Laplacian and divergence and gradient

The differential of a function
\[ df = f_{,i} \, dx^i \]
\[ f_{,i} = \frac{\partial f}{\partial x^i} \quad (\text{coord. frame}) \]
\[ df = f_{,i} \, e^i \]
\[ f_{,i} = e^i \cdot f = df(e_i) \quad (\text{arb. frame}) \]
is a covector field or 1-form field, or simply a 1-form, in the standard terminology.

In Cartesian coordinates on \( \mathbb{R}^n \), the gradient \( \nabla f = \text{grad} f \) is a vector field whose components are the corresponding partial derivatives of \( f \)
\[ \nabla f = \nabla f = \delta^{ij} f_{,j} \hat{x}^i = df^# . \]
The Kronecker delta is necessary to respect index positioning and tells us that we are actually using the Euclidean metric to raise the index on the 1-form \( df \) to obtain a vector field \( \nabla f \).

The same relation can be used to evaluate the gradient in non-Cartesian coordinates or in a frame for the Euclidean metric or any other metric
\[ \text{grad} f = \nabla f = (df)^# = g^{ij} f_{,i} \, e^j . \]
While the differential \( df \) is completely independent of a metric, the gradient only can be defined with the use of a metric.

For a function, covariant and ordinary differentiation commute, so one can also write
\[ [\nabla f]^i = g^{ij} f_{,j} \equiv f^i = \nabla^i f . \]
In other words, the operator \( \nabla = \# \circ \nabla \) consists of covariant differentiation followed by raising the derivative index when acting on functions.
If a vector field \( \mathbf{X} \) is tangent to a level hypersurface \( f = \text{const} \) of the function \( f \), then the derivative of \( f \) along \( \mathbf{X} \) is zero which implies that \( \mathbf{X} \) is orthogonal to \( \nabla f \), or turning it around, the gradient is orthogonal to the space of tangent vectors which are in fact tangent to the level surface of \( f \) at each point — it is a normal to the tangent plane. Without a metric one only has the differential \( df \) whose hyperplanes in the tangent space describe the linear approximation to the increment of \( f \) away from each point, and no normal.

**Exercise.** Evaluate \( \nabla f \) for \( f = x^2 - y^2 \) on \( \mathbb{R}^3 \) in both cylindrical and spherical coordinates.

In cartesian coordinates on \( \mathbb{R}^n \), the divergence of a vector field \( \mathbf{X} \) is given by:

\[
\text{div} \mathbf{X} = \frac{\partial X_i}{\partial x^i} = X_i^j i = X^j_i j. 
\]

Since covariant & ordinary differentiation coincide here, by the definition of covariant differentiation, \( \text{div} \mathbf{X} \) has the expression

\[
\text{div} \mathbf{X} = X^i_i = X^i_i + \Gamma^i_{jk} X^j
\]

in any frame or coordinate system.

**Exercise**

Let \( \mathbf{X} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \) be \( \rho \frac{\partial}{\partial \rho} + \sin \theta \frac{\partial}{\partial \theta} + \cos \theta \frac{\partial}{\partial \phi} \).

Why? Check this result for spherical coordinates.

But before checking that \( \text{div} \mathbf{X} = 2 \) in cylindrical coordinates, more theory.
\[ \Pi_{i;k} = \frac{1}{2} g^{ie} [g_{ek} g_{i;k} - g_{ik} g_{e;k} + g_{ik} g_{e;k}] + \frac{1}{2} (C^{k}_{;i} - C^{k}_{i} + C^{i}_{k}) \]
\[ = \frac{1}{2} g^{ie} g_{ek} g_{i;k} - \frac{1}{2} g^{ie} g_{ik} g_{e;k} + \frac{1}{2} g^{ie} g_{ik} g_{e;k} \]
\[ = \frac{1}{2} C^{k}_{;i} - \frac{1}{2} C^{i}_{k} + \frac{1}{2} C^{i}_{k} g^{ie} g_{ek} g_{i;k} \]
\[ = \frac{1}{2} (C^{k}_{e} g_{i} - C^{i}_{e} g_{k}) g^{ie} g_{ek} g_{i;k} \]

To get a simpler expression for the metric derivative term, we need an aside on the determinant function.
Aside (Part 1 revised)

Differential of \( \det A \)

Recall on page 54 of part 1 the formula for the determinant of a matrix \( A \)

\[
\det A = \epsilon_{i_1 \cdots i_n} A_{j_1}^{i_1} \cdots A_{j_n}^{i_n}
\]

\[
\det A \frac{\epsilon_{i_2 \cdots i_n}}{i_1} = \epsilon_{i_1 j_1 \cdots j_n} A_{j_1}^{i_1} \cdots A_{j_n}^{i_n}
\]

So

\[
\det A = \frac{1}{n!} \delta_{i_1 j_1 \cdots j_n} A_{i_1}^{j_1} \cdots A_{i_n}^{j_n}
\]

Define the cofactor matrix

\[
\epsilon(A)^{i_1 \cdots i_n}_{j_1 \cdots j_n} = \frac{1}{(n-1)!} \epsilon_{i_1 j_1 \cdots j_n} A_{j_1}^{i_1} \cdots A_{j_n}^{i_n}
\]

so that

\[
\det A = \epsilon(A)^{i_1 \cdots i_n}_{i_1 \cdots i_n} = \text{Tr} \epsilon(A) A
\]

Then

\[
\epsilon(A)^{i_1 \cdots i_n}_{i_1 \cdots i_n} A_{j_1}^{i_1} \cdots A_{j_n}^{i_n} = \frac{1}{(n-1)!} \epsilon_{i_1 j_1 \cdots j_n} A_{j_1}^{i_1} \cdots A_{j_n}^{i_n}
\]

at summand

\[
\delta_{i_1 j_1 \cdots j_n} \det A = \text{from above de}f \text{of determinant}
\]

\[
\frac{1}{(n-1)!} \epsilon_{i_1 j_1 \cdots j_n} A_{j_1}^{i_1} \cdots A_{j_n}^{i_n}
\]

i.e.

\[
\epsilon(A) A = (\det A) I
\]

or if \( \det A \neq 0 \):

\[
(\det A)^{-1} \epsilon(A) A = I
\]

We have derived the formula for the inverse of \( A \).

[Actually, \( \epsilon(A)^T \) is the cofactor of \( A^T \); i.e., differs by a transpose from the matrix of cofactors.]
Now by the product rule
\[ \frac{d}{dt} \det(A) = \frac{1}{n!} \sum_{i_1 \cdots i_n} \frac{d}{dt} \left( A_{i_1, j_1} \cdots A_{i_n, j_n} \right) \]

\[ = \frac{1}{n!} \sum_{i_1 \cdots i_n} A_{i_1, j_1} \cdots A_{i_n, j_n} \frac{dA_{i_1, j_1}}{dt} + \frac{dA_{i_1, j_1}}{dt} A_{i_1, j_1} \cdots A_{i_n, j_n} \frac{dA_{i_n, j_n}}{dt} \]

\[ = \frac{1}{n!} \sum_{i_1 \cdots i_n} A_{i_1, j_1} \cdots A_{i_n, j_n} \left( \frac{dA_{i_1, j_1}}{dt} + \frac{dA_{i_n, j_n}}{dt} \right) \]

\[ = \frac{1}{n!} \sum_{i_1 \cdots i_n} A_{i_1, j_1} \cdots A_{i_n, j_n} \left( \frac{dA_{i_1, j_1}}{dt} + \frac{dA_{i_n, j_n}}{dt} \right) \]

\[ = \text{Tr} \, \text{C}(A) \, dA \]

So if \( \det(A) \neq 0 \)

\[ \frac{d}{dt} \det(A) = \text{Tr} \left( \frac{dA}{dt} \right) A^{-1} dA \]

or

\[ \frac{d}{dt} \ln(\det(A)) = \text{Tr} \, A^{-1} dA \]

Now replace the matrix \( A \) by the matrix \( g = (g_{ij}) \)

\[ \frac{d}{dt} \ln(\det(g)) = \text{Tr} \, g^{-1} d(g) = g^{ij} d(g_{ij}) = g^{ij} d(g_{ij}) \]

and

\[ \frac{d}{dt} \ln(\det(g))^{1/2} = \frac{1}{2} \frac{d}{dt} \ln(\det(g)) = \frac{1}{2} g^{ij} d(g_{ij}) \]

\[ \frac{d}{dt} \ln(\det(g))^{1/2} = \frac{1}{2} g^{ij} d(g_{ij}) = \frac{1}{2} g^{ij} d(g_{ij}) \]

This is the formula we need for divergences. If you believe it, you can forget its derivation for now.

**Exercise:** The matrix \( g^{-1} = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \), \( dA \cdot dA^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} d\phi 

If the exercise on p. 50 (worked on p. 51) is an orthogonal matrix with unit determinant, so what is \( \text{Tr} \, A \cdot dA^{-1} \)? (Let \( A = g^{-1} \) in above formula.)
Returning to the self-contraction of the components of the covariant derivative

\[ \nabla_{i; k} = \frac{1}{2} g^{ik} g_{\ell i, k} - C^{i}_{\ell i} \]

\[ \frac{1}{2} \text{Tr} g^{-1} \, d g (e_k) \]

\[ = \frac{d}{d [\ln(\det g)]} (e_k) \]

\[ = [\ln(\det g)]_{, k} \]

So

\[ \text{div} \ X = X_{, i}^{i} + \nabla_{i; k} X^{k} - C^{i}_{\ell i} X^{k} \]

\[ = \ln(\det g)^{1/2} R_{X}^{2} \]

\[ = (\det g)^{-1/2} \left[ (\det g)^{1/2} X^{i} \right]_{, i} \]

(since by product rule = \( (\det g)^{1/2} \left[ (\det g)^{1/2} X^{i} \right]_{, i} + \left[ (\ln(\det g)^{1/2} \right]_{, i} X^{i} \)

\[ = X^{i}_{, i} + \left[ (\ln(\det g)^{1/2} \right]_{, i} X^{i} \)

\[ \text{div} \ X = (\det g)^{-1/2} \left[ (\det g)^{1/2} X^{i} \right]_{, i} = X^{i}_{, i} \]

\[ \text{frame} \]

\[ \text{coordinate frame} \]

\[ \text{frame} \]

Exercise. Use this formula to compute \( \text{div} \ X \) in polar and spherical coordinates for \( X \) given on page 66.

Notice that the divergence operator only involves the metric through the factor \( (\det g)^{1/2} = \gamma_{12...n} \) which is the component of the...
unit volume n-form associated with the metric. It does not care about the individual metric components. Any metrics whose unit volume forms coincide will yield the same divergence operator for vector fields.

Second covariant derivatives: notation

$$T'_{;i;\;k;\;\ell} = \left[\nabla\nabla T\right]_{;i;\;k;\;\ell}$$

is abbreviated to $$T'_{;i;\;k;\;\ell}$$;
in other words, the semi-colon is used to separate the additional covariant derivative indices from the original tensor indices, no matter how many extra derivative indices are added.

For a function $$\nabla f = \delta f = f_{;i} \omega^i$$ is the first covariant derivative and

$$\nabla \delta f = f_{;i;j} \omega^i \omega^j$$

is the second covariant derivative. The same notation is extended to the comma and ordinary differentiation:

$$f_{,;i;j} = f_{,i;j}$$

df function.

The Laplacian is defined in Cartesian coordinates on $$\mathbb{R}^n$$ by

$$\nabla^2 f = \text{div grad } f = \delta_{ij} f_{;i;j} = \delta_{ij} f_{;i} \omega^i \omega^j = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} + \frac{\partial^2 f}{\partial x^j \partial x^i}\right)$$

Therefore, in any frame or coordinate system, one has

$$\nabla^2 f = \text{div grad } f = g^{ij} f_{;i;j} = (g^{ij} f_{;i})_{;j};$$

since $$g^{ij}_{;k;\;l} = 0$$

Since both the metric & inverse metric are covariant constants, raising the first derivative index and then differentiating again is equivalent to differentiating twice and then contracting with the inverse metric.
Using the formula for the divergence, letting \( X = \nabla f = \nabla \phi \), we get

\[
\nabla^2 f = \text{div} \nabla f = (\det g)^{-1} \left[ (\det g) g^{ij} \partial_i \phi, \partial_j \phi \right] - C^k_{\ i j} g^{ij} \partial_k f
\]

vanishes for coordinate frame.

exerc | \n\n**Exercise** \n\n\[ \nabla^2 (x^2 - y^2) = \frac{\partial^2}{\partial x^2} (2x) - \frac{\partial^2}{\partial y^2} (2y) = 2 - 2 = 0 \quad \text{on } \mathbb{R}^2 \]

\[ f = x^2 - y^2 = \rho^2 (\cos^2 \phi - \sin^2 \phi) = \rho^2 \cos 2\phi \]

\[ = r^2 \sin^2 \theta \cos 2\phi \]

Confirm that \( \nabla^2 f = 0 \) in cylindrical and spherical coordinates.

exerc | \n\n**Exercise** \n\nVerify the divergence formulas (since \( \det g \) is \{ \rho \text{ cylindrical}, r \text{ spherical} \})

\[
\nabla^2 = \frac{\rho^2}{\rho} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}
\]

\[ = r^{-2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \]
exercise: Suppose \( \{ x^i \} \) are orthogonal coordinates on \( \mathbb{R}^3 \):

\[
\begin{align*}
g &= (h_1)^2 dx^1 dx^1 + (h_2)^2 dx^2 dx^2 + (h_3)^2 dx^3 dx^3 & (h_1 h_2 h_3 > 0) \\
g^{-1} &= (h_1)^{-2} \delta^2_1 dx^1 \delta^1_1 + (h_2)^{-2} \delta^2_2 dx^2 \delta^2_2 + (h_3)^{-2} \delta^3_3 dx^3 \delta^3_3 \\
\mathcal{N} &= h_1 h_2 h_3 \ dx^1 dx^2 dx^3.
\end{align*}
\]

Let \( e^i = \frac{\partial}{\partial x^i} \) and \( e^i = \frac{1}{h_i} \frac{\partial}{\partial x^i} \) be the coordinate frame and its associated normalized orthonormal frame, with \( \omega^i = dx^i \) and \( \omega^0 = h_1 h_2 h_3 \) (no sum on i).

Verify the formulas:

\[
\nabla^2 f = \frac{1}{h_1 h_2 h_3} \left( \frac{\partial}{\partial x^1} \left( h_1 h_3 \frac{\partial f}{\partial x^1} \right) + \frac{\partial}{\partial x^2} \left( h_2 h_3 \frac{\partial f}{\partial x^2} \right) + \frac{\partial}{\partial x^3} \left( h_1 h_2 \frac{\partial f}{\partial x^3} \right) \right)
\]

\[
\nabla \times \mathbf{X} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial x^1} \left( h_2 h_3 \frac{\partial X^2}{\partial x^1} \right) + \frac{\partial}{\partial x^2} \left( h_1 h_3 \frac{\partial X^3}{\partial x^2} \right) + \frac{\partial}{\partial x^3} \left( h_1 h_2 \frac{\partial X^1}{\partial x^3} \right) \right]
\]

What about the curl?

For this we need to explore another derivative: the "exterior derivative" which generalizes the differential of a function to an operator on p-forms or "differential forms" (antisymmetric covariant tensor fields).
More Practice Evaluating Components of the Covariant Derivative

The orthonormal frame associated with the orthogonal spherical coordinate frame is related to the orthonormal Cartesian coordinate frame by a rotation

\[
\begin{pmatrix}
e_x & e_y & e_z \\
\end{pmatrix} = \begin{pmatrix}
\sin \theta \cos \phi & \cos \theta & -\sin \phi \\
\sin \theta \sin \phi & \cos \theta & \cos \phi \\
\cos \theta & -\sin \theta & 0 \\
\end{pmatrix}
\]

The columns of the orthogonal matrix \( A^{-1} \) are the Cartesian coordinate frame components of the new orthonormal frame vectors and are obtained by normalizing the columns of the matrix \( A^{-1} \) on page 36 which represent the Cartesian coordinate components of the spherical coordinate frame vectors.

One can understand the matrix \( A^{-1} \) as resulting from the following sequence of simpler transformations, using an obvious shorthand for trig functions:

\[
\begin{pmatrix}
e_\phi & e_\theta & e_\rho \\
\end{pmatrix} \bigg|_{(r, \phi, \theta)} = \begin{pmatrix}
e_x & e_y & e_z \end{pmatrix} \bigg|_{(r, \theta = 0, \phi)} \begin{pmatrix}
n_\phi & -n_\theta & 0 \\
n_\theta & n_\phi & 0 \\
0 & 0 & 1 \\
\end{pmatrix} \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
e_\phi & e_\theta & e_\rho \\
\end{pmatrix} \bigg|_{(r, \theta = 0, \phi)} \xrightarrow{\text{rotate to angle } \phi} \begin{pmatrix}
e_\phi & e_\theta & e_\rho \\
\end{pmatrix} \bigg|_{(r, \phi, \theta)} \xrightarrow{\text{permute}} \begin{pmatrix}
e_\theta & e_\phi & e_\rho \\
\end{pmatrix} \bigg|_{(r, \phi, \theta)} \xrightarrow{\text{rotate to angle } \theta}
\]

\[
\begin{pmatrix}
e_\phi & e_\theta & e_\rho \\
\end{pmatrix} \bigg|_{(r, \phi, \theta)} \xrightarrow{\text{rotation by angle } \phi \text{ about } e_\rho \text{ in plane of } e_x \text{ and } e_y} \begin{pmatrix}
e_\phi & e_\theta & e_\rho \\
\end{pmatrix} \bigg|_{(r, \phi, \theta)} \xrightarrow{\text{rotation by angle } \theta \text{ about } e_\phi \text{ in plane of } e_\theta \text{ and } e_\rho}
\]

\[
74
\]
Check that the matrix product of these three factor matrices is \( A^{-1} \).

Since \( A \) is an orthogonal matrix, \( A^{-1} = A^T \) or \((A^*)^T = A\), so

\[
A = \begin{pmatrix}
\sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\
\cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\
-\sin\phi & \cos\phi & 0
\end{pmatrix}
\]

Now it is a straightforward problem to evaluate the matrix

\[
\hat{\mathbf{a}} = A \hat{\mathbf{a}} A^{-1} = \left( \begin{pmatrix}
\hat{\mathbf{r}}^A \hat{\mathbf{r}}^A \\
\hat{\mathbf{q}}^A \hat{\mathbf{q}}^A \\
\hat{\mathbf{p}}^A \hat{\mathbf{p}}^A
\end{pmatrix}
\right)
\]

in the spherical orthonormal frame.

\[
= \begin{pmatrix}
\cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\
\sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\
-\sin\phi & \cos\phi & 0
\end{pmatrix}
\begin{pmatrix}
\hat{\mathbf{r}}^A \\
\hat{\mathbf{q}}^A \\
\hat{\mathbf{p}}^A
\end{pmatrix}
\begin{pmatrix}
\cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\
\sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\
-\sin\phi & \cos\phi & 0
\end{pmatrix}^{-1}
\]

\[
= \begin{pmatrix}
\cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\
\sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\
-\sin\phi & \cos\phi & 0
\end{pmatrix}
\begin{pmatrix}
\hat{\mathbf{r}}^A & \hat{\mathbf{q}}^A & \hat{\mathbf{p}}^A
\end{pmatrix}
\begin{pmatrix}
\cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\
\sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\
-\sin\phi & \cos\phi & 0
\end{pmatrix}^{-1}
\]

\[
= \begin{pmatrix}
\cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\
\sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\
-\sin\phi & \cos\phi & 0
\end{pmatrix}
\begin{pmatrix}
\hat{\mathbf{r}}^A & \hat{\mathbf{q}}^A & \hat{\mathbf{p}}^A
\end{pmatrix}
\begin{pmatrix}
\cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\
\sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\
-\sin\phi & \cos\phi & 0
\end{pmatrix}^{-1}
\]

\[
= \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\hat{\mathbf{r}}^A \\
\hat{\mathbf{q}}^A \\
\hat{\mathbf{p}}^A
\end{pmatrix}
\begin{pmatrix}
\cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\
\sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\
-\sin\phi & \cos\phi & 0
\end{pmatrix}^{-1}
\]

\[
= \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\hat{\mathbf{r}}^A \\
\hat{\mathbf{q}}^A \\
\hat{\mathbf{p}}^A
\end{pmatrix}
\begin{pmatrix}
\cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\
\sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\
-\sin\phi & \cos\phi & 0
\end{pmatrix}^{-1}
\]

So \( \hat{\mathbf{r}}^B = \hat{\mathbf{r}}^A \), \( \hat{\mathbf{q}}^B = \hat{\mathbf{q}}^A \), \( \hat{\mathbf{p}}^B = \hat{\mathbf{p}}^A \), and

\[
\hat{\mathbf{r}}^B = \hat{\mathbf{r}}^A = \hat{\mathbf{r}}^A = \hat{\mathbf{r}}^A
\]

\[
\hat{\mathbf{q}}^B = \hat{\mathbf{q}}^A = \hat{\mathbf{q}}^A = \hat{\mathbf{q}}^A
\]

\[
\hat{\mathbf{p}}^B = \hat{\mathbf{p}}^A = \hat{\mathbf{p}}^A = \hat{\mathbf{p}}^A
\]

gives the first nonzero components of the covariant derivative.

Six (vii)

We can also derive these results from the metric formula. For an orthonormal frame \( \{ \mathbf{e}^i \} \), then \( g_{ij} = g(\mathbf{e}^i, \mathbf{e}^j) = \delta_{ij} \), i.e., the components of the metric are constants so the metric component derivative terms in the formula (vanish), leaving only the structure function terms

\[
\Gamma^A_{ik} = \frac{1}{2} \left( \frac{\partial C^{Ak}_{ik}}{\partial q^k} + \frac{\partial C^{Ak}_{ik}}{\partial q^i} \right) = \frac{1}{2} \left( C_{ik}^{Ak} - C_{ik}^{Ak} + C_{ik}^{Ak} \right)
\]

On page 53 these are given (incorrectly): (since index shifting trivial in ON frame)

\[
C_{ik}^{Ak} = -\frac{1}{2} = -C_{ik}^{Ak}, \quad C_{ik}^{Ak} = -\frac{1}{2} = -C_{ik}^{Ak}, \quad C_{ik}^{Ak} = -\frac{1}{2} \cot \theta = -C_{ik}^{ Ak}.
\]
The nonzero components of the connection must have indices which are
permutation of the index positions on the structure functions.
Forgetting for a moment that we know which components of the
connection are nonzero, we can use the following reasoning to avoid
evaluating the formula for many components which turn out to be
zero.

On page 60 we saw that the covariant constancy of the metric
means
\[ 0 = g_{ij,k} = g_{ij,k} - g_{lj} \Gamma^{l}_{ki} - g_{ik} \Gamma^{k}_{lj} \]
\[ = g_{ij,k} - \Gamma^{l}_{ki} + \Gamma^{k}_{lj} \]

or
\[ g_{ij,k} = \Gamma^{l}_{ki} + \Gamma^{k}_{lj} \]
For an orthonormal frame \( g_{ij} = \delta_{ij} \) and \( g_{ij,k} = 0 \) (constant components)
so
\[ \Gamma^{l}_{ki} = - \Gamma^{k}_{li} \]
\[ \text{i.e., the components of the covariant derivative are antisymmetric in} \]
\[ \text{their outer indices. This remains true when we raise the index since} \]
\[ \text{the metric component matrix is the identity matrix and explains why} \]
\[ \text{the matrix} \quad \Omega = (\Gamma^{k}_{ij} \omega^{k}) \quad \text{evaluated above for the spherical} \]
\[ \text{orthonormal frame is antisymmetric — its matrix indices are the} \]
\[ \text{outer pair of indices on the components of the covariant derivative} \]
\[ \text{[Go back and look at the result for} \quad \Omega \text{ on the previous page} \]
\[ \text{and see that it is antisymmetric.]} \]

Thus the outer pair of indices on \( \Gamma^{k}_{ij} \) must be distinct
\text{and antisymmetry tells us the value of one index ordering in terms of}
\text{the other ordering. Given the three nonzero independent structure}
\text{functions (6 by antisymmetry), there are only three nonzero independent}
\text{components of the connection (6 by antisymmetry) that we can}
\text{write down.} \]
\[
\begin{align*}
C^\theta_{\phi\phi} &= \frac{1}{2} (C^\theta_{\phi\phi} - C^\phi_{\phi\theta} + C^\theta_{\phi\theta}) = \frac{1}{2} (C^\theta_{\phi\phi} + C^\theta_{\phi\phi} + C^\phi_{\phi\theta}) \\
-\Gamma^\theta_{\phi\phi} &= \frac{1}{2} (C^\phi_{\theta\phi} - C^\phi_{\phi\theta} + C^\theta_{\theta\phi}) = \frac{1}{2} (C^\phi_{\theta\phi} + C^\phi_{\theta\phi} + C^\theta_{\theta\phi}) \\
C^\phi_{\phi\theta} &= \frac{1}{2} (C^\theta_{\phi\theta} - C^\phi_{\phi\theta} + C^\theta_{\phi\theta}) = \frac{1}{2} (C^\phi_{\phi\theta} + C^\phi_{\phi\theta} + C^\theta_{\phi\theta}) \\
-\Gamma^\phi_{\phi\theta} &= \frac{1}{2} (C^\theta_{\phi\phi} - C^\phi_{\phi\theta} + C^\theta_{\phi\theta}) \\
\end{align*}
\]

So in fact we did not do any work to verify that certain components are zero.

What is the significance of the antisymmetry property of the matrix \( \omega = (\Gamma^i_{jk}) \) in an orthonormal frame? Well, the definition:

\[
\nabla_X e_i = \Gamma^j_{ik} e_j, \quad \omega^j_i(x)
\]

\[
\nabla_X e_i = \nabla_X [\epsilon_X e_i] = \frac{d}{dX} \nabla e_i = \Gamma^j_{ik} e_j
\]

directional covariant derivative linear in direction:

\[
\nabla_X Y^i = Y^j_{\mid j} X^i
\]

So

\[
\nabla_X e_i = \omega^j_i(x) e_j
\]

The value of the 1-form \( \omega \) on \( X \) gives the matrix of the linear transformation of the frame vectors which describes their covariant derivative in that direction, i.e., how they change relative to a Cartesian frame as we move in that direction. The fact that this matrix is antisymmetric tells us that in 3-dimensions it can be represented by the cross-product of a vector. (77)
Aside on Orthogonal Matrices

For the Euclidean inner product on \( \mathbb{R}^n \), the components of the inner product are just
\[
g(e_i, e_j) = e_i \cdot e_j = \delta_{ij}
\]
in the standard basis or in any orthonormal basis. If
\[
\bar{e}_i = A^{ij} e_j, \quad e_i = A_{ij} \bar{e}_j
\]
is a transformation relating any two orthonormal bases, then the inner product transforms in the following way
\[
\delta_{ij} = A^{im} A^{nj} \delta_{mn}
\]
or in matrix form
\[
\begin{bmatrix} \delta_{ij} \end{bmatrix} = A^T \delta_{mn} A
\]
Thus \( A^T = A^{-1} \) describes the matrix of linear transformations between orthonormal bases. The condition \( A^T A = I \) just states that the column vectors of \( A \) are an orthonormal set of vectors in \( \mathbb{R}^n \). Such matrices are called orthogonal matrices. They represent rotations and reflections of \( \mathbb{R}^n \) into itself.

Suppose \( A \) depends on a parameter \( \lambda \) so we get a family of orthogonal matrices. Then
\[
\frac{d}{d\lambda} \left[ A^T A = I \right]
\]
\[
\left( \frac{d}{d\lambda} A \right)^T A + A^T \frac{d}{d\lambda} A = 0
\]
and \( A^T = A^{-1} \) so
\[
\frac{d}{d\lambda} A^{-1} + \left( \frac{d}{d\lambda} A \right)^T = 0
\]
This just says that the matrix \( B = A^{-1} \frac{d}{d\lambda} A = -B^T \frac{d}{d\lambda} \) is antisymmetric.
The same thing is true if we take the differential
\[ A^i dA = A^i dA d\lambda \]
instead of the derivative. This explains why
\[ \hat{\Omega} = \hat{A} dA^{-1} = (A^{-i})^{-1} d(A^{-i}) \]
is antisymmetric. We are differeniating an orthogonal matrix \( A^{-i} \). The matrix \( \hat{\Omega}(X) \) for a given vector field \( X \) tells us the rate of change of the rotation which the orthonormal frame undergoes as we move in the direction of \( X \). The rate of change of a rotation can be described by an angular velocity.

To understand this, suppose a point of \( \mathbb{R}^3 \) undergoes an active rotation
\[ X'(t) = A_1^j(\theta) X^j(0) \]
position at \( t = 0 \)

Then
\[ \frac{dX}{dt} = \frac{dA}{dt} \frac{X}{A} \frac{dX}{dA} = \frac{dA}{dt} \frac{X}{A} \frac{dX}{dA} \]

\[ A A^{-1} = I \]

\[ \frac{dA}{dt} A^{-1} + A \frac{dA^{-1}}{dt} = 0 \]

so
\[ \frac{dX}{dt} = -i \kappa \Omega \times X \]

\[ \frac{dX}{dt} = \hat{\Omega} \times X \]

describes instantaneous rate of change of angle
\[ \frac{dX}{dt} = \| \hat{\Omega} \| \]

\[ \text{Eighth rule 3f to } \hat{X}, \text{ thumb out} \]

\[ \text{instantaneous rate of change of angle} \]

\[ \frac{dX}{dt} = \| \hat{\Omega} \| \]

\[ \text{Eighth rule 3f to } \hat{X}, \text{ thumb out} \]
In other words by taking the dual of the antisymmetric matrix \( B \) we get an angular velocity vector which describes the direction about which a rotation is occurring (instantaneously) and the rate of change of the angle about that direction.

Anyway, this is more of an aside than I wanted to get into. I just wanted to give you a feeling for the antisymmetric matrix of covariant derivatives of the frame vectors.

Okay, so you didn’t do rotation and angular velocity in your physics courses, or maybe you never understood the righthand rule, or maybe you’re just not patient enough to read this stuff about derivatives of orthogonal matrices and duals of antisymmetric matrices — okay, it doesn’t matter. The cross-product & right hand rule only work in 3 dimensions where a pair of antisymmetric indices can be swapped for a single index by the duality operation. In any other dimension, you are stuck with a 2-plane in which a rotation takes place, so it is enough to look at rotations of \( \mathbb{R}^2 \) to understand how they work.
By trigonometry
\[
\vec{e}_1 = \cos \theta \vec{e}_1 + \sin \theta \vec{e}_2 \\
\vec{e}_2 = -\sin \theta \vec{e}_1 + \cos \theta \vec{e}_2
\]

\[
\frac{d}{d\theta} (\vec{e}_1, \vec{e}_2) = (e_1, e_2) \begin{pmatrix} -\sin \theta & -\cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix}
\]

\[
\frac{d}{d\theta} (e_1, e_2) = (e_1, e_2) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

Incremental change in \(\vec{e}_1, \vec{e}_2\) for small change \(d\theta\) in \(\theta\) at \(\theta = 0\)

The interpretation of this is that as you begin to rotate the basis vectors through a small angle \(d\theta\), \(\vec{e}_1\) begins to rotate toward \(\vec{e}_2\) and \(\vec{e}_2\) towards \(-\vec{e}_1\) explaining the antisymmetry of the matrix \(B = \frac{dA}{d\theta}\) at \(\theta = 0\).

Now look at 
\[
\omega = A dA^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} d\theta + \begin{pmatrix} 0 & -\sin \theta \\ \sin \theta & 0 \end{pmatrix} d\phi
\]

which tells us how the spherical orthonormal frame vectors begin to change as we make small increments \(d\theta\) and \(d\phi\) in the angular variables, or alternatively, tells us the rate at which these frame vectors are rotating as we change the angular coordinates. The fact that these \(1\)-forms have no component along \(d\theta\) means that they don't rotate as we change \(\theta\), i.e., as we move radially, and that is exactly right.
If we hold \( \phi \) fixed and increase \( \Theta \), 
\( E_\Theta \) remains fixed while \( (E_\phi, E_\theta) \) rotate by exactly the increment of \( \Theta \) in their 2-plane in the usual counterclockwise sense, so the 2\times2 part of the matrix with \( \phi, \theta \) indices is exactly the matrix of our two-dimensional discussion.

If we hold \( \Theta \) fixed and increase \( \phi \), what happens depends on the value of \( \Theta \).

For \( \Theta = \frac{\pi}{2} \) we are in the x-y plane and 
\( E_\phi \) remains equal to \( E_\theta \) as we change \( \phi \) but \( (E_\phi, E_\theta) \) undergoes the same 2-dimensional rotation by exactly the increment in \( \phi \).

This is just what
\[
\Delta \left|_{\phi=\pi/2} \right. = \left. \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right| d\phi
\]
describes.

At the other extreme \( \Theta \to 0 \) we end upon the z-axis almost, where \( E_\phi \approx E_\theta \) remains fixed and
\[
(E_\phi, E_\theta) \] rotate by exactly the increment in \( \phi \) which is what
\[
\Delta \left|_{\phi=0} \right. = \left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \right| d\phi
\]
describes. Are you convinced?
What is the interpretation of \( \mathbf{\Omega} = \mathbf{A} \mathbf{dA}^{-1} \) for the spherical coordinate frame which is not orthonormal? From page 36:

\[
\begin{bmatrix}
S \omega_{\phi} & S \omega_{\theta} & C \omega_{\phi} \\
\frac{r \omega_{\phi}}{r} - r & -S \omega_{\phi} & -r \omega_{\phi} \\
0 & 0 & 0
\end{bmatrix} \mathbf{d}
\begin{bmatrix}
S \omega_{\phi} & S \omega_{\theta} & C \omega_{\phi} \\
\frac{r \omega_{\phi}}{r} & -S \omega_{\phi} & -r \omega_{\phi} \\
0 & 0 & 0
\end{bmatrix}
\]

\[
= \ldots
\begin{bmatrix}
0 & 0 & 0 \\
0 & \frac{1}{r} & 0 \\
0 & 0 & \frac{1}{r}
\end{bmatrix}
\mathbf{d}
\begin{bmatrix}
0 & -1 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix}
\]

Thus \( \Gamma^\theta_{\phi \theta} = \Gamma^\phi_{\phi \phi} = r^{-1} \):

\[
\Gamma^\theta_{\phi \theta} = r^{-1}, \quad \Gamma^\phi_{\theta \phi} = r^{-1} \quad; \quad \Gamma^\phi_{\phi \phi} = \cot \theta
\]

\[
\Gamma^\theta_{\phi \phi} = -S \omega_{\phi}, \quad \Gamma^\phi_{\theta \phi} = \cot \theta
\]

\[\text{[Check these.]}\]

The appearance of \( r \) in the \( \theta \) and \( \phi \) components of the 1-form \( \mathbf{\Omega} \) just takes into account the fact that for fixed \( r \), \( \epsilon_\theta \) and \( \epsilon_\phi \) are not unit vectors, so the existing non-zero components of the covariant derivative in the associated orthonormal frame are simply rescaled by factors of \( r \), except for the additional component \( \Gamma^\theta_{\phi \phi} = \cot \theta \cdot \mathbf{\Omega} (\epsilon_\phi, \epsilon_\phi) \). This describes the change in the length of \( \epsilon_\phi \) as we change \( \theta \). Similarly, the extra components \( \Gamma^\theta_{\phi \phi} \) and \( \Gamma^\theta_{\phi \phi} \) describe the change in the length of \( \epsilon_\theta \) and \( \epsilon_\phi \) as we change \( r \).
\[ [X, Y]^\hat{=} = Y^\hat{=} \hat{\rho} X^\hat{=} + Y^\hat{=} \hat{\rho} X^\hat{=} - X^\hat{=} \hat{\rho} Y^\hat{=} + \frac{\partial}{\partial \hat{\rho}} \frac{\partial}{\partial \hat{\rho}} X^\hat{=} Y^\hat{=} \]
\[ = \hat{\rho} \frac{\partial}{\partial \hat{\rho}} (p \sin 2\varphi) + \frac{1}{2} \frac{\partial}{\partial \rho} (p \cos 2\varphi) - \frac{\partial}{\partial \rho} (p \sin 2\varphi) \cdot \hat{\rho} \]
\[ = p \sin 2\varphi - p \sin 2\varphi = 0 \]

\[ [X, Y]^\hat{=} = Y^\hat{=} \hat{\rho} X^\hat{=} + Y^\hat{=} \hat{\rho} X^\hat{=} - X^\hat{=} \hat{\rho} Y^\hat{=} + C^\hat{=} \hat{\rho} \hat{\rho} X^2 Y^\hat{=} + C^\hat{=} \hat{\rho} \hat{\rho} X^\hat{=} Y^2 + C^\hat{=} \hat{\rho} \hat{\rho} X^\hat{=} Y^\hat{=} \]
\[ = - \frac{\partial}{\partial \rho} (p \cos 2\varphi) \cdot \hat{\rho} + p \cos 2\varphi = 0 \]

\[ [X, Y]^2 = \frac{\gamma^2 (\gamma + 3)}{3} \hat{\rho} X^\hat{=} x^\hat{=} + C^\hat{=} \hat{\rho} \hat{\rho} x^3 y^\hat{=} = 0 \]

so \[ [X, Y] = 0 \]

Compare

\[ [X, Y] = [y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}] \]
\[ = y \frac{\partial}{\partial x} \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \frac{\partial}{\partial y} - x \frac{\partial}{\partial x} \frac{\partial}{\partial y} - y \frac{\partial}{\partial y} \frac{\partial}{\partial x} \]
\[ = y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} - x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} = 0. \]
exercise on page 57 worked

Preliminary remark. In any frame we have the definition
\[ \delta_{m}^{i} = \delta_{m}^{i} \delta_{n}^{j} - \delta_{n}^{i} \delta_{m}^{j}. \]

If we differentiate this equation,
\[ \delta_{mn}^{i} = \frac{\delta_{m}^{j} \delta_{n}^{i} \delta_{n}^{j}}{\delta_{m}^{i}} + \frac{\delta_{n}^{j} \delta_{m}^{i} \delta_{m}^{j}}{\delta_{n}^{i}} - \frac{\delta_{n}^{j} \delta_{n}^{i} \delta_{m}^{j}}{\delta_{m}^{i}} - \frac{\delta_{m}^{j} \delta_{m}^{i} \delta_{n}^{j}}{\delta_{n}^{i}} = 0, \]
then \( \nabla \delta_{m}^{i} = 0 \) follows from \( \nabla \delta = 0 \) and the product rule.

However, just using the formula in the barred frame
\[ \delta_{mn}^{i} = \frac{\delta_{m}^{j} \delta_{n}^{i} \delta_{n}^{j}}{\delta_{m}^{i}} + \frac{\delta_{n}^{j} \delta_{m}^{i} \delta_{m}^{j}}{\delta_{n}^{i}} - \frac{\delta_{n}^{j} \delta_{n}^{i} \delta_{m}^{j}}{\delta_{m}^{i}} - \frac{\delta_{m}^{j} \delta_{m}^{i} \delta_{n}^{j}}{\delta_{n}^{i}} = 0, \]
\[ \left( \delta_{km}^{i} - \delta_{km}^{n} \right) + \left( \delta_{kn}^{i} - \delta_{kn}^{n} \right) \]
\[ - \left( \delta_{km}^{n} - \delta_{kn}^{n} \right) - \left( \delta_{km}^{i} - \delta_{kn}^{i} \right) = 0. \]
Exercise on page 55 worked

Remember only $\Gamma^0_{\rho\sigma} = -\rho_{\sigma}, \Gamma^0_{\sigma\rho} = -\rho_{\sigma}, \Gamma^0_{\rho\sigma} = -\rho_{\sigma}$ are nonzero.

$X, Y$ have no $Z$ components, no components depend on $Z$, so this is basically a 2-dimensional problem (anything with a $Z$ index vanishes).

So:

$\nabla_{\rho} \phi = \frac{1}{\rho} \frac{\partial \phi}{\partial \varphi} = -\frac{\partial \phi}{\rho \sin \theta}$

write down only nonzero terms:

$\nabla_{\rho} = \frac{1}{\rho} \frac{\partial}{\partial \varphi} \quad \nabla_{\theta} = \frac{1}{\rho \sin \theta} \frac{\partial}{\partial \theta}$

$\nabla_{\rho} (x^0 + x^2) = \frac{1}{\rho} \frac{\partial}{\partial \varphi} (x^0 + x^2) = \frac{1}{\rho^2} \frac{\partial}{\partial \theta} (x^0 + x^2) = 0$

nothing new to be gained by doing $\nabla \phi = 0$ so let's move on:

$\nabla^2 (x^0 + x^2) = \frac{1}{\rho^2} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} (x^0 + x^2) = 0$

nothing new from $\nabla \delta y = 0$

Finally $[dz]_\rho = 0 = [dz]_\theta$, $[dz]_\varphi = 1$ so

$[dz]_{ij} = [dz]_{ij} - [\Gamma_{ij}^k (dz)_k]_\varphi = 0$

$\delta y = 0$

$\nabla^2 (x^0 + x^2) = -\nabla^2 (x^0 + x^2) = 0$

$\delta y - \rho \cos \theta \delta y = 0$
Exercises on page 62 worked

\[ \Gamma_{ijk} = \frac{1}{2} g^{le} \left( \delta_{rk,e} - \delta_{rk,i} + \delta_{rk,j} \right) \]

but \( g_{ij} = g_{ji} \) so

\[ \Gamma_{ijk} = \frac{1}{2} g^{le} \left( \delta_{rk,e} - \delta_{rk,i} + \delta_{rk,j} \right) = \Gamma_{ijk} \]

\[ \tilde{g}_{ik} = \frac{1}{2} \left( g_{ik,j} - g_{kj,i} + g_{ij,k} \right) \]

[Note this is symmetric in \( i, j, k \) for the same reason]

\[ \tilde{g}_{pp} = \tilde{g}_{xx} = p^2 \]

At least two indices have to be the same to get a diagonal metric component to differentiate, otherwise you differentiate an off-diagonal metric component which is zero. Finally, the only diagonal component with a nonzero derivative is \( g_{pp} = p^2 \) so the indices have to be some permutation of \( (p, p, p) \) to get a nonzero result.

\[ \Gamma_{pnp} = \frac{1}{2} \left( g_{pp,e} - g_{en,p} + g_{ep,n} \right) = -\frac{1}{2} \left( 2p \right) = -p \]

\[ \Gamma_{ppp} = \frac{1}{2} \left( g_{pp,e} - g_{pp,n} + g_{pp,p} \right) = \frac{1}{2} \left( 2p \right) = p \]

\[ \Gamma_{opp} = \frac{1}{2} \left( \ldots \right) = \Gamma_{defenses} = \rho^2 \] since symmetric in last two indices in coordinate frame.

Now raise first index:

\[ \nabla^p \rho^p = g^{pp} \Gamma_{ppp} = -p \]

\[ \nabla^p \rho^p = g^{pp} \Gamma_{ppp} = \rho^2 \left( p \right) = \frac{1}{\rho} = \Gamma^p \rho^p \]

Done.

Interpretation:

\[ \nabla e^p = -e^p \rho^p \]

\[ e^p \rho^p \]

\[ e^p \rho^p \] in radial direction.

\[ e^p \] has length \( p \). Translate its value at \((p, q, 0, 0)\) back to \((0, 0)\) so has same initial point as \( e^p \) at \((0, 0)\).

Difference is \( \rho \cdot e^p \) in radial direction.

Try interpreting another.
last exercise on page 62 worked

Preliminary remark. Whatever symmetries a tensor has, its covariant derivative has the same symmetries.

Example: \[ T_{ij} = T_{ji} \] is symmetric so

\[ T_{ijk} = T_{ij,k} = \Gamma^k_{ij} T_{ij} - \Gamma^i_{kj} T_{ij} - \Gamma^j_{ki} T_{ij} \]

But really this is a 2-dimensional problem because nothing depends on \( z \) and no \( z \) components are nonzero, so no \( z \)-component of \( T_{ijk} \) is nonzero, so we have 3 independent components of \( T_{ij} \) times 2 for its covariant derivative for a grand total of 6. Not too bad.

Recall \( \Gamma^p_{qq} = \delta^p_q \), \( \Gamma^p_{pe} = \Gamma^p_{eq} = \delta^p_q \)

\[ T_{pp;p} = T_{pp} \frac{\partial p}{\partial p} - \Gamma^i_{pp} T_{pi} - \Gamma^i_{pp} T_{pi} = -2 \cot \gamma + 2 \sin \gamma = 0 \]

\[ T_{pq;p} = T_{pq} \frac{\partial p}{\partial p} - \Gamma^i_{pq} T_{qi} - \Gamma^i_{pq} T_{qi} = 2 \cot \gamma + 2 \sin \gamma = 0 \]

\[ T_{pp;q} = T_{pp} \frac{\partial q}{\partial p} - \Gamma^i_{pp} T_{qi} - \Gamma^i_{pp} T_{qi} = 2 \cot \gamma - 2 \sin \gamma = 0 \]

\[ T_{pq;q} = T_{pq} \frac{\partial q}{\partial p} - \Gamma^i_{pq} T_{qi} - \Gamma^i_{pq} T_{qi} = 0 \]

\[ T_{pp;p} = T_{pp} \frac{\partial p}{\partial p} - \Gamma^i_{pp} T_{pi} - \Gamma^i_{pp} T_{pi} = - \sin \gamma + \sin \gamma = 0 \]

\[ T_{pq;p} = T_{pq} \frac{\partial p}{\partial p} - \Gamma^i_{pq} T_{qi} - \Gamma^i_{pq} T_{qi} = \frac{\partial ( \cot \gamma - \sin \gamma )}{\partial p} = 0 \]

\[ T_{pp;q} = T_{pp} \frac{\partial q}{\partial p} - \Gamma^i_{pp} T_{qi} - \Gamma^i_{pp} T_{qi} = \frac{\partial ( \cot \gamma - \sin \gamma )}{\partial p} = 0 \]

\[ T_{pq;q} = T_{pq} \frac{\partial q}{\partial p} - \Gamma^i_{pq} T_{qi} - \Gamma^i_{pq} T_{qi} = \frac{\partial ( \cot \gamma - \sin \gamma )}{\partial p} = 0 \]
\[ \Pi^{i j k} = \left[ \xi^{i j k} \right] + \frac{1}{2} \left( C^{i j k} - C^{j i k} + C^{k i j} \right) \]

since already antisymmetric in these indices

\[ = \frac{1}{2} \left( C^{i j k} - C^{k i j} \right) \]

by antisymmetry of last pair of indices

\[ = \frac{1}{2} \left( C^{i j k} - C^{k i j} \right) = 0 \]

cancel in pairs

\[ = \frac{1}{2} C^{i j k} \checkmark \]

If you don't believe it:

\[ C^{i j} = g^i m_j g^m n_j \]

\[ = -g^i m_j g^m n_j = -C^{i j} \checkmark \]

etc.

On page 53 we found

\[ C^{\phi \phi} = -\frac{1}{\rho} = -C^{\phi \phi} \]

(only nonzero structure function)

So to get a nonzero component of \( \Pi^{i j k} \) the indices must be a permutation of \( (\phi \phi \phi) \):

\[ \Gamma^{\phi \phi \phi} = \frac{1}{2} \left( C^{\phi \phi \phi} - C^{\phi \phi \phi} + C^{\phi \phi \phi} \right) = \frac{1}{2} \left( C^{\phi \phi \phi} - C^{\phi \phi \phi} + C^{\phi \phi \phi} \right) = 0 \]

\[ \Gamma^{\phi \phi \phi} = \frac{1}{2} \left( C^{\phi \phi \phi} - C^{\phi \phi \phi} + C^{\phi \phi \phi} \right) = \frac{1}{2} \left( C^{\phi \phi \phi} - C^{\phi \phi \phi} + C^{\phi \phi \phi} \right) = 0 \]

\[ \Gamma^{\phi \phi \phi} = \frac{1}{2} \left( C^{\phi \phi \phi} - C^{\phi \phi \phi} + C^{\phi \phi \phi} \right) = \frac{1}{2} \left( C^{\phi \phi \phi} - C^{\phi \phi \phi} + C^{\phi \phi \phi} \right) = 0 \]

Compare with page 58 andoops! I forgot to normalize!

\[ \bar{\omega} = \left( \Pi^{\omega \omega} \bar{\omega} \right) = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \bar{\omega} = \left( \begin{array}{cc} 0 & -\rho^{-1} \\ 1 & 0 \end{array} \right) \bar{\omega} \]

so

\[ \Pi^{\phi \phi} = -\rho^{-1}, \quad \Pi^{\phi \phi} = \rho^{-1} \quad \text{agreement} \checkmark \]
Following up the first exercise on page 66, consider instead the function
\[ f = xy = \rho^2 \sin \theta \cos \phi = \frac{1}{2} \rho^2 \sin \theta \sin 2\phi \]

Then
\[ df = y \, dx + x \, dy = \mathbf{X}^t \]
yields our friend \( \mathbf{X} \) from pages 33, 37, 38, 39

\[ \nabla f = (df)^t = y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} = \mathbf{X} \]

where we saw that
\[ \mathbf{X} = \rho \sin \theta \cos \phi \frac{\partial}{\partial \rho} + \cos \theta \frac{\partial}{\partial \phi} = \sin \theta \sin 2\phi \left( \rho \sin \theta \cos \phi + \cos \phi \right) + \cos \theta \frac{\partial}{\partial \phi} \]

\[ \mathbf{X}^t = \rho \sin \theta \sin 2\phi \frac{\partial}{\partial \rho} + \rho^2 \cos \theta \frac{\partial}{\partial \phi} = \sin \theta \sin 2\phi \left( \rho \sin \theta \cos \phi + \cos \phi \right) + \rho^2 \cos \theta \frac{\partial}{\partial \phi} \]

\[
\left[ \frac{\partial}{\partial r} \right] = g_{ij} \left( \frac{\partial}{\partial r} \right) = g_{ir} \quad \Rightarrow \quad \left[ \frac{\partial}{\partial r} \right] = g_{ir} \, \partial x^i = g_{ir} \, dr = dr
\]

Similarly
\[
\left[ \frac{\partial}{\partial \theta} \right] = g_{ij} \left( \frac{\partial}{\partial \theta} \right) = g_{i\theta} \, \partial x^i = g_{i\theta} \, d\theta = r \sin \theta \, d\theta
\]

In general
\[
e_i^* = g_{ik} e_i^* \Theta_k = g_{i\theta} \omega^i
\]

so that
\[
\mathbf{X}^* = (X^i e_i^*)^* = X^i e_i^* \Omega = X^i \omega^i
\]

Similarly
\[
\left[ \Omega^i \right]^* = g_{ij} e_j
\]

for an orthogonal frame, interchanging the frame vectors and dual frame anchors yields the corresponding basis vector or vector multiplied by the diagonal metric component or its reciprocal.

Compute \( df \) and \( \nabla f = \nabla f \) in cylindrical coordinates and verify that you get our previous results quoted above.