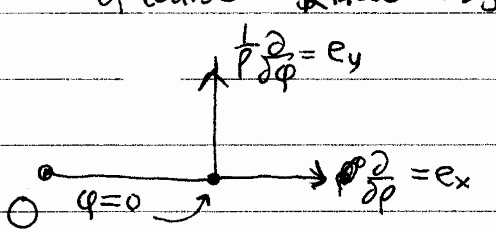
Identify (λ, u^1, u^2) with (θ, x, y) for solution. (and of course $u^3 = \text{const.}$)

$$\left.\begin{aligned} u^1 &= (\cos\lambda)u^1_0 + (\sin\lambda)u^2_0 \\ u^2 &= -\sin\lambda\, u^1_0 + \cos\lambda\, u^2_0 \\ u^3 &= u^3_0 \end{aligned}\right\} \rightarrow$$

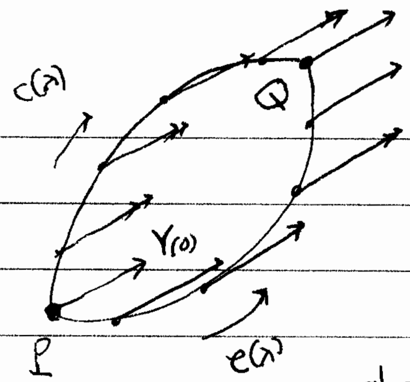
convert back to Y^ρ, Y^φ, Y^z to get solution which represents the solution for initial conditions $Y^\rho_{(0)}, Y^\varphi_{(0)}, Y^z_{(0)}$.

Of course we know this just represents the components of the vector field

$$Y = Y^\rho_{(0)}\frac{\partial}{\partial x} + \underbrace{\rho_0 Y^\varphi_{(0)}}_{\text{physical component.}}\frac{\partial}{\partial y} + Y^z_{(0)}\frac{\partial}{\partial z}.$$



Anyway, this is just to give you an idea how this can be done in principle.



Suppose we have two points P and Q and two curves $c(\lambda)$ and $\mathcal{C}(\lambda)$ with

$$c(0) = \mathcal{C}(0) = P \quad \text{and} \quad c(\lambda_1) = \mathcal{C}(\lambda_2) = Q.$$

If we transport a tangent vector $Y(0)$ at P along each curve to Q, of course we'll get the same result. This <u>path independence</u> of parallel transport on $\mathbb{R}^n$ is a feature of its <u>flat geometry</u>.

"<u>Parallel transport</u>" is called "parallel" transport because at each point on the curve we move the vector to the next tangent space so that it remains parallel to itself (and also has constant length). This operation provides a "connection" between any two tangent spaces to points connected to each other by a simple curve. Every vector in the first tangent space can be transported along the curve to the tangent space at the second point (indeed any other point on the curve), establishing a vector space isomorphism between them. (This mapping also preserves lengths & angles so it maps the Euclidean geometry of the first onto that of the second.) For this reason a covariant derivative on a space is often called a "connection", and $\Gamma^i{}_{jk}$ are called the "components of the connection".

Inner products (and so all lengths and relative angles) of tangent vectors are preserved under this operation since the metric is covariant constant

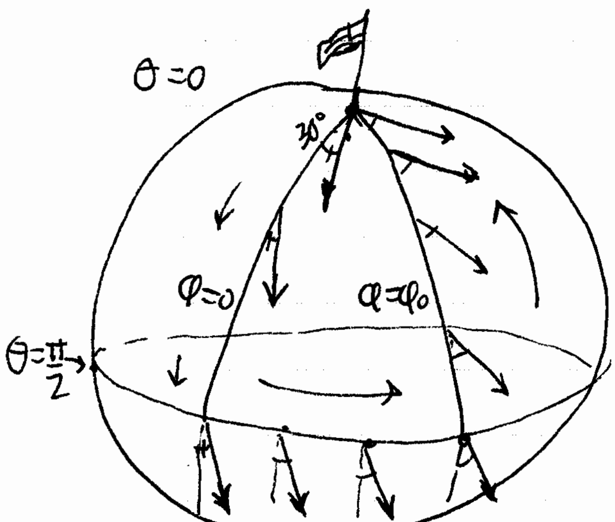
$$g_{ij;k} = 0 \to \frac{Dg_{ij}}{d\lambda} = [\nabla_{c'(\lambda)}\, g]_{ij} = g_{ij;k}\, c^{k\prime}(\lambda) = 0$$

$$\frac{d}{d\lambda}(\underbrace{g_{ij}X^iY^j}_{\text{function}}) = \frac{D}{d\lambda}(g_{ij}X^iY^j) = \underbrace{\left(\frac{Dg_{ij}}{d\lambda}\right)}_{=0} X^iY^j + g_{ij}\frac{DX^i}{d\lambda}Y^j + g_{ij}X^i\frac{DY^j}{d\lambda}.$$

Thus if X and Y are parallel transported along the curve $\frac{DX^i}{d\lambda} = 0 = \frac{DY^i}{d\lambda}$, we get $\frac{d}{d\lambda}(g_{ij}X^iY^j) = 0$, ie their inner product is a constant along the curve. Thus $(X=Y)$ lengths are preserved and inner products of unit vectors (angles) also.

All of these properties are obvious if we work in terms of Cartesian coordinates where parallel transport amounts to defining a tensor along a curve whose Cartesian coordinate frame components are constants, but they are not obvious working in a non Cartesian coordinate system or in a general frame where everything depends on position.

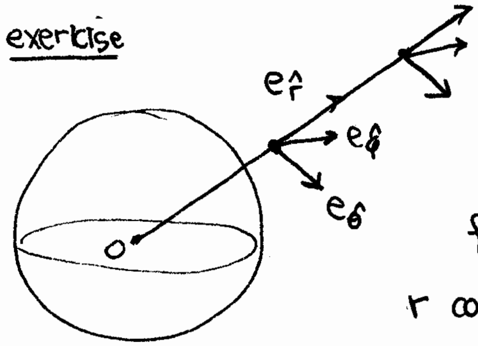
Everybody's favorite example of a curved surface is a sphere in $\mathbb{R}^3$, which we can take to be a coordinate sphere $r=r_0$ in spherical coordinates, leaving the angles θ and φ to serve as coordinates on the 2-dimensional space of points on that sphere. Their coordinate lines are the mesh of lines of longitude and latitude, although for the latter we measure the latitude as an angle from the equator instead of the North Pole.



Suppose we take a unit vector at the North pole tangent to the sphere and making an angle $30°$ with the $\varphi=0$ coordinate line. If we move this vector down the line of longitude $\varphi=0$, so that it remains tangent to the sphere, but maintaining a $30°$ angle with respect to the direction in which we are moving, then of course the direction of the vector has to change in order to remain in the 2-dimensional tangent plane to the sphere at each point, but apart from this necessary rotation no further unnecessary rotation occurs. Keep on going around the equator as shown and then come back up to the North pole, where it will now be at an angle of $30°$ to the line of longitude $\varphi=\varphi_0$. BUT it will be rotated by an angle φ_0 with respect to its initial direction!

This is the manifestation of curvature. If you transport a vector around a closed curve in a curved space, it will undergo a rotation in general. [On some curves it may not — for example on any great circle, the above exercise will not change the initial vector upon completion of one revolution.]

exercise



It is clear from the geometry of spherical coordinates that the orthonormal frame $\{e_{\hat r}, e_{\hat\theta}, e_{\hat\varphi}\}$ is parallel transported along the r coordinate lines. These curves may be parametrized by

$$\left.\begin{array}{ll} r = c^r(\lambda) = \lambda & c^{r\prime}(\lambda) = 1 \\ \theta = c^\theta(\lambda) = \theta_0 & c^{\theta\prime}(\lambda) = 0 \\ \varphi = c^\varphi(\lambda) = \varphi_0 & c^{\varphi\prime}(\lambda) = 0 \end{array}\right\} \quad c'(\lambda) = \bar c^{i\prime}(\lambda) \frac{\partial}{\partial \bar x^i}\Big|_{c(\lambda)} = \frac{\partial}{\partial r}\Big|_{c(\lambda)} = e_{\hat r}\Big|_{c(\lambda)}.$$

$\{\bar x^i\} = \{r, \theta, \varphi\}$

Since $e_{\hat r}$ is itself the unit tangent vector, the covariant derivative along this curve expressed in the orthonormal frame is

$$\frac{DX}{d\lambda}^{\hat i} = \frac{dX}{d\lambda}^{\hat i} + \Gamma^{\hat i}{}_{\hat k \hat j} c'^{\hat k} X^{\hat j} = \frac{dX}{d\lambda}^{\hat i} + \Gamma^{\hat i}{}_{\hat r \hat j} X^{\hat j}$$

Letting X be one of the vectors $\{e_{\hat r}, e_{\hat\theta}, e_{\hat\varphi}\}$ results in constant components $X^{\hat i}$ (for example $[e_{\hat r}]^{\hat i} = \delta^{\hat i}{}_{\hat r}$), so the term $dX^{\hat i}/d\lambda$ vanishes.

The components $\Gamma^{\hat i}{}_{\hat j \hat k}$ are given on page 75:

$$\Gamma^{\hat r}{}_{\hat\theta\hat\theta} = -r^{-1} = -\Gamma^{\hat\theta}{}_{\hat\theta\hat r}, \quad \Gamma^{\hat r}{}_{\hat\varphi\hat\varphi} = -r^{-1} = -\Gamma^{\hat\varphi}{}_{\hat\varphi\hat r}, \quad \Gamma^{\hat\theta}{}_{\hat\varphi\hat\varphi} = -r^{-1}\cot\theta = -\Gamma^{\hat\varphi}{}_{\hat\varphi\hat\theta}.$$

Thus for example

$$\frac{D}{d\lambda}[e_{\hat r}]^{\hat i} = \frac{d}{d\lambda}[e_{\hat r}]^{\hat i} + \Gamma^{\hat i}{}_{\hat r\hat j}[e_{\hat r}]^{\hat j} = \Gamma^{\hat i}{}_{\hat r\hat r} = 0. \rightarrow \frac{D}{d\lambda} e_{\hat r} = 0$$

says that $e_{\hat r}$ is parallel transported along this curve.

Verify that $e_{\hat\theta}$ is also parallel transported along r.

a) First use the orthonormal frame as in the example.

b) Next use the coordinate frame, where $c'(\lambda) = e_{\hat r} = e_r$ and $[e_{\hat\theta}]^r = 0$, $[e_{\hat\theta}]^\theta = \frac{1}{r}$, $[e_{\hat\theta}]^\varphi = 0$ and from page 83:

$$\Gamma^\theta{}_{r\theta} = \Gamma^\varphi{}_{r\varphi} = r^{-1}, \quad \Gamma^r{}_{\theta\theta} = -r, \quad \Gamma^\theta{}_{\theta r} = r^{-1}, \quad \Gamma^r{}_{\varphi\varphi} = -r\sin^2\theta, \quad \Gamma^\varphi{}_{\varphi r} = r^{-1}$$

$$\Gamma^\theta{}_{\varphi\varphi} = -\cos\theta\sin\theta, \quad \Gamma^\varphi{}_{\varphi\theta} = \cot\theta \quad \text{and} \quad \frac{DX^i}{d\lambda} = \frac{dX^i}{d\lambda} + \Gamma^i{}_{rj} X^j.$$