Laplacian and divergence and gradient

The differential of a function

$$df = f_{,i}\,dx^i \qquad f_{,i} = \frac{\partial f}{\partial x^i} = df\left(\frac{\partial}{\partial x^i}\right) \qquad \text{(coord frame)}$$

$$df = f_{,i}\,\omega^i \qquad f_{,i} = e_i f = df(e_i) \qquad \text{(arb. frame)}$$

is a covector field or 1-form field or simply a 1-form in the standard terminology.

In Cartesian coordinates on $\mathbb{R}^n$, the gradient $\vec{\nabla} f \equiv \text{grad}\, f$ is a vector field whose components are the corresponding partial derivatives of f

$$[\vec{\nabla} f]^i = \delta^{ij} f_{,j}$$

$$\text{grad} f \equiv \vec{\nabla} f = \delta^{ij} f_{,j} \frac{\partial}{\partial x^i} = df^{\#} .$$

The Kronecker delta is necessary to respect index positioning and tells us that we are actually using the Euclidean metric to raise the index on the 1-form df to obtain a vector field $\vec{\nabla} f$.

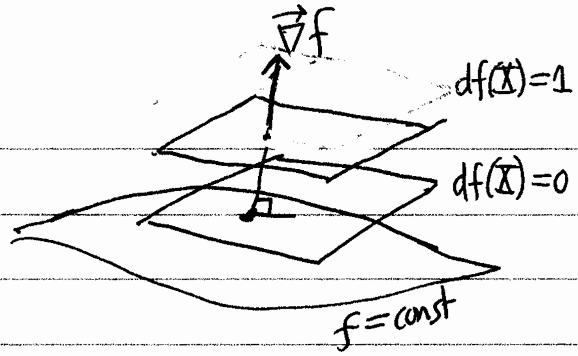
The same relation can be used to evaluate the gradient in nonCartesian coordinates or in a frame for the Euclidean metric or any other metric

$$\text{grad} f = \vec{\nabla} f = (df)^{\#} = g^{ij} f_{,j}\, e_i .$$

While the differential df is completely independent of a metric, the gradient only can be defined with the use of a metric.

For a function, covariant and ordinary differentiation coincide, so one can also write $\quad [\vec{\nabla} f]^i = g^{ij} f_{;j} \equiv f^{;i} = \nabla^i f$.

In other words the operator $\vec{\nabla} = \# \circ \nabla$ consists of covariant differentiation followed by raising the derivative index when acting on functions.



$$X\cdot\vec{\nabla} f = g_{ij}X^i\nabla^j f = X^i\nabla_i f = \begin{cases} Xf \\ \nabla_X f\end{cases}$$

If a vector field X is tangent to a level hypersurface $f=\text{const}$ of the function f, then the derivative of f along X is zero which implies that X is orthogonal to $\vec{\nabla} f$, or turning it around, the gradient is orthogonal to the space of tangent vectors which are in fact tangent to the level surface of f at each point — it is a normal to the tangent plane. Without a metric one only has the differential df whose hyperplanes in the tangent space describe the linear approximation to the increment of f away from each point, and no <u>normal</u>!

<u>exercise</u>. Evaluate $\vec{\nabla} f$ for $f = x^2-y^2$ on $\mathbb{R}^3$ in both cylindrical and spherical coordinates.

In Cartesian coordinates on $\mathbb{R}^n$, the divergence of a vector field is given by

$$\operatorname{div} X = \frac{\partial X^i}{\partial x^i} = X^i{}_{,i} = X^i{}_{;i}\,.$$

Since covariant & ordinary differentiation coincide here, by the definition of covariant differentiation, $\operatorname{div} X$ has the expression

$$\operatorname{div} X = X^i{}_{;i} = X^i{}_{,i} + \Gamma^i{}_{ij}X^j$$

in any frame or coordinate system.

<u>exercise</u> oops, checking was necessary

Let $X = x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y} = \rho\frac{\partial}{\partial\rho} \underset{\uparrow}{=} r\sin\theta\left[\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \cancel{-\frac{\sin 2\phi}{r\sin\theta}\frac{\partial}{\partial\theta}}\right]$

why? check this result for spherical coordinates

But before checking that $\operatorname{div} X = 2$ in cylindrical coordinates, more theory:

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{il} \left[g_{lj,k} - g_{jk,l} + g_{kl,j} \right] + \frac{1}{2} \left(C^i{}_{jk} - C_{jk}{}^i + C_k{}^i{}_j \right)$$

$$\Gamma^i{}_{ik} = \frac{1}{2} g^{il} \left[g_{li,k} - g_{ik,l} + g_{kl,i} \right] + \frac{1}{2} \left(C^i{}_{ik} - C_{ik}{}^i + C_k{}^i{}_i \right)$$

$$= \frac{1}{2} g^{il} g_{li,k} \underbrace{- \frac{1}{2} g^{il} g_{ki,l} + \frac{1}{2} \underbrace{g^{li} g_{kl,i}}_{g^{il} g_{ki,l}}}_{0} \underbrace{- \frac{1}{2} C^i{}_{ki} - \frac{1}{2} C^i{}_{ki}}_{-C^i{}_{ki}} + \frac{1}{2} C_{kji} g^{ji}$$

$$\frac{1}{2} \underbrace{\underbrace{C_{kji}}_{\text{antisymmetric}} \underbrace{g^{ji}}_{\text{symmetric}}}_{} = \frac{1}{4} \left(C_{kji} - C_{kij} \right) g^{ji} = \frac{1}{4} \left(C_{kji} g^{ji} - C_{kij} g^{ij} \right) = 0$$

$$\boxed{\Gamma^i{}_{ik} = \frac{1}{2} g^{il} g_{li,k} - C^i{}_{ki}}$$

To get a simpler expression for the metric derivative term we need an aside on the determinant function.

Aside (Part 1 revisited)

Differential of det

Recall on page 54 of part 1 the formula for the determinant of a matrix $\underline{A}$

$$\left[\det \underline{A}\ \epsilon_{j_1\cdots j_n} = \epsilon_{i_1\cdots i_n} A^{i_1}{}_{j_1}\cdots A^{i_n}{}_{j_n}\right]\epsilon^{j_1\cdots j_n}$$

$$\det \underline{A}\ \underbrace{\epsilon_{j_1\cdots j_n}\epsilon^{j_1\cdots j_n}}_{n!} = \underbrace{\epsilon^{j_1\cdots j_n}\epsilon_{i_1\cdots i_n}}_{\equiv\, \delta^{j_1\cdots j_n}{}_{i_1\cdots i_n}} A^{i_1}{}_{j_1}\cdots A^{i_n}{}_{j_n}$$

so $$\boxed{\det \underline{A} = \frac{1}{n!}\,\delta^{j_1\cdots j_n}{}_{i_1\cdots i_n} A^{i_1}{}_{j_1}\cdots A^{i_n}{}_{j_n}} \equiv \frac{1}{n}\mathcal{C}(\underline{A})^{j_n}{}_{i_n} A^{i_n}{}_{j_n}$$

Define the cofactor matrix

$$\mathcal{C}(\underline{A})^{j_n}{}_{i_n} = \frac{1}{(n-1)!}\,\delta^{j_1\cdots j_{n-1}}{}_{i_1\cdots i_{n-1}} A^{i_1}{}_{j_1}\cdots A^{i_{n-1}}{}_{j_{n-1}}$$

so that $$\det \underline{A} = \mathcal{C}(\underline{A})^{j_n}{}_{i_n} A^{i_n}{}_{j_n} = \operatorname{Tr} \underline{\mathcal{C}(A)}\ \underline{A}.$$

Then $$\mathcal{C}(\underline{A})^{j_n}{}_{i_n} A^{i_n}{}_{k} = \frac{1}{n}\,\delta^{j_1\cdots j_{n-1} j_n}{}_{i_1\cdots i_{n-1} i_n} A^{i_1}{}_{j_1}\cdots A^{i_{n-1}}{}_{j_{n-1}} A^{i_n}{}_{k}$$

$$= \underbrace{\frac{1}{(n-1)!}\epsilon^{j_1\cdots j_{n-1} j_n}\underbrace{\underbrace{\epsilon_{i_1\cdots i_{n-1} i_n}}_{\text{all summed}} A^{i_1}{}_{j_1}\cdots A^{i_{n-1}}{}_{j_{n-1}} A^{i_n}{}_{k}}_{\epsilon_{j_1\cdots j_{n-1} k}\ \det A \quad \text{from above def of determinant}}}$$

$$\frac{1}{(n-1)!}\,\delta^{j_1\cdots j_{n-1} j_n}{}_{j_1\cdots j_{n-1} k} = \delta^{j_n}{}_k \quad \text{by contraction formula on page 56 part 1.}$$

$$= (\det A)\,\delta^{j_n}{}_k .$$

i.e. $$\underline{\mathcal{C}(A)}\ \underline{A} = (\det \underline{A})\underline{I}$$

or if $\det \underline{A} \neq 0$: $$\underbrace{(\det A)^{-1}\,\underline{\mathcal{C}}(\underline{A})}_{\therefore\ \underline{A}^{-1}}\ \underline{A} = \underline{I}$$

We have derived the formula for the inverse of $\underline{A}$.

[actually $\mathcal{C}(\underline{A})^i{}_j$ is the cofactor of $A^j{}_i$ — ie differs by a transpose from the matrix of cofactors.]

Now by the product rule

$$d(\det \underline{A}) = \frac{1}{n!}\,\delta^{j_1\cdots j_n}{}_{i_1\cdots i_n}\, \underbrace{d\,(A^{i_1}{}_{j_1}\cdots A^{i_n}{}_{j_1})}_{\underbrace{dA^{i_1}{}_{j_1}A^{i_2}{}_{j_2}\cdots + A^{i_1}{}_{j_1}dA^{i_2}{}_{j_2}\cdots + A^{i_1}{}_{j_1}\cdots A^{i_{n-1}}{}_{j_{n-1}}dA^{i_n}{}_{j_n}}_{n \text{ terms}}}$$

$$= \frac{1}{(n-1)!}\,\delta^{j_1\cdots j_n}{}_{i_1\cdots i_n}\,A^{i_1}{}_{j_1}\cdots A^{i_{n-1}}{}_{j_{n-1}}\,dA^{i_n}{}_{j_n}$$

(By cyclic property all n terms collapse to n times last term)

$$= \mathcal{C}(A)^{j_n}{}_{i_n}\,dA^{i_n}{}_{j_n}$$

$$= \mathrm{Tr}\,\underline{\mathcal{C}}(\underline{A})\,d\underline{A}$$

So if $\det \underline{A} \neq 0$

$$d\,(\det A) = \mathrm{Tr}\{(\det \underline{A})\,A^{-1}\,d\underline{A}\}$$

or

$$\boxed{d(\ln \det \underline{A}) = \mathrm{Tr}\,\underline{A}^{-1}d\underline{A}}$$

Now replace the matrix $\underline{A}$ by the matrix $\underline{g} = (g_{ij})$

$$d\,\ln \det \underline{g} = \mathrm{Tr}\,\underline{g}^{-1}d\underline{g} = g^{ij}dg_{ji} = g^{ij}dg_{ij}$$

and

$$d\,\ln(\det g)^{1/2} = \tfrac{1}{2}\,d\ln(\det \underline{g}) = \tfrac{1}{2}\,g^{ij}\,dg_{ii}$$

$$\boxed{d\ln(\det g)^{1/2} = \tfrac{1}{2}\,g^{ij}\,dg_{ji} = \tfrac{1}{2}\,g^{ij}\,dg_{ij}}$$

This is the formula we need for divergences. If you believe it, you can forget its derivation for now.

exercise The matrix $\underline{a}^{-1} = \begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}$, $a\,d\underline{a}^{-1} = \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}d\varphi$

of the exercise on p 50 (worked on p. 58), is an orthogonal matrix with unit determinant, so what is $\mathrm{Tr}\,\underline{a}\,d\underline{a}^{-1}$ (let $\underline{A} = \underline{a}^{-1}$ in above formula) ?

Returning to the self-contraction of the components of the covariant derivative

$$\Gamma^i{}_{ik} = \underbrace{\underbrace{\tfrac{1}{2} g^{il} g_{li,k}}_{\tfrac{1}{2}\,\mathrm{Tr}\,\underline{g}^{-1} \underbrace{d\underline{g}(e_k)}_{(\equiv \underline{g},_k)}}}_{d[\ln(\det \underline{g})^{1/2}](e_k) \;=\; [\ln(\det \underline{g})^{1/2}],_k} - C^i{}_{ki}$$

so
$$\mathrm{div}\, X = \underbrace{X^i{}_{,i} + \underbrace{\Gamma^i{}_{ik} X^k}_{[\ln(\det g)^{1/2}],_k X^k}}_{(\det \underline{g})^{-1/2}\,[(\det \underline{g})^{1/2} X^i],_i} - C^i{}_{ki} X^k$$

relable to i

(since by product rule $= (\det \underline{g})^{-1/2} [(\det g)^{1/2} X^i{}_{,i} + ((\det g)^{1/2}],_i X^i]$
$= X^i{}_{,i} + [\ln(\det g)^{1/2}],_i X^i$)

$$\boxed{\mathrm{div}\, X = (\det \underline{g})^{-1/2} [(\det \underline{g})^{1/2} X^i],_i = X^i{}_{;i}}$$ (coordinate frame)

$$\boxed{= (\det \underline{g})^{-1/2} [(\det \underline{g})^{1/2} X^i],_i - C^k{}_{ik} X^i}$$ (arb. frame)

<u>exercise</u> Use this formula to compute div X in polar and spherical coordinates for X given on page 66.

Notice that the divergence operator <u>only</u> involves the metric through the factor $(\det \underline{g})^{1/2} = \eta_{1\ldots n}$ which is the component of the unit volume n-form associated with the metric. It does not care about the individual metric components. Any metrics whose unit volume forms coincide will yield the same divergence operator for vector fields.

Second covariant derivatives: notation

$$T^i{}_{j;k;\ell} \equiv [\nabla\nabla T]^i{}_{jk\ell}$$

is abbreviated to $\quad T^i{}_{j;k\ell}$,

in other words the semi-colon is used to separate the additional covariant derivative indices from the original tensor indices, no matter how many extra derivative indices are added.

For a function $\quad \nabla f = df = f_{;i}\,\omega^i \quad$ is the first covariant derivative and

$$\nabla\nabla f = f_{;ij}\,\omega^i \otimes \omega^j .$$

is the second covariant derivative. The same notation is extended to the comma and ordinary differentiation of a function: $\quad f_{,i,j} \equiv f_{,ij}$

The Laplacian is defined in Cartesian coordinates on $\mathbb{R}^n$ by

$$\nabla^2 f = \text{div grad} f = \delta^{ij} f_{,ij} = \delta^{ij} f_{;ij}. \quad \left(= \frac{\partial^2 f}{(\partial x^1)^2} + \dots + \frac{\partial^2 f}{(\partial x^n)^2}\right)$$

Therefore in any frame or coordinate system one has

$$\boxed{\nabla^2 f = \text{div grad} f = g^{ij} f_{;ij}} = \underbrace{(g^{ij} f_{;i})_{;j}}_{\text{since } g^{ij}{}_{;k}=0}$$

since both the metric & inverse metric are covariant constant, raising the first derivative index and then differentiating again is equivalent to differentiating twice and then contracting with the inverse metric.

Using the formula for the divergence, letting $X = \vec{\nabla} f = \text{grad} f$; we get

$$\boxed{\nabla^2 f = \text{div}\,\text{grad} f = (\det g)^{-1/2} \left[(\det g)^{1/2} g^{ij} f_{;j} \right]_{,i} - \underbrace{C^k{}_{ik} g^{kl} f_{,l}}_{\text{vanishes for coordinate frame}}}$$

<u>exercise</u> $\nabla^2 (x^2 - y^2) = \frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial y}(2y) = 2 - 2 = 0$ on $\mathbb{R}^3$.

$$f = x^2 - y^2 = \rho^2(\cos^2\varphi - \sin^2\varphi) = \rho^2 \cos 2\varphi$$
$$= r^2 \sin^2\theta \cos 2\varphi .$$

Confirm that $\nabla^2 f = 0$ in cylindrical and spherical coordinates.

<u>exercise</u> Verify the divergence formulas (since $(\det g)^{1/2} = \begin{cases} \rho & \text{cyl coords} \\ r^2 \sin\theta & \text{sph coords} \end{cases}$)

$$\nabla^2 = \rho^{-1} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$
$$= r^{-2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \varphi^2}$$

<u>exercise</u> Suppose $\{x^i\}$ are orthogonal coordinates on $\mathbb{R}^3$:

$$g = (h_1)^2 dx^1\otimes dx^1 + (h_2)^2 dx^2\otimes dx^2 + (h_3)^2 dx^3\otimes dx^3 \qquad (h_1, h_2, h_3 > 0)$$

$$g^{-1} = (h_1)^{-2} \frac{\partial}{\partial x^1}\otimes\frac{\partial}{\partial x^1} + (h_2)^{-2}\frac{\partial}{\partial x^2}\otimes\frac{\partial}{\partial x^2} + (h_3)^{-2}\frac{\partial}{\partial x^3}\otimes\frac{\partial}{\partial x^3}$$

$$\eta = h_1h_2h_3\, dx^1\wedge dx^2\wedge dx^3.$$

Let $e_i = \frac{\partial}{\partial x^i}$ and $e_{\hat{i}} = \frac{1}{h_i}\frac{\partial}{\partial x^i}$ be the coordinate frame and its associated normalized orthonormal frame, with $\omega^i = dx^i$ and $\omega^{\hat{i}} = h_i\omega^i$ (no sum on i).

Verify the formulas:

$$\vec{\nabla} f = \frac{1}{h_1}\frac{\partial f}{\partial x^1} e_{\hat{1}} + \frac{1}{h_2}\frac{\partial f}{\partial x^2} e_{\hat{2}} + \frac{1}{h_3}\frac{\partial f}{\partial x^3} e_{\hat{3}}$$

$$\operatorname{div} X = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}(X^{\hat{1}}h_2h_3) + \frac{\partial}{\partial x^2}(X^{\hat{2}}h_3h_1) + \frac{\partial}{\partial x^3}(X^{\hat{3}}h_1h_2)\right]$$

$$\nabla^2 f = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial x^3}\right)\right]$$

What about the curl?

For this we need to explore another derivative: the "exterior derivative" d which generalizes the differential of a function to an operator on p-forms or "differential forms" (antisymmetric covariant tensor fields).

More Practice Evaluating Components of the Covariant Derivative

The orthonormal frame associated with the orthogonal spherical coordinate frame is related to the orthonormal Cartesian coordinate frame by a rotation

$$(e_{\hat r}\ e_{\hat\theta}\ e_{\hat\varphi}) = (e_x\ e_y\ e_z) \underbrace{\begin{pmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}}_{\equiv \underline{a}^{-1}}$$

The columns of the orthogonal matrix $\underline{a}^{-1}$ are the Cartesian coordinate frame components of the new orthonormal frame vectors and are obtained by normalizing the columns of the matrix $\underline{A}^{-1}$ on page 36 which represent the Cartesian coordinate components of the spherical coordinate frame vectors.

One can understand the matrix $\underline{a}^{-1}$ as resulting from the following sequence of simpler transformations, using an obvious shorthand for trig functions

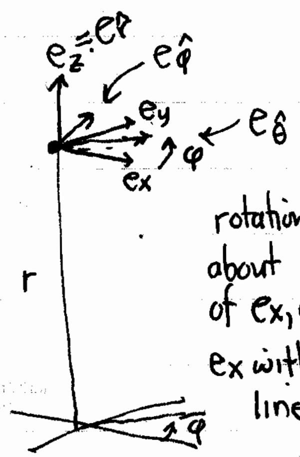
start on z-axis

$$(e_{\hat r}\, e_{\hat\theta}\, e_{\hat\varphi})\big|_{(r,\theta,\varphi)} = \underbrace{\underbrace{\underbrace{(e_x, e_y, e_z)\big|_{(r,\theta=0,\varphi=0)} \begin{pmatrix} C_\varphi & -S_\varphi & 0\\ S_\varphi & C_\varphi & 0 \\ 0&0&1\end{pmatrix}}_{(e_{\hat\theta}\, e_{\hat\varphi}\, e_{\hat r})|_{(r,\theta=0,\varphi)}} \begin{pmatrix}C_\theta&0&S_\theta\\0&1&0\\-S_\theta&0&C_\theta\end{pmatrix}}_{(e_{\hat\theta}\, e_{\hat\varphi}\, e_{\hat r})|_{(r,\theta,\varphi)}} \underbrace{\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}}_{\text{cyclic permutation to make } e_{\hat r} \text{ first}}$$

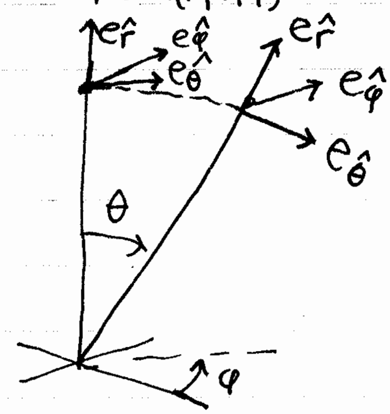
rotate to angle φ

rotate to angle θ

$$(e_{\hat r}\ e_{\hat\theta}\ e_{\hat\varphi})\big|_{(r,\theta,\varphi)}$$



rotation by angle φ about e_z in plane of e_x, e_y to align e_x with direction of θ coordinate line in φ direction



rotation by angle $+\theta$ about $e_{\hat\varphi}$ in plane of $e_{\hat\theta}$ and $e_{\hat r}$

Check that the matrix product of these three factor matrices is $\underline{a}^{-1}$!

Since $\underline{a}$ is an orthogonal matrix, $\underline{a}^{-1} = \underline{a}^{T}$ or $(\underline{a}^{-1})^{T} = \underline{a}$, so

$$\underline{a} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix}$$

Now it is a straightforward problem to evaluate the matrix

just a reminder of ON frame

$$\underline{\hat{\omega}} = \underline{a}\, d\, \underline{a}^{-1} = \left(\Gamma^{\hat{i}}{}_{\hat{k}\hat{j}}\, \omega^{\hat{k}} \right) \quad \text{in the spherical orthonormal frame:}$$

$$= \begin{pmatrix} S_\theta C_\varphi & S_\theta S_\varphi & C_\theta \\ C_\theta C_\varphi & C_\theta S_\varphi & -S_\theta \\ -S_\varphi & C_\varphi & 0 \end{pmatrix} \left\{ \begin{pmatrix} C_\theta C_\varphi & -S_\theta C_\varphi & 0 \\ C_\theta S_\varphi & -S_\theta S_\varphi & 0 \\ -S_\theta & -C_\theta & 0 \end{pmatrix} d\theta + \begin{pmatrix} -S_\theta S_\varphi & -C_\theta S_\varphi & -C_\varphi \\ S_\theta C_\varphi & C_\theta C_\varphi & -S_\varphi \\ 0 & 0 & 0 \end{pmatrix} d\varphi \right\}$$

$$= \cdots = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\theta + \begin{pmatrix} 0 & 0 & -S_\theta \\ 0 & 0 & -C_\theta \\ S_\theta & C_\theta & 0 \end{pmatrix} d\varphi$$

$$= \begin{pmatrix} 0 & -r^{-1} & 0 \\ r^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \omega^{\hat{\theta}} + \begin{pmatrix} 0 & 0 & -r^{-1} \\ 0 & 0 & -r^{-1}\cot\theta \\ r^{-1} & r^{-1}\cot\theta & 0 \end{pmatrix} \omega^{\hat{\varphi}}$$

haste leads to errors!

$$\Gamma^{\hat{2}}{}_{\hat{\varphi}\hat{3}} = \boxed{\Gamma^{\hat{\theta}}{}_{\hat{\varphi}\hat{\varphi}} = -r^{-1}\cot\theta}$$
$$\Gamma^{\hat{3}}{}_{\hat{\varphi}\hat{2}} = \boxed{\Gamma^{\hat{\varphi}}{}_{\hat{\varphi}\hat{\theta}} = r^{-1}\cot\theta}$$

So
$$\Gamma^{\hat{1}}{}_{\hat{\theta}\hat{2}} = \boxed{\Gamma^{\hat{r}}{}_{\hat{\theta}\hat{\theta}} = -r^{-1}}, \quad \Gamma^{\hat{1}}{}_{\hat{\varphi}\hat{3}} = \boxed{\Gamma^{\hat{r}}{}_{\hat{\varphi}\hat{\varphi}} = -r^{-1}}, \text{ and}$$
$$\Gamma^{\hat{2}}{}_{\hat{\theta}\hat{1}} = \boxed{\Gamma^{\hat{\theta}}{}_{\hat{\theta}\hat{r}} = r^{-1}}, \quad \Gamma^{\hat{3}}{}_{\hat{\varphi}\hat{1}} = \boxed{\Gamma^{\hat{\varphi}}{}_{\hat{\varphi}\hat{r}} = r^{-1}}$$

gives the ~~four~~ six (oops) nonzero components of the covariant derivative.

We can also derive these results from the metric formula. For an orthonormal frame $\{e_{\hat{i}}\}$, then $g_{\hat{i}\hat{j}} = g(e_{\hat{i}}, e_{\hat{j}}) = \delta_{ij}$, i.e., the components of the metric are constants so the metric component derivative terms in the formula vanish, leaving only the structure function terms

$$\Gamma^{\hat{i}}{}_{\hat{j}\hat{k}} = \tfrac{1}{2}\left(C^{\hat{i}}{}_{\hat{j}\hat{k}} - C_{\hat{j}\hat{k}}{}^{\hat{i}} + C_{\hat{k}}{}^{\hat{i}}{}_{\hat{j}} \right) = \tfrac{1}{2}\left(C^{\hat{i}}{}_{\hat{j}\hat{k}} - C^{\hat{j}}{}_{\hat{k}\hat{i}} + C^{\hat{k}}{}_{\hat{i}\hat{j}} \right)$$

(since index shifting trivial in ON frame)

On page 53 these are given (incorrectly):

$$C^{\hat{\theta}}{}_{\hat{r}\hat{\theta}} = -\tfrac{1}{r} = -C^{\hat{\theta}}{}_{\hat{\theta}\hat{r}}, \quad C^{\hat{\varphi}}{}_{\hat{r}\hat{\varphi}} = -\tfrac{1}{r} = -C^{\hat{\varphi}}{}_{\hat{\varphi}\hat{r}}, \quad C^{\hat{\varphi}}{}_{\hat{\theta}\hat{\varphi}} = -\tfrac{1}{r}\cot\theta = -C^{\hat{\varphi}}{}_{\hat{\varphi}\hat{\theta}}.$$

The nonzero components of the connection must have indices which are at most a permutation of the index positions on the nonzero structure functions. Forgetting for a moment that we know which ~~four~~ six components of the connection are nonzero, we can use the following reasoning to avoid evaluating the formula for many components which turn out to be zero.

On page 60 we saw that the covariant constancy of the metric means

$$0 = g_{ij;k} = g_{ij,k} - g_{\ell j}\Gamma^{\ell}{}_{ki} - g_{i\ell}\Gamma^{\ell}{}_{kj}$$
$$= g_{ij,k} - \Gamma_{jki} - \Gamma_{ikj}$$

or $$g_{ij,k} = \Gamma_{jki} + \Gamma_{ikj}$$

For an orthonormal frame $g_{ij} = \delta_{ij}$ and $g_{ij,k} = 0$ (constant components)

so $$\Gamma_{jki} = -\Gamma_{ikj},$$

ie. the components of the covariant derivative are antisymmetric in their outer indices. This remains true when we raise the index since the metric component matrix is the identity matrix and explains why the matrix $\underline{\omega} = (\Gamma^{i}{}_{kj}\,\omega^{k})$ evaluated above for the spherical orthonormal frame is antisymmetric — its matrix indices are the outer pair of indices on the components of the covariant derivative [Go back and look at the result for $\underline{\omega}$ on the previous page and see that it is antisymmetric.]

Thus the outer pair of indices on $\Gamma^{\hat{i}}{}_{\hat{k}\hat{j}}$ must be distinct and antisymmetry tells us the value of one index ordering in terms of the other ordering. Given the three nonzero independent structure functions (6 by antisymmetry), there are only three nonzero independent components of the connection (6 by antisymmetry) that we can write down.

antisymmetry of components of covariant derivative

antisymmetry of structure functions

distinct

$$C^{\hat\theta}{}_{\hat r\hat\theta} = -C^{\hat\theta}{}_{\hat\theta\hat r} \to \Gamma^{\hat\theta}{}_{\hat\theta\hat r} = \tfrac12\left(C^{\hat\theta}{}_{\hat\theta\hat r} - C_{\hat\theta\hat r}{}^{\hat\theta} + C_{\hat r}{}^{\hat\theta}{}_{\hat\theta}\right) = \tfrac12\left(C^{\hat\theta}{}_{\hat\theta\hat r} + C^{\hat\theta}{}_{\hat r\hat\theta} + \cancelto{0}{C^{\hat r}{}_{\hat\theta\hat\theta}}\right)$$

$$\Gamma^{\hat\theta}{}_{\hat\theta\hat r} = -\Gamma^{\hat r}{}_{\hat\theta\hat\theta} \qquad = C^{\hat\theta}{}_{\hat\theta\hat r} = r^{-1}$$

distinct

$$C^{\hat\varphi}{}_{\hat r\hat\varphi} = -C^{\hat\varphi}{}_{\hat\varphi\hat r} \to \Gamma^{\hat\varphi}{}_{\hat\varphi\hat r} = \tfrac12\left(C^{\hat\varphi}{}_{\hat\varphi\hat r} - C_{\hat\varphi\hat r}{}^{\hat\varphi} + C_{\hat r}{}^{\hat\varphi}{}_{\hat\varphi}\right) = \tfrac12\left(C^{\hat\varphi}{}_{\hat\varphi\hat r} + C^{\hat\varphi}{}_{\hat\varphi\hat r} + \cancelto{0}{C^{\hat r}{}_{\hat\varphi\hat\varphi}}\right)$$

$$\Gamma^{\hat\varphi}{}_{\hat\varphi\hat r} = -\Gamma^{\hat r}{}_{\hat\varphi\hat\varphi} \qquad = C^{\hat\varphi}{}_{\hat\varphi\hat r} = r^{-1}$$

distinct

$$C^{\hat\varphi}{}_{\hat\theta\hat\varphi} = -C^{\hat\varphi}{}_{\hat\varphi\hat\theta} \to \Gamma^{\hat\varphi}{}_{\hat\varphi\hat\theta} = \tfrac12\left(C^{\hat\varphi}{}_{\hat\varphi\hat\theta} - C_{\hat\varphi\hat\theta}{}^{\hat\varphi} + C_{\hat\theta}{}^{\hat\varphi}{}_{\hat\varphi}\right) = \tfrac12\left(C^{\hat\varphi}{}_{\hat\varphi\hat\theta} + C^{\hat\varphi}{}_{\hat\varphi\hat\theta} + \cancelto{0}{C^{\hat\theta}{}_{\hat\varphi\hat\varphi}}\right)$$

$$\Gamma^{\hat\varphi}{}_{\hat\varphi\hat\theta} = -\Gamma^{\hat\theta}{}_{\hat\varphi\hat\varphi} \qquad = C^{\hat\varphi}{}_{\hat\varphi\hat\theta} = r^{-1}\cot\theta$$

So in fact we did no extra useless work to verify certain components are zero.

What is the significance of the antisymmetry property of the matrix $\underline{\omega} = (\Gamma^i{}_{kj}\omega^k)$ in an orthonormal frame? Well, the definition

$$\nabla_{e_k} e_i = \Gamma^j{}_{ki} e_j$$

$$\nabla_X e_i = \underbrace{\nabla_{X^k e_k} e_i}_{\substack{\text{directional covariant}\\ \text{derivative linear in}\\ \text{direction.}}} = X^k \nabla_{e_k} e_i = \underbrace{\Gamma^j{}_{ki}\overbrace{X^k}^{\omega^k(X)}}_{\omega^j{}_i(X)} e_j$$

$$\nabla_X Y^i = Y^i{}_{;j} X^j = X^j \nabla_{e_j} Y^i$$

so $\boxed{\nabla_X e_i = \omega^j{}_i(X)\, e_j}$.

The value of the 1-form $\underline{\omega}$ on X gives the matrix of the linear transformation of the frame vectors which describes their covariant derivative in that direction, i.e. how they change relative to a Cartesian frame as we move in that direction. The fact that this matrix is antisymmetric tells us that in 3-dimensions it can be represented by the cross-product of a vector. (77)

ASIDE ON ORTHOGONAL MATRICES

For the Euclidean inner product on $\mathbb{R}^n$, the components of the inner product are just $\quad g(e_i, e_j) = e_i \cdot e_j = \delta_{ij}$
in the standard basis or in any orthonormal basis. If

$$\bar{e}_i = A^{-1\,j}{}_i\, e_j \quad , \quad e_i = A^j{}_i\, \bar{e}_j$$

is a transformation relating any two orthonormal bases, then the inner product transforms in the following way

$$\delta_{ij} = A^{-1\,m}{}_i\, A^{-1\,n}{}_j\, \delta_{mn} \quad \text{or} \quad \delta_{ij} = A^m{}_i\, A^n{}_j\, \delta_{mn}$$

or in matrix form

$$\delta_{ij} = A^m{}_i\, \delta_{mn}\, A^n{}_j$$
$$\underline{\underline{I}} = \underline{A}^T\, \underline{\underline{I}}\, \underline{A} = \underline{A}^T \underline{A}$$

Thus $\underline{A}^T = \underline{A}^{-1}$ describes the matrix of linear transformations between orthonormal bases. The condition $\underline{A}^T\underline{A} = \underline{\underline{I}}$ just states that the column vectors of A are an orthonormal set of vectors in $\mathbb{R}^n$. Such matrices are called $\boxed{\text{ORTHOGONAL matrices}}$.
They represent rotations and reflections of $\mathbb{R}^n$ into itself.

Suppose $\underline{A}$ depends on a parameter λ so we get a family of orthogonal matrices. Then

$$\frac{d}{d\lambda}\left[\underline{A}^T\underline{A} = \underline{\underline{I}}\right]$$

$$\underbrace{\left(\frac{d\underline{A}}{d\lambda}\right)^T}_{\left[\underline{A}^T \frac{d\underline{A}}{d\lambda}\right]^T} \underline{A} + \underline{A}^T \frac{d\underline{A}}{d\lambda} = 0$$

$\boxed{\text{using properties} \quad (\underline{A}\,\underline{B})^T = \underline{B}^T\underline{A}^T, \quad (\underline{A}^T)^T = \underline{A}}$

and $\underline{A}^T = \underline{A}^{-1}$ so

$$\underbrace{\underline{A}^{-1}\frac{d\underline{A}}{d\lambda} + \left[\underline{A}^{-1}\frac{d\underline{A}}{d\lambda}\right]^T = 0}$$

This just says that the matrix $\underline{B} \equiv \underline{A}^{-1}\dfrac{d\underline{A}}{d\lambda} = -\underline{B}^T$

is antisymmetric.

The same thing is true if we take the differential

$$\underline{A}^{-1} d\underline{A} = \underline{A}^{-1} \frac{d\underline{A}}{d\lambda} d\lambda$$

instead of the derivative. This explains why

$$\underline{\hat{\omega}} = \underline{a}\, d\underline{a}^{-1} = (\underline{a}^{-1})^{-1} d(\underline{a}^{-1})$$

is antisymmetric. We are differentiating an orthogonal matrix $\underline{a}^{-1}$.

The matrix $\underline{\omega}(X)$ for a given vector field X tells us the rate of change of the rotation which the orthonormal frame undergoes as we move in the direction of X. The rate of change of a rotation can be described by an angular velocity.

To understand this suppose a point of $\mathbb{R}^3$ undergoes an active rotation

$$x^i(t) = A^i{}_j(t) \underbrace{x^j(0)}_{\text{position at } t=0}$$

Then

$$\frac{dx^i}{dt}(t) = \frac{dA^i{}_j}{dt}(t) \underbrace{x^j(0)}_{A^{-1\,j}{}_k(t) x^k(t)} = \underbrace{\frac{dA^i{}_j}{dt}(t) A^{-1\,j}{}_k(t)\, x^k(t)}_{= -A^i{}_j \frac{dA^{-1\,j}{}_k}{dt} x^k(t)}$$

$$\underbrace{-A^i{}_j \frac{dA^{-1\,j}{}_k}{dt}}_{\equiv \underbrace{B^i{}_k}_{\equiv \epsilon_{ikm} \Omega^m(t)}} \quad B^i{}_k \text{ antisymmetric}$$

(index level unimportant in orthonormal frame)

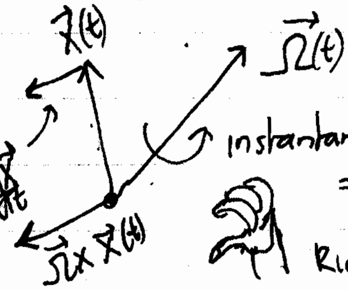
$$A A^{-1} = I$$

$$\left(\frac{dA}{dt}\right) A^{-1} + A \frac{dA^{-1}}{dt} = 0$$

so

$$\frac{dx^i}{dt}(t) = \underbrace{-\epsilon_{ikm}}_{+\epsilon_{imk}} \Omega^m(t) x^k(t) = \underbrace{\epsilon_{imk} \Omega^m(t) x^k(t)}_{[\vec{\Omega}(t) \times \vec{x}(t)]_i}$$

$$\frac{d\vec{x}(t)}{dt} = \underbrace{\vec{\Omega}(t)}_{\text{describes angular velocity}} \times \vec{x}(t)$$



instantaneous rate of change of angle $= \|\Omega(t)\|$

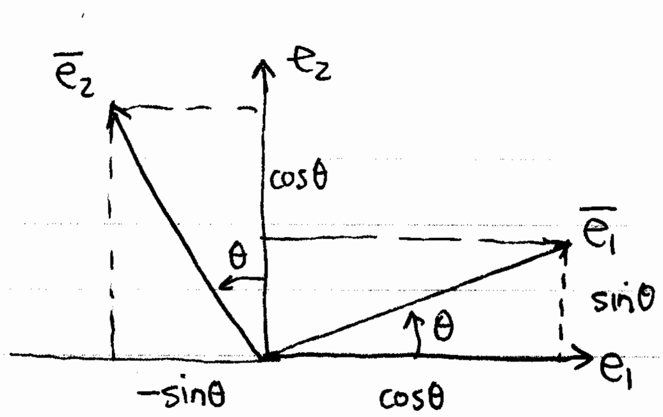
Righthandrule $\vec{\Omega}$ to $\vec{x}$, thumbs it!

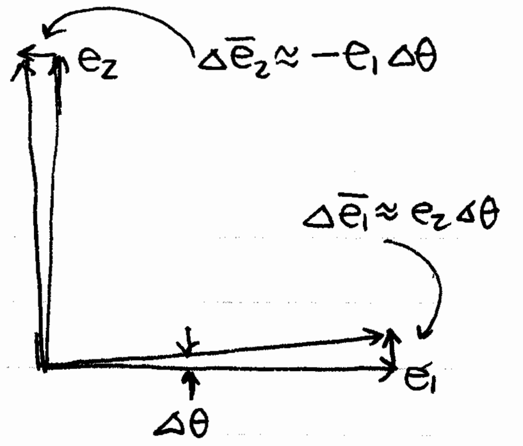
In other words by taking the dual of the antisymmetric matrix B we get an angular velocity vector which describes the direction about which a rotation is occuring (instantaneously) and the rate of change of the angle about that direction.

Anyway, this is more of an aside than I wanted to get into. I just wanted to give you a feeling for the antisymmetric matrix of covariant derivatives of the frame vectors.

Okay, so you didnt do rotation and angular velocity in your physics courses, or maybe you never understood the righthand rule, or maybe youre just not patient enough to read this stuff about derivatives of orthogonal matrices and duals of antisymmetric matrices — OKAY, it doesn't matter.

The cross-product & right hand rule only work in 3 dimensions where a pair of antisymmetric indices can be swapped for a single index by the duality operation. In any other dimension, you are stuck with a 2-plane in which a rotation takes place, so it is enough to look at rotations of $\mathbb{R}^2$ to understand how they work.





By trigonometry

$$\bar{e}_1 = \cos\theta\, e_1 + \sin\theta\, e_2$$
$$\bar{e}_2 = -\sin\theta\, e_1 + \cos\theta\, e_2$$

or $$(\bar{e}_1\,\bar{e}_2) = (e_1\,e_2)\underbrace{\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}}_{\equiv \underline{A}}$$

$$\frac{d}{d\theta}(\bar{e}_1,\bar{e}_2) = (e_1,e_2)\begin{pmatrix}-\sin\theta & -\cos\theta\\ \cos\theta & -\sin\theta\end{pmatrix}$$

$$\left.\frac{d}{d\theta}\right|_{\theta=0}(\bar{e}_1\,\bar{e}_2) = (e_1\,e_2)\underbrace{\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}}_{\underline{B}} \longrightarrow (\Delta\bar{e}_1,\Delta\bar{e}_2) \approx (e_1\,e_2)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\Delta\theta$$

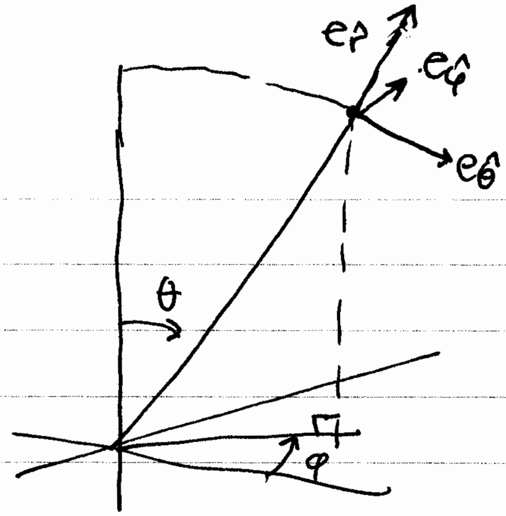
incremental change in $\bar{e}_1\ \bar{e}_2$ for small change $\Delta\theta$ in θ at $\theta=0$

The interpretation of this is that as you begin to rotate the basis vectors through a small angle $\Delta\theta$, $\bar{e}_1$ begins to rotate toward e_2 and $\bar{e}_2$ towards $-e_1$, explaining the antisymmetry of the matrix $\underline{B} = \left.\frac{d\underline{A}}{d\theta}\right|_{\theta=0}$

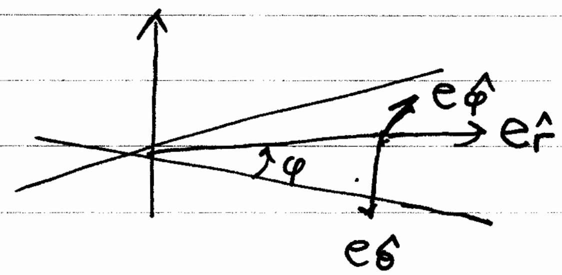
Now look at

$$\underline{\hat{\omega}} = \underline{a}\, d\underline{a}^{-1} = \begin{array}{c|ccc} & \hat{r} & \hat{\theta} & \hat{\varphi}\\ \hline \hat{r} & 0 & -1 & 0\\ \hat{\theta} & 1 & 0 & 0\\ \hat{\varphi} & 0 & 0 & 0\end{array}\, d\theta + \begin{pmatrix}0 & 0 & -\sin\theta\\ 0 & 0 & -\cos\theta\\ \sin\theta & \cos\theta & 0\end{pmatrix} d\varphi$$

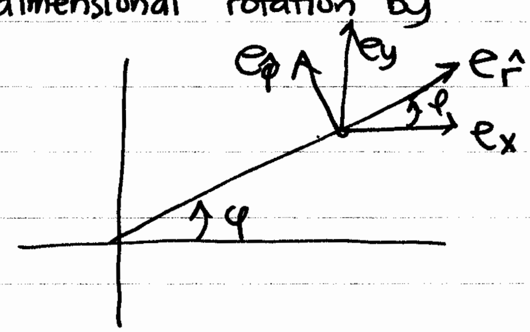
which tells us how the spherical orthonormal frame vectors begin to change as we make small increments $\Delta\theta$ and $\Delta\varphi$ in the angular variables, or alternately, tells us the rate at which these frame vectors are rotating as we change the angular coordinates. The fact that these 1-forms have no component along dr means that they don't rotate as we change r, i.e. as we move radially, and that is exactly right.



If we hold φ fixed and increase θ, $e_{\hat\varphi}$ remains fixed while $(e_{\hat r}, e_{\hat\theta})$ rotate by exactly the increment of θ in their 2-plane in the usual counterclockwise sense, so the 2x2 part of the matrix with $\hat r, \hat\theta$ indices is exactly the matrix of our two dimensional discussion.



If we hold θ fixed and increase φ, what happens depends on the value of θ.
For $\theta = \frac{\pi}{2}$ we are in the x-y plane and $e_{\hat\theta}$ remains equal to $-e_z$ as we change φ but $(e_{\hat r}, e_{\hat\varphi})$ undergoes the same 2-dimensional rotation by exactly the increment in φ.

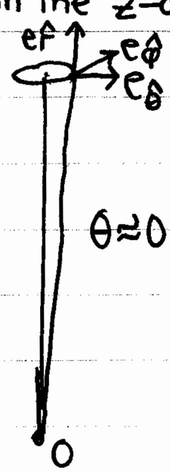


This is just what

$$\underset{\sim}{\hat\omega}\Big|_{\substack{\theta=\pi/2 \\ d\theta=0}} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} d\varphi$$

describes.

At the other extreme $\theta \to 0$ we end up on the z-axis almost, where $e_{\hat r} \approx e_z$ remains fixed and



$(e_{\hat\theta}, e_{\hat\varphi})$ rotate by exactly the increment in φ which is what

$$\underset{\sim}{\hat\omega}\Big|_{\substack{\theta\approx 0 \\ d\theta=0}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} d\varphi$$

describes. Are you convinced?

What is the interpretation of $\underline{\omega} = \underline{A}\, d\underline{A}^{-1}$ for the spherical coordinate frame which is not orthonormal? From page 36:

$$\underline{\omega} = \begin{pmatrix} s_\theta c_\varphi & s_\theta s_\varphi & c_\theta \\ r^{-1} c_\theta c_\varphi & r^{-1} c_\theta s_\varphi & -r^{-1} s_\theta \\ -r^{-1} \frac{s_\varphi}{s_\theta} & r^{-1}\frac{c_\varphi}{s_\theta} & 0 \end{pmatrix} \underbrace{d \begin{pmatrix} s_\theta c_\varphi & r c_\theta c_\varphi & -r s_\theta s_\varphi \\ s_\theta s_\varphi & r c_\theta s_\varphi & r s_\theta c_\varphi \\ c_\theta & -r s_\theta & 0 \end{pmatrix}}$$

$$\begin{pmatrix} 0 & c_\theta c_\varphi & -s_\theta s_\varphi \\ 0 & c_\theta s_\varphi & s_\theta c_\varphi \\ 0 & -s_\theta & 0 \end{pmatrix} dr + \begin{pmatrix} +c_\theta c_\varphi & -r s_\theta c_\varphi & -r c_\theta s_\varphi \\ c_\theta s_\varphi & -r s_\theta s_\varphi & r c_\theta c_\varphi \\ -s_\theta & -r c_\theta & 0 \end{pmatrix} d\theta + \begin{pmatrix} -s_\theta s_\varphi & -r c_\theta s_\varphi & -r s_\theta c_\varphi \\ s_\theta c_\varphi & r c_\theta c_\varphi & -r s_\theta s_\varphi \\ 0 & 0 & 0 \end{pmatrix} d\varphi$$

$$\underset{(\text{exercise})}{= \dots} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & r^{-1} & 0 \\ 0 & 0 & r^{-1} \end{pmatrix} dr + \begin{pmatrix} 0 & -r & 0 \\ r^{-1} & 0 & 0 \\ 0 & 0 & \cot\theta \end{pmatrix} d\theta + \begin{pmatrix} 0 & 0 & -r s_\theta \\ 0 & 0 & -c_\theta \\ r^{-1} s_\theta & c_\theta & 0 \end{pmatrix} d\varphi$$

Thus
$$\Gamma^\theta{}_{r\theta} = \Gamma^\varphi{}_{r\varphi} = r^{-1}$$
$$\Gamma^r{}_{\theta\theta} = -r, \quad \Gamma^\theta{}_{\theta r} = r^{-1} \quad ; \qquad \Gamma^\varphi{}_{\theta\varphi} = \cot\theta$$
$$\Gamma^r{}_{\varphi\varphi} = -r s_\theta^2, \quad \Gamma^\varphi{}_{\varphi r} = r^{-1}$$
$$\Gamma^\theta{}_{\varphi\varphi} = -c_\theta s_\theta, \quad \Gamma^\varphi{}_{\varphi\theta} = \cot\theta \,.$$
[Check these.]

The appearance of r in the θ and φ components of the 1-form $\underline{\omega}$ just takes into account the fact that for fixed r, e_θ and e_φ are not unit vectors, so the existing nonzero components of the covariant derivative in the associated orthonormal frame are simply rescaled by factors of r, except for the additional component $\Gamma^\varphi{}_{\theta\varphi} = \cot\theta = \omega^\varphi(\nabla_{e_\theta} e_\varphi)$. This describes the change in the length of e_φ as we change θ. Similarly the extra components $\Gamma^\theta{}_{r\theta}$ and $\Gamma^\varphi{}_{r\varphi}$ describe the change in the length of e_θ and e_φ as we change r.

<u>exercise on page 40c worked</u>

$$[X,Y]^{\hat\rho} = Y^{\hat\rho}{}_{,\hat\rho}X^{\hat\rho} + Y^{\hat\rho}{}_{,\hat\varphi}X^{\hat\varphi} - X^{\hat\rho}{}_{,\hat\rho}Y^{\hat\rho} + \overbrace{C^{\hat\rho}{}_{\hat\jmath\hat k}}^{=0}X^{\hat\jmath}Y^{\hat k}$$

$$= \frac{\partial}{\partial\rho}(\rho)\cdot(\rho\sin 2\varphi) + \frac{1}{\rho}\underbrace{\frac{\partial}{\partial\varphi}(\rho)}_{=0}\cdot(\rho\cos 2\varphi) - \frac{\partial}{\partial\rho}(\rho\sin 2\varphi)\cdot\rho$$

$$= \rho\sin 2\varphi - \rho\sin 2\varphi = 0$$

$$[X,Y]^{\hat\varphi} = \underbrace{Y^{\hat\varphi}}_{=0}{}_{,\hat\rho}X^{\hat\rho} + \underbrace{Y^{\hat\varphi}}_{=0}{}_{,\hat\varphi}X^{\hat\varphi} - X^{\hat\varphi}{}_{,\hat\rho}Y^{\hat\rho} + C^{\hat\varphi}{}_{\hat\rho\hat\varphi}X^{\hat\rho}\underbrace{Y^{\hat\varphi}}_{=0} + \underbrace{C^{\hat\varphi}{}_{\hat\varphi\hat\rho}}_{\frac{1}{\rho}}\underbrace{X^{\hat\varphi}}_{(\rho\cos 2\varphi)}\underbrace{Y^{\hat\rho}}_{\rho}$$

$$= -\frac{\partial}{\partial\rho}(\rho\cos 2\varphi)\cdot\rho + \rho\cos 2\varphi = 0$$

$$[X,Y]^{\hat z} = \underbrace{Y^{\hat z}}_{=0}{}_{,\hat\imath}X^{\hat\imath} - \underbrace{X^{\hat z}}_{=0}{}_{,\hat\imath}X^{\hat\imath} + \underbrace{C^{\hat z}{}_{\hat\jmath\hat k}}_{=0}X^{\hat\jmath}Y^{\hat k} = 0$$

so $[X,Y] = 0$.

Compare

$$[X,Y] = \left[y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},\ x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right]$$

$$= y\frac{\partial}{\partial x}(x)\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}(y)\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}(x)\frac{\partial}{\partial y} - y\frac{\partial}{\partial y}(y)\frac{\partial}{\partial x}$$

$$= y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = 0.$$

exercise on page 57 worked

Preliminary remark. In any frame we have the definition

$$\delta^{ij}_{mn} = \delta^i_m \delta^j_n - \delta^i_n \delta^j_m .$$

If we differentiate this equation

$$\delta^{ij}_{mn;k} = \underbrace{\delta^i_{m;k}}_{=0}\delta^j_n + \delta^i_m \underbrace{\delta^j_{n;k}}_{=0} - \underbrace{\delta^i_{n;k}}_{=0}\delta^j_m - \delta^i_m \underbrace{\delta^j_{m;k}}_{=0} . \qquad = 0$$

then $\nabla \delta^{(2)} = 0$ follows from $\nabla \delta = 0$ and the product rule.

However, just using the formula in the barred frame

$$\bar\delta^{ij}_{mn;k} = \underbrace{\bar\delta^{ij}_{mn,k}}_{=0} + \bar\Gamma^i_{k\ell}\underbrace{\bar\delta^{\ell j}_{mn}}_{\bar\delta^\ell_m \bar\delta^j_n - \bar\delta^\ell_n \bar\delta^j_m} + \bar\Gamma^j_{k\ell}\underbrace{\bar\delta^{i\ell}_{mn}}_{\bar\delta^i_m \bar\delta^\ell_n - \bar\delta^i_n \bar\delta^\ell_m} - \bar\Gamma^\ell_{km}\underbrace{\bar\delta^{ij}_{\ell n}}_{\bar\delta^i_\ell \bar\delta^j_n - \bar\delta^i_n \bar\delta^j_\ell} - \bar\Gamma^\ell_{kn}\underbrace{\bar\delta^{ij}_{m\ell}}_{\bar\delta^i_m \bar\delta^j_\ell - \bar\delta^i_\ell \bar\delta^j_m}$$

$$= \left(\bar\Gamma^i{}_{km}\bar\delta^j_n - \bar\Gamma^i{}_{kn}\bar\delta^j_m\right) + \left(\bar\Gamma^j{}_{kn}\bar\delta^i_m - \bar\Gamma^j{}_{km}\bar\delta^i_n\right)$$
$$- \left(\Gamma^i{}_{km}\delta^j_n - \Gamma^j{}_{km}\delta^i_n\right) - \left(\bar\Gamma^j{}_{kn}\delta^i_m - \bar\Gamma^i{}_{kn}\bar\delta^j_m\right)$$

$$= 0.$$

exercise on page 59 worked

Remember only $\Gamma^{\rho}{}_{\varphi\varphi} = -\rho$, $\Gamma^{\varphi}{}_{\varphi\rho} = \rho^{-1}$, $\Gamma^{\varphi}{}_{\rho\varphi} = \rho^{-1}$ are nonzero

X, Y have no z components, no components depend on z, so this is basically a 2-dimensional problem. (anything with a z index vanishes).

So:

$X^{\rho} = \cos\varphi \quad X^{\varphi} = -\frac{\sin\varphi}{\rho}$ write down only nonzero terms:

$$X^{\rho}{}_{;\rho} = \underbrace{X^{\rho}{}_{,\rho}}_{=0} + \underbrace{\Gamma^{\rho}{}_{\rho i}}_{=0} X^{i} = 0$$

$$X^{\rho}{}_{;\varphi} = \underbrace{X^{\rho}{}_{,\varphi}}_{-\sin\varphi} + \underbrace{\Gamma^{\rho}{}_{\varphi\varphi}}_{-\rho} \underbrace{X^{\varphi}}_{\left(-\frac{\sin\varphi}{\rho}\right)} = -\sin\varphi + \sin\varphi = 0$$

$$X^{\rho}{}_{;z} = \underbrace{X^{\rho}{}_{,z}}_{=0} + \underbrace{\Gamma^{\rho}{}_{zi}}_{=0} X^{i} = 0$$

$$X^{\varphi}{}_{;\rho} = \underbrace{X^{\varphi}{}_{,\rho}}_{\frac{1}{\rho^2}\sin\varphi} + \underbrace{\Gamma^{\varphi}{}_{\rho\varphi}}_{\rho^{-1}} \underbrace{X^{\varphi}}_{\left(-\frac{\sin\varphi}{\rho}\right)} = \frac{1}{\rho^2}\sin\varphi - \frac{1}{\rho^2}\sin\varphi = 0$$

$$X^{\varphi}{}_{;\varphi} = \underbrace{X^{\varphi}{}_{,\varphi}}_{-\frac{\cos\varphi}{\rho}} + \underbrace{\Gamma^{\varphi}{}_{\varphi\rho}}_{\rho^{-1}} \underbrace{X^{\rho}}_{\cos\varphi} = 0$$

$$X^{\varphi}{}_{;z} = \underbrace{X^{\varphi}{}_{,z}}_{=0} + \underbrace{\Gamma^{\varphi}{}_{zi}}_{=0} X^{i}$$

$$X^{z}{}_{;i} = \underbrace{X^{z}{}_{,i}}_{=0} + \underbrace{\Gamma^{z}{}_{ij}}_{=0} X^{j} = 0$$

so $X^{i}{}_{;i} = 0$, i.e., $\nabla X = 0$.

nothing new to be gained by doing $\nabla\left[\frac{\partial}{\partial v}\right] = 0$ so lets move on:

$[dx]_{\rho} = \cos\varphi \quad [dx]_{\varphi} = -\rho\sin\varphi, \quad [dx]_{i;j} = [dx]_{i,j} - \Gamma^{\ell}{}_{ji}[dx]_{\ell}$

$$[dx]_{i;z} = \underbrace{[dx]_{i,z}}_{=0} - \underbrace{\Gamma^{\ell}{}_{zi}}_{=0}[dx]_{\ell} = 0, \quad [dx]_{z;i} = \underbrace{[dx]_{z,i}}_{=0} - \underbrace{\Gamma^{\ell}{}_{iz}}_{=0}[dx]_{\ell} = 0$$

$$[dx]_{\rho;\rho} = \underbrace{[dx]_{\rho,\rho}}_{=0} - \underbrace{\Gamma^{i}{}_{\rho\rho}}_{=0}[dx]_{i} = 0$$

$$[dx]_{\rho;\varphi} = \underbrace{[dx]_{\rho,\varphi}}_{-\sin\varphi} - \underbrace{\Gamma^{\varphi}{}_{\varphi\rho}}_{\rho^{-1}} \underbrace{[dx]_{\varphi}}_{-\rho\sin\varphi} = 0$$

$$[dx]_{\varphi;\rho} = \underbrace{[dx]_{\varphi,\rho}}_{-\sin\varphi} - \underbrace{\Gamma^{\varphi}{}_{\rho\varphi}}_{\rho^{-1}} \underbrace{[dx]_{\varphi}}_{(-\rho\sin\varphi)} = 0$$

$$[dx]_{\varphi;\varphi} = \underbrace{[dx]_{\varphi,\varphi}}_{-\rho\cos\varphi} - \underbrace{\Gamma^{\rho}{}_{\varphi\varphi}}_{-\rho} \underbrace{[dx]_{\rho}}_{\cos\varphi} = 0$$

nothing new from $\nabla dy = 0$

Finally $[dz]_{\rho} = 0 = [dz]_{\varphi}$, $[dz]_{z} = 1$ so

$$[dz]_{i;j} = \underbrace{[dz]_{i,j}}_{0} - \Gamma^{z}{}_{ji} \underbrace{[dz]_{z}}_{1} = -\Gamma^{z}{}_{ji} = 0.$$

exercises on page 62 worked

- $\bar\Gamma^i{}_{jk} = \frac{1}{2}\bar g^{il}(\bar g_{lj,k} - \bar g_{jk,l} + \bar g_{kl,j})$ but $\bar g_{ij} = \bar g_{ji}$ so

$$\bar\Gamma^i{}_{kj} = \frac{1}{2}\bar g^{il}(\underbrace{\bar g_{lk,j}}_{g_{kl,j}} - \underbrace{\bar g_{kj,l}}_{g_{jk,l}} + \underbrace{\bar g_{jl,k}}_{g_{lj,k}}) = \bar\Gamma^i{}_{jk} \ \checkmark$$

- $\bar\Gamma_{ijk} = \frac{1}{2}(\bar g_{ij,k} - \bar g_{jk,i} + \bar g_{ki,j})$ [Note this is symmetric in (jk) for the same reason]

$\bar g_{\rho\rho} = 1 = \bar g_{zz}$, $\bar g_{\varphi\varphi} = \rho^2$.

At least two indices have to be the same to get a diagonal metric component to differentiate, otherwise you differentiate an off diagonal metric component which is zero. Finally the only diagonal component with a nonzero derivative is $g_{\varphi\varphi} = \rho^2$ so the indices have to be some permutation of $(\varphi\varphi\rho)$ to get a nonzero result.

$$\Gamma_{\rho\varphi\varphi} = \frac{1}{2}(\cancel{g_{\rho\varphi,\varphi}} - g_{\varphi\varphi,\rho} + \cancel{g_{\varphi\rho,\varphi}}) = -\frac{1}{2}(2\rho) = -\rho$$

$$\Gamma_{\varphi\rho\varphi} = \frac{1}{2}(\cancel{g_{\varphi\rho,\varphi}} - \cancel{g_{\rho\varphi,\varphi}} + g_{\varphi\varphi,\rho}) = \frac{1}{2}(2\rho) = \rho$$

$\Gamma_{\varphi\varphi\rho} = \frac{1}{2}(\ldots) = \Gamma_{\varphi\rho\varphi} = \rho$ since symmetric in last two indices in coordinate frame.

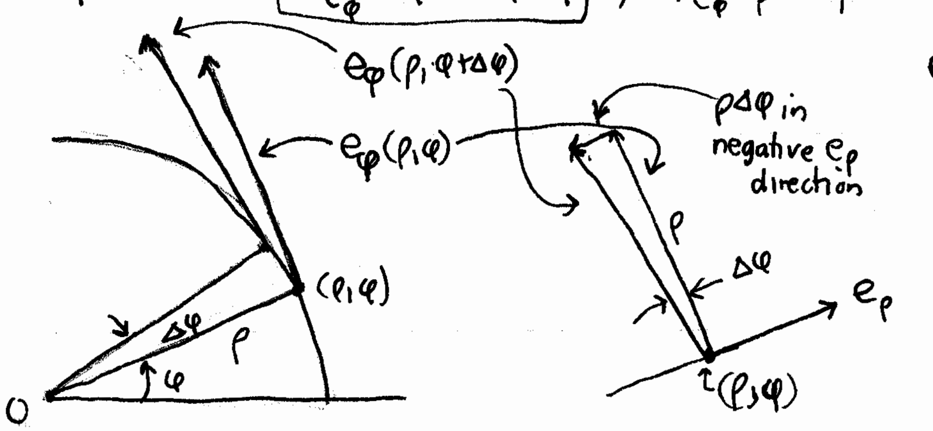
Now raise first index:

$$\Gamma^\rho{}_{\varphi\varphi} = g^{\rho\rho}\Gamma_{\rho\varphi\varphi} = -\rho$$

$$\Gamma^\varphi{}_{\rho\varphi} = g^{\varphi\varphi}\Gamma_{\varphi\rho\varphi} = \rho^{-2}(\rho) = \frac{1}{\rho} = \Gamma^\varphi{}_{\varphi\rho}.$$

Done.

Interpretation: $\boxed{\nabla_{e_\varphi} e_\varphi = -\rho e_\rho}$, $\nabla_{e_\varphi} e_\rho = \frac{1}{\rho} e_\varphi = \nabla_{e_\rho} e_\varphi$



e_φ has length ρ. Translate its value at $(\rho,\varphi+\Delta\varphi)$ back to (ρ,φ) so has same initial point as e_φ at (ρ,φ).
Difference is $\approx -\rho\Delta\varphi$ in radial direction.
Try interpreting another.

last exercise on page 62 worked

Preliminary remark. Whatever symmetries a tensor has, its covariant derivative has the same symmetries.

example: $T_{ij} = T_{ji}$ is symmetric so

$$T_{ij;k} = T_{ij,k} - \Gamma^{\ell}{}_{ki} T_{\ell j} - \Gamma^{\ell}{}_{kj} T_{i\ell}$$

$$T_{ji;k} = T_{ji,k} - \Gamma^{\ell}{}_{kj} T_{\ell i} - \Gamma^{\ell}{}_{ki} T_{j\ell} = T_{ij,k} - \Gamma^{\ell}{}_{ki} T_{\ell j} - \Gamma^{\ell}{}_{kj} T_{i\ell}$$

$$= T_{ij;k}.$$

A symmetric 2-index object in 3-dimensions has 6 independent components. Its covariant derivative has $6 \times 3 = 18$ in general (still a lot, no?). But really this is a 2-dimensional problem because nothing depends on z and no z components are nonzero so no z-component of $T_{ij;k}$ is nonzero. So we have 3 independent components of T_{ij} times 2 for its covariant derivate for a grand total of 6. Not too bad.

Recall $\Gamma^{\rho}{}_{\varphi\varphi} = -\rho$, $\Gamma^{\varphi}{}_{\rho\varphi} = \Gamma^{\varphi}{}_{\varphi\rho} = \rho^{-1}$

$$T_{\rho\rho;\rho} = \underbrace{T_{\rho\rho,\rho}}_{=0} - \underbrace{\Gamma^{i}{}_{\rho\rho}}_{0} T_{i\rho} - \underbrace{\Gamma^{i}{}_{\rho\rho}}_{0} T_{\rho i}$$

$$T_{\rho\rho;\varphi} = \underbrace{T_{\rho\rho,\varphi}}_{-2\cos\varphi\sin\varphi} - \underbrace{\Gamma^{i\to\varphi}{}_{\varphi\rho}}_{\rho^{-1}} \underbrace{T_{i\rho}}_{(-\rho\sin\varphi\cos\varphi)} - \underbrace{\Gamma^{i\to\varphi}{}_{\varphi\rho}}_{\rho^{-1}} \underbrace{T_{\rho i}}_{(-\rho\sin\varphi\cos\varphi)} = -2\cos\varphi\sin\varphi + 2\sin\varphi\cos\varphi = 0$$

$$T_{\varphi\varphi;\rho} = \underbrace{T_{\varphi\varphi,\rho}}_{2\rho\sin^2\varphi} - \underbrace{\Gamma^{i\to\varphi}{}_{\rho\varphi}}_{\rho^{-1}} \underbrace{T_{i\varphi}}_{(\rho^2\sin^2\varphi)} - \underbrace{\Gamma^{i\to\varphi}{}_{\rho\varphi}}_{\rho^{-1}} \underbrace{T_{\varphi i}}_{(\rho^2\sin^2\varphi)} = 2\rho\sin^2\varphi + 2\rho\sin^2\varphi = 0$$

$$T_{\varphi\varphi;\varphi} = \underbrace{T_{\varphi\varphi,\varphi}}_{2\rho^2\sin\varphi\cos\varphi} - \underbrace{\Gamma^{i\to\rho}{}_{\varphi\varphi}}_{-\rho} \underbrace{T_{i\varphi}}_{(-\rho\sin\varphi\cos\varphi)} - \underbrace{\Gamma^{i\to\rho}{}_{\varphi\varphi}}_{-\rho} \underbrace{T_{\varphi i}}_{(-\rho\sin\varphi\cos\varphi)} = 2\rho^2\sin\varphi\cos\varphi - 2\rho^2\sin\varphi\cos\varphi = 0$$

$$T_{\rho\varphi;\rho} = \underbrace{T_{\rho\varphi,\rho}}_{-\sin\varphi\cos\varphi} - \underbrace{\Gamma^{i}{}_{\rho\rho}}_{=0} T_{i\varphi} - \underbrace{\Gamma^{i\to\varphi}{}_{\rho\varphi}}_{\rho^{-1}} \underbrace{T_{\rho i}}_{(-\rho\sin\varphi\cos\varphi)} = -\sin\varphi\cos\varphi + \sin\varphi\cos\varphi = 0$$

$$T_{\rho\varphi;\varphi} = \underbrace{T_{\rho\varphi,\varphi}}_{-\frac{\partial}{\partial\varphi}\left(\frac{\rho}{2}\sin 2\varphi\right)} - \underbrace{\Gamma^{i\to\varphi}{}_{\varphi\rho}}_{\rho^{-1}} \underbrace{T_{i\varphi}}_{(\rho^2\sin^2\varphi)} - \underbrace{\Gamma^{i\to\rho}{}_{\varphi\varphi}}_{-\rho} \underbrace{T_{\rho i}}_{\cos^2\varphi} = -\rho(\cos^2\varphi - \sin^2\varphi) - \rho\sin^2\varphi + \rho\cos^2\varphi = 0 \checkmark$$

$$= -\rho\cos 2\varphi = -\rho(\cos^2\varphi - \sin^2\varphi)$$

exercises on page 64 worked

$$\Gamma^i{}_{[jk]} = \underbrace{\left\{ {i \atop [jk]} \right\}}_{=0} + \frac{1}{2} \Big(\underbrace{C^i{}_{[jk]}}_{C^i{}_{jk}} - \underbrace{C_{[jk]}{}^i}_{-\frac{1}{2}(C_{jk}{}^i - C_{kj}{}^i)} + \underbrace{C_{[k}{}^i{}_{j]}}_{\frac{1}{2}(C_k{}^i{}_j - C_j{}^i{}_k)} \Big)$$

since already antisymmetric in these indices

by antisymmetry of last pair of indices

$$\frac{1}{2}(C_k{}^i{}_j - C_j{}^i{}_k) = -\frac{1}{2}(C_{kj}{}^i - C_{jk}{}^i)$$

$$\underbrace{-\frac{1}{2}(C_{jk}{}^i - C_{kj}{}^i) - \frac{1}{2}(C_{kj}{}^i - C_{jk}{}^i)}_{=0 \text{ cancel in pairs}}$$

$$= \frac{1}{2} C^i{}_{jk} \quad ✓$$

[If you dont believe it:
$$C_k{}^i{}_j = g_{km} g^{in} C^m{}_{nj} = -g_{km} g^{in} C^m{}_{jn} = -C_{kj}{}^i$$
etc.]

On page 53 we found $\quad C^{\hat\varphi}{}_{\hat\rho\hat\varphi} = -\frac{1}{\rho} = -C^{\hat\varphi}{}_{\hat\varphi\hat\rho}$ (only nonzero structure function)

So to get a nonzero component of $\Gamma^{\hat i}{}_{\hat j\hat k}$ the indices must be a permutation of $(\hat\varphi\hat\varphi\hat\rho)$.

$$\Gamma^{\hat\rho}{}_{\hat\varphi\hat\varphi} = \frac{1}{2}(C^{\hat\rho}{}_{\hat\varphi\hat\varphi} - C_{\hat\varphi\hat\varphi}{}^{\hat\rho} + C_{\hat\varphi}{}^{\hat\rho}{}_{\hat\varphi}) = \frac{1}{2}(\underbrace{C^{\hat\rho}{}_{\hat\varphi\hat\varphi}}_{=0} \underbrace{- C^{\hat\varphi}{}_{\hat\varphi\hat\rho}}_{+C^{\hat\varphi}{}_{\hat\rho\hat\varphi}} + C^{\hat\varphi}{}_{\hat\rho\hat\varphi})$$
$$= C^{\hat\varphi}{}_{\hat\rho\hat\varphi} = -\frac{1}{\rho}$$

$$\Gamma^{\hat\varphi}{}_{\hat\rho\hat\varphi} = \frac{1}{2}(C^{\hat\varphi}{}_{\hat\rho\hat\varphi} - C_{\hat\rho\hat\varphi}{}^{\hat\varphi} + C_{\hat\varphi}{}^{\hat\varphi}{}_{\hat\rho}) = \frac{1}{2}(C^{\hat\varphi}{}_{\hat\rho\hat\varphi} - \underbrace{C^{\hat\rho}{}_{\hat\varphi\hat\varphi}}_{=0} + \underbrace{C^{\hat\varphi}{}_{\hat\varphi\hat\rho}}_{-C^{\hat\varphi}{}_{\hat\rho\hat\varphi}}) = 0$$

$$\Gamma^{\hat\varphi}{}_{\hat\varphi\hat\rho} = \frac{1}{2}(C^{\hat\varphi}{}_{\hat\varphi\hat\rho} - C_{\hat\varphi\hat\rho}{}^{\hat\varphi} + C_{\hat\rho}{}^{\hat\varphi}{}_{\hat\varphi}) = \frac{1}{2}(C^{\hat\varphi}{}_{\hat\varphi\hat\rho} \underbrace{- C^{\hat\varphi}{}_{\hat\rho\hat\varphi}}_{+C^{\hat\varphi}{}_{\hat\varphi\hat\rho}} + \underbrace{C^{\hat\rho}{}_{\hat\varphi\hat\varphi}}_{=0})$$
$$= C^{\hat\varphi}{}_{\hat\varphi\hat\rho} = \frac{1}{\rho}$$

Compare with page 58 and oops! I forgot to normalize!

$$\underline{\hat\omega} = (\bar\Gamma^{\hat i}{}_{\hat j\hat k}\,\bar\omega^{\hat j}) = \begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix} d\varphi = \begin{pmatrix}0&-\rho^{-1}&0\\\rho^{-1}&0&0\\0&0&0\end{pmatrix}\omega^{\hat\varphi}$$

so $\bar\Gamma^{\hat\rho}{}_{\hat\varphi\hat\varphi} = -\rho^{-1}$, $\bar\Gamma^{\hat\varphi}{}_{\hat\varphi\hat\rho} = \rho^{-1}$ agreement ✓

exercise

Following up the first exercise on page 66,
consider instead the function

$$f = xy = \rho^2 \sin\varphi \cos\varphi = \tfrac{1}{2}\rho^2 \sin 2\varphi = \tfrac{1}{2} r^2 \sin^2\theta \sin 2\varphi$$

Then

$$df = y\,dx + x\,dy = X^\flat \quad \text{yields our friend } X \text{ from pages 33, 37, 38, 39}$$

$$\vec{\nabla} f = [df]^\sharp = y\frac{\partial}{\partial x} + x \frac{\partial}{\partial y} = X$$

where we saw that

$$X = \rho \sin 2\varphi \frac{\partial}{\partial \rho} + \cos 2\varphi \frac{\partial}{\partial \varphi} = \sin\theta \sin 2\varphi \left(r\sin\theta \frac{\partial}{\partial r} + \cos\theta \frac{\partial}{\partial \theta}\right) + \cos 2\varphi \frac{\partial}{\partial \varphi}$$

$$X^\flat = \rho \sin 2\varphi \, d\rho + \rho^2 \cos 2\varphi \, d\varphi = \sin\theta \sin 2\varphi \left(r\sin\theta \, dr + r^2 \cos\theta \, d\theta\right) + r^2 \sin^2\theta \cos 2\varphi \, d\varphi$$

$$\left[\frac{\partial}{\partial r}\right]_i = g_{ij} \underbrace{\left[\frac{\partial}{\partial r}\right]^j}_{\delta^j{}_r} = g_{ir} \;\rightarrow\; \left[\frac{\partial}{\partial r}\right]^\flat = g_{ir}\, d\bar{x}^i = g_{rr}\, dr = dr$$

similarly $\left[\frac{\partial}{\partial \varphi}\right]^\flat = g_{\varphi i}\, d\bar{x}^i = g_{\varphi\varphi}\, d\varphi = r^2 \sin^2\theta \, d\varphi$

$\left[\frac{\partial}{\partial \theta}\right]^\flat = g_{\theta i}\, d\bar{x}^i = g_{\theta\theta}\, d\theta = r^2 d\theta$

In general $\quad e_i{}^\flat = g_{kj} \underbrace{e^j{}_i}_{\delta^k{}_i} \omega^k = g_{ik} \omega^k$

so that $\quad X^\flat = (X^i e_i)^\flat = X^i e_i{}^\flat = X^i g_{ik} \omega^k = X_k \omega^k$

Similarly $\quad [\omega^i]^\sharp = g^{ij} e_j$

for an orthogonal frame, index shifting the frame vectors and dual frame covectors yields the corresponding basis covector or vector multiplied by the diagonal metric component or its reciprocal

Compute df and $\operatorname{grad} f = \vec{\nabla} f$ in cylindrical coordinates and verify that you get our previous results quoted above.