Covariant Derivatives on $\mathbb{R}^n$ with Euclidean metric

It is clear what we mean by a "constant tensor field" on $\mathbb{R}^n$ — namely one whose Cartesian coordinate components are constant in the standard Cartesian coordinate system on $\mathbb{R}^n$ (or in fact for any Cartesian coordinates). There is a 1-1 correspondence between tensors over the vector space $\mathbb{R}^n$ and such constant tensor fields on $\mathbb{R}^n$.

$$S = \underbrace{S^{i_1\cdots i_p}{}_{j_1\cdots j_q}}_{\text{constants}}\, e_{i_1}\otimes\cdots\otimes\omega^{j_1}\otimes\cdots \in T^{(p,q)}(\mathbb{R}^n)$$

$$\updownarrow$$

$$\tilde{S} = S^{i_1\cdots i_p}{}_{j_1\cdots j_q}\,\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes dx^{j_1}\otimes\cdots = \overset{\text{constant}}{\binom{p}{q}}\text{-tensor field on } \mathbb{R}^n.$$

Such a constant tensor field is also characterized by the vanishing of the Cartesian coordinate derivatives of its Cartesian coordinate components

$$\frac{\partial}{\partial x^k} S^{i_1\cdots i_p}{}_{j_1\cdots j_q} \equiv S^{i_1\cdots i_p}{}_{j_1\cdots j_q, k} = 0 .$$

This is an important concept for the geometry of $\mathbb{R}^n$, since if we take a tangent vector at the origin and translate it all over space, we obtain a constant vector field. In other words constant tensor fields tell us something about how to move the tangent and cotangent spaces around in space without changing length or orientation information.

The cartesian coordinate frame and dual frame consist of such constant tensor fields, so constant linear combinations of them and the corresponding constant basis tensor fields are consequently constant. However if we use a general coordinate system, the new ~~basis~~ frame vector fields <u>will not be</u> constant, and hence a constant tensor field must have nonconstant components. How can we test for constancy of a tensor field in a general coordinate system?

[The expressions for $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, dx, dy, dz\}$ in cylindrical or spherical coordinates are a good example.]

Example: $$\underset{\bar X^i{}_{,j}}{\underset{\parallel}{\frac{\partial \bar X^i}{\partial \bar x^j}}} = \underbrace{\left(\frac{\partial x^n}{\partial \bar x^j}\frac{\partial}{\partial x^n}\right)}_{\frac{\partial}{\partial \bar x^j}}\left(\underbrace{\frac{\partial \bar x^i}{\partial x^m} X^m}_{\bar X^i}\right) = \underbrace{\frac{\partial x^n}{\partial \bar x^j}\frac{\partial \bar x^i}{\partial x^m} X^m{}_{,n}}_{\substack{\text{term for transformation}\\ \text{as though } X^m{}_{,n}\\ \text{defined components of the}\\ \text{same } \binom{1}{1}\text{-tensor field in}\\ \text{every coordinate system}}} + \underbrace{\frac{\partial x^n}{\partial \bar x^j}\frac{\partial^2 \bar x^i}{\partial x^n \partial x^m} X^m}_{\substack{\text{extra}\\ \text{nonhomogeneous}\\ \text{term}\\ \text{breaking}\\ \text{tensor}\\ \text{transformation}\\ \text{law}}}$$

If X is a constant vector field, i.e., $X^m{}_{,n} = 0$ in the Cartesian coordinates, then $\bar X^i{}_{,j} = \frac{\partial x^n}{\partial \bar x^j}\frac{\partial^2 \bar x^j}{\partial x^n \partial x^m} X^m$

only if $\bar x^i = \underbrace{A^i{}_j}_{\text{constants}} x^j$ define new Cartesian coordinates (in general nonorthonormal)

so that $\frac{\partial \bar x^i}{\partial x^m} = A^i{}_m$, $\frac{\partial^2 \bar x^i}{\partial x^n \partial x^m} = 0$, then $\bar X^i{}_{,j} = 0$. Otherwise the new partial derivatives of the new components will not be identically zero, i.e. the vector will have nonconstant components in the new coordinates.

We can do the following manipulation of the extra term

$$\frac{\partial x^n}{\partial \bar x^j}\frac{\partial^2 \bar x^i}{\partial x^n \partial x^m} X^m = \frac{\partial x^n}{\partial \bar x^j}\frac{\partial^2 \bar x^i}{\partial x^n \partial x^m}\frac{\partial x^m}{\partial x^\ell}\bar X^\ell = - \underbrace{\left[\frac{\partial \bar x^i}{\partial x^m}\frac{\partial^2 x^m}{\partial \bar x^j \partial \bar x^\ell}\right]}\bar X^\ell$$

$$\equiv - \bar\Gamma^i{}_{j\ell}\,\bar X^\ell$$

$$\frac{\partial}{\partial x^n}\left[\frac{\partial \bar x^i}{\partial x^m}\frac{\partial x^m}{\partial \bar x^\ell} = \delta^i{}_\ell\right]$$

[Note $\bar\Gamma^i{}_{j\ell} = A^i{}_m A^{-1m}{}_{\ell,\bar j}$] $\nearrow \frac{\partial}{\partial \bar x^j}$

$$\rightarrow \quad \frac{\partial^2 \bar x^i}{\partial x^n \partial x^m}\frac{\partial x^m}{\partial \bar x^\ell} + \frac{\partial \bar x^i}{\partial x^m}\underbrace{\frac{\partial^2 x^m}{\partial x^n \partial \bar x^\ell}}_{\frac{\partial \bar x^p}{\partial x^n}\frac{\partial^2 x^m}{\partial \bar x^p \partial \bar x^\ell}} = 0$$

$$\underbrace{\bar X^\ell \frac{\partial x^n}{\partial \bar x^j}\left[\frac{\partial^2 \bar x^i}{\partial x^n \partial x^m}\frac{\partial x^m}{\partial \bar x^\ell}\right.}_{\text{extra term}} = \left. - \frac{\partial \bar x^i}{\partial x^m}\frac{\partial \bar x^p}{\partial x^n}\frac{\partial^2 x^m}{\partial \bar x^p \partial \bar x^\ell}\right].$$

$$= \frac{\partial \bar x^i}{\partial x^m}\left(-\delta^p{}_j \frac{\partial^2 x^m}{\partial \bar x^p \partial \bar x^\ell}\bar X^\ell\right) = - \frac{\partial \bar x^i}{\partial x^m}\left(\frac{\partial^2 x^m}{\partial \bar x^j \partial \bar x^\ell}\bar X^\ell\right)$$

Solving for the first term on the right leads to :

$$\underbrace{\frac{\partial x^n}{\partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial x^m}\boldsymbol{X}^m{}_{,n}}_{\substack{\text{new components of}\\ \text{tensor field whose}\\ \text{Cartesian components}\\ \text{are } \boldsymbol{X}^m{}_{,n}\\ \text{call this } \binom{1}{1}\text{-tensor field}\\ \nabla \boldsymbol{X}}} = \bar{\boldsymbol{X}}^i{}_{,j} + \bar{\Gamma}^i{}_{j\ell}\bar{\boldsymbol{X}}^\ell \equiv \overline{[\nabla \boldsymbol{X}]}^i{}_j \equiv \bar{\boldsymbol{X}}^i{}_{;j}$$

$$\underbrace{\equiv \bar{\nabla}_j \bar{\boldsymbol{X}}^i}_{\substack{\text{new components of}\\ \text{" covariant derivative of } \boldsymbol{X}\text{"},\\ \text{namely } \nabla \boldsymbol{X}.}}$$

In other words we define a tensor field by transforming the tensor field $\nabla \boldsymbol{X} = \boldsymbol{X}^m{}_{,n}\frac{\partial}{\partial x^m}\otimes dx^n$ from Cartesian coordinates to any other coordinates. The result will vanish any time the Cartesian components vanish, ie, it will be zero in any coordinate system. Constant vector fields have $\nabla \boldsymbol{X} = 0$ in any coordinate system.

The additional term in the new components of $\nabla \boldsymbol{X}$ is a correction term to compensate for the nonconstant frame vector fields. It will be interpreted below in terms of the derivatives of these nonconstant frame vector fields.

In fact

$$\frac{\partial}{\partial \bar{x}^i} = \frac{\partial x^m}{\partial \bar{x}^i}\frac{\partial}{\partial x^m} \equiv \underbrace{A^{-1m}{}_i}_{\text{old components of new frame vectors}}\frac{\partial}{\partial x^m}$$

$$\bar{\Gamma}^i{}_{j\ell} = \frac{\partial \bar{x}^i}{\partial x^m}\frac{\partial^2 x^m}{\partial \bar{x}^j \partial \bar{x}^\ell} = A^i{}_m A^{-1m}{}_{\ell\underset{\substack{\uparrow\\ \text{understood as new coordinate}\\ \text{derivative .}}}{,j}} = \left[\underline{A}\,\frac{\partial}{\partial \bar{x}^j}\underline{A}^{-1}\right]^i{}_\ell$$

So the correction term comes from the new derivatives of the new frame vectors.

Notation for Covariant Derivatives

Given any $\binom{p}{q}$-tensor field $\quad S = S^{i_1\cdots i_p}{}_{j_1\cdots j_q}\, e_{i_1}\otimes\cdots\otimes\omega^{j_q}$

on $\mathbb{R}^n$ expressed in any frame $\{e_i\}$ with dual frame $\{\omega^i\}$,

define the following two tensor fields, if $X = X^i e_i$ is any vector field:

$\binom{p}{q+1}$-tensor field $\quad \nabla S = [\nabla S]^{i_1\cdots i_p}{}_{j_1\cdots j_q k}\, e_{i_1}\otimes\cdots\otimes\omega^{j_q}\otimes\omega^k \quad$ " covariant derivative of S "

$\binom{p}{q}$-tensor field $\quad \nabla_X S = [\nabla_X S]^{i_1\cdots i_p}{}_{j_1\cdots j_q} \quad$ " covariant derivative of S along X "

where

$$[\nabla S]^{i_1\cdots i_p}{}_{j_1\cdots j_q k} \equiv S^{i_1\cdots i_p}{}_{j_1\cdots j_q;k} \equiv [\nabla_{e_k} S]^{i_1\cdots i_p}{}_{j_1\cdots j_q}$$

$$[\nabla_X S]^{i_1\cdots i_p}{}_{j_1\cdots j_q} \equiv S^{i_1\cdots i_p}{}_{j_1\cdots j_q;k} X^k$$

are alternate more convenient notations for these fields.

They are defined so that their components reduce to ordinary derivatives in a Cartesian coordinate system

$$S^{i_1\cdots i_p}{}_{j_1\cdots j_q;k} = S^{i_1\cdots i_p}{}_{j_1\cdots j_q,k}\,.$$

A tensor field which vanishes in a single frame, by definition vanishes in every frame, ie. zero components in one frame define a zero tensor field which must have zero components in every frame.

Constant tensor fields have vanishing covariant derivative and will be called "covariant constant."

The components in any non Cartesian coordinate frame may be calculated in two ways: 1) by transformation from Cartesian coords 2) by being clever.

First notice that the covariant derivative obeys obvious product rules which are inherited from partial derivatives in Cartesian coordinates.

In Cartesian coordinates $(S^{i\ldots}{}_{j\ldots}T^{k\ldots}{}_{\ell\ldots})_{,m} = S^{i\ldots}{}_{j\ldots,m}T^{k\ldots}{}_{\ell\ldots} + S^{i\ldots}{}_{j\ldots}T^{k\ldots}{}_{\ell\ldots,m}$

so in any frame $(S^{i\ldots}{}_{j\ldots}T^{k\ldots}{}_{\ell\ldots})_{;m} = S^{i\ldots}{}_{j\ldots;m}T^{k\ldots}{}_{\ell\ldots} + S^{i\ldots}{}_{j\ldots}T^{k\ldots}{}_{\ell\ldots;m}$

since a direct transformation of the first line yields the second line in the new frame. Thus, dropping indices

$$\nabla(S\otimes T) = \nabla S\otimes T + S\otimes\nabla T,$$

or

$$\nabla_X(S\otimes T) = \nabla_X S\otimes T + S\otimes\nabla_X T.$$

The same product rule holds for any number of contractions of up/down index pairs in this equation. For example

$$(S^i{}_j T^j)_{;k} = S^i{}_{j;k}T^j + S^i{}_j T^j{}_{;k}$$

$$(S^i{}_j T^j)_{;i} = S^i{}_{j;i}T^j + S^i{}_j T^j{}_{;i}.$$

For a $\binom{0}{0}$-tensor, i.e., a function f we get

$$f_{;k} = f_{,k} \quad \text{in cartesian coords}$$

$$\bar{f}_{,k} = \frac{\partial x^\ell}{\partial\bar{x}^k} f_{,\ell} = \frac{\partial x^\ell}{\partial\bar{x}^k}\frac{\partial f}{\partial x^\ell} = \frac{\partial f}{\partial\bar{x}^k}$$

i.e., the covariant derivative equals the covector whose components are the corresponding partial derivatives of the function in any coordinate system, i.e., the differential of the function:

$$\nabla f = \bar{f}_{;k}\,d\bar{x}^k = \underset{\substack{\uparrow\\ \text{to indicate } \frac{\partial f}{\partial\bar{x}^k}}}{f_{,\bar{k}}}\,d\bar{x}^k = df$$

$$\nabla_{\mathbb{X}} f = \bar{f}_{;k}\bar{\mathbb{X}}^k = df(\mathbb{X}) = \mathbb{X}f$$

The covariant derivative of a function along $\mathbb{X}$ is just the ordinary derivative of f by $\mathbb{X}$.

Suppose in a given frame $\{e_i\}$, we define

$$\underbrace{\nabla_{e_i} e_j}_{\substack{\text{covariant derivative of } e_j \\ \text{along } e_i \;=\; \text{vector field} \\ \text{which can be expressed} \\ \text{in terms of same frame}}} = \underbrace{\Gamma^k{}_{ij}}_{=\,\omega^k(\nabla_{e_i} e_j) \,=\, \text{kth component of vector field } \nabla_{e_i} e_j .} e_k$$

From these we get the covariant derivatives of the dual frame covector fields

since $$\omega^j(e_k) = \underbrace{\delta^j{}_k}_{\substack{\text{constant function for each } (j,k) \\ \text{since = evaluation of covector field on vector field}}} \qquad \text{(duality condition)}$$

{ derivatives of constant function are zero

$$0 = \nabla_{e_i}(\delta^j_k) = \nabla_{e_i}\left[\omega^j(e_k)\right] = \underbrace{(\nabla_{e_i}\omega^j)(e_k)}_{\substack{\text{kth component} \\ \text{of covector field} \\ \nabla_{e_i}\omega^j}} + \underbrace{\omega^j(\nabla_{e_i} e_k)}_{\Gamma^j{}_{ik}} \qquad \left(\text{product rule}\right)$$

so $$\nabla_{e_i}\omega^j = -\Gamma^j{}_{ik}\omega^k .$$

Now use product rule on arbitrary tensor fields

$$S = S^{i_1\cdots i_p}{}_{j_1\cdots j_q}\, e_{i_1}\otimes\cdots\otimes\omega^{j_1}\otimes\cdots\otimes\omega^{j_p}$$

$$\begin{aligned}
\nabla_{e_k} S = {} & \underbrace{(\nabla_{e_k} S^{i_1\cdots i_p}{}_{j_1\cdots j_q})}_{e_k S^{i_1\cdots i_p}{}_{j_1\cdots j_q}}\, e_{i_1}\otimes\cdots\otimes\omega^{j_1}\otimes\cdots\otimes\omega^{j_p} \\
& + S^{i_1\cdots i_p}{}_{j_1\cdots j_q}\, \overbrace{(\nabla_{e_k} e_{i_1})}^{\Gamma^\ell{}_{k i_1} e_\ell}\otimes\cdots\otimes\omega^{j_1}\otimes\cdots\otimes\omega^{j_q} + \cdots \\
& + S^{i_1\cdots i_p}{}_{j_1\cdots j_q}\, e_{i_1}\otimes\cdots\otimes \underbrace{\nabla_{e_k}\omega^{j_1}}_{-\Gamma^{j_1}{}_{k\ell}\omega^\ell}\otimes\cdots\otimes\omega^{j_q} + \cdots
\end{aligned}$$

$$= \left[e_k S^{i_1\cdots i_p}{}_{j_1\cdots j_p} + \Gamma^{i_1}{}_{km} S^{m\cdots i_p}{}_{j_1\cdots j_q} + \cdots - \Gamma^m{}_{kj_1} S^{i_1\cdots i_p}{}_{m\cdots j_q} - \cdots\right] e_{i_1}\otimes\cdots\otimes\omega^{j_p}$$

$$\equiv S^{i_1\cdots i_p}{}_{j_1\cdots j_p;k}\, e_{i_1}\otimes\cdots\otimes\omega^{j_p}$$

or $$S^{i_1\cdots i_p}{}_{j_1\cdots j_p;k} = \underbrace{e_k S^{i_1\cdots i_p}{}_{j_1\cdots j_p}}_{\text{ordinary derivative along } e_k} + \underbrace{\overbrace{\Gamma^{i_1}{}_{km} S^{m\cdots i_p}{}_{j_1\cdots j_p} + \cdots}^{\substack{\text{one such term for} \\ \text{each up index}}} - \overbrace{\Gamma^m{}_{kj} S^{i_1\cdots i_p}{}_{m\cdots j_q} - \cdots}^{\substack{\text{one such term} \\ \text{for each down index}}}}_{\substack{\text{correction terms for covariant derivatives} \\ \text{of frame and dual frame.}}}$$

The correction terms are all zero in a Cartesion cordinate frame where the frame and dual frame vectors and covectors are covariant constant and so the covariant derivative reduces to the ordinary derivative

So once we calculate the components of the covariant derivatives of the frame vector fields, i.e., $\Gamma^k{}_{ij}$, we can evaluate any covariant derivative. These were evaluated for general cordinate frames on page 46 in terms of the transformation from Cartesian coordinates. The only difference for a general frame is, in that notation

$$\bar{e}_i = A^{-1\,j}{}_i \frac{\partial}{\partial x^j} \quad ,$$

$$\bar{\Gamma}^i{}_{jl} = \qquad\qquad = [\underline{A}\, \bar{e}_j \underline{A}^{-1}]^i{}_l$$

exercise. a) Calculate the nonzero components $\bar{\Gamma}^i{}_{jl}$ in cylindrical coordinates. b) Do the same in the associated orthonormal frame $\{e_{\hat\rho}, e_{\hat\varphi}, e_{\hat z}\} = \{\frac{\partial}{\partial\rho}, \frac{1}{\rho}\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial z}\}$, with dual frame $\{\omega^{\hat\rho}, \omega^{\hat\varphi}, \omega^{\hat z}\} = \{d\rho, \rho\, d\varphi, dz\}$.

Recall $\bar{\omega}^i = d\bar{x}^i = A^i{}_j dx^j$

$$\bar{\omega}^{\hat i} = (\bar{g}_{ii})^{1/2}\bar{\omega}^i = \underbrace{(\bar{g}_{ii})^{1/2} A^i{}_j}_{a^i{}_j} dx^j$$

$a^i{}_j$ = normalize rows of $\underline{A}$ by their length

$$\underbrace{\underline{A} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\frac{1}{\rho}\sin\varphi & \frac{1}{\rho}\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{} \qquad \underbrace{\underline{a} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{}$$

use for coord frame calculation (taken from page 34)

rotation matrix of rotation from orthonormal Cartesian coordinate frame to the orthonormal normalized cylindrical coordinate frame

$$\underline{a}^{-1} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow$$

substitute $\underline{A}$ by $\underline{a}$ for this calculation.

BUT, before you do, next page please.

Matrix methods when applicable are much more efficient than working with individual components. The n^3 components $\Gamma^i{}_{jk}$ of the covariant derivative operator, or "connection" as it is called for reasons to become clear later, may be packaged in a more usefriendly format by introducing the matrix of connection covector fields for a given frame $\{e_i\}$

$$\boxed{\omega^i{}_k \equiv \Gamma^i{}_{jk}\,\omega^j \quad , \quad \underline{\omega} \equiv (\omega^i{}_k).}$$

Each entry in this matrix $\underline{\omega}$ is a covector field or "1-form" as covector fields are usual called.

The differential of a function $df = f_{,i}\,dx^i$ in a coordinate frame is a 1-form which has the expression

$$df = \underbrace{df(e_i)}_{\substack{\text{value of 1-form on } e_i \\ (\text{component of } df \text{ wrt } \{e_i\})}}\,\omega^i = f_{,i}\,\omega^i$$

$= e_i f$ by definition of differential

$\equiv f_{,i}$ extend comma notation to frame derivatives of functions

Thus returning to our barred notation on the previous page

~~$\bar{\Gamma}^i{}_{jk} = [\underline{A}\;\underline{A}^{-1}{}_{,j}]^i{}_k$~~

$$\bar{\omega}^i{}_k = [\underline{A}\;\underline{A}^{-1}{}_{,j}\,\bar{\omega}^j]^i{}_k = [\underline{A}\;d\underline{A}^{-1}]^i{}_k$$

$$\boxed{\underline{\bar{\omega}} = \underline{A}\;d\underline{A}^{-1}}$$

This makes it easy to calculate $\underline{\bar{\omega}}$. Once found the (i,k) entry gives the 1-form $\bar{\omega}^i{}_k = \bar{\Gamma}^i{}_{jk}\,\omega^j$ and the j-th component of this 1-form is the connection component $\bar{\Gamma}^i{}_{jk}$.

Now try the above exercise.

exercise on page 39 worked:

$$Y = r\frac{\partial}{\partial r} + \frac{\partial}{\partial \varphi} = \cancel{(x^2+y^2+z^2)^{1/2}}\left[\frac{x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}}{\cancel{(x^2+y^2+z^2)^{1/2}}}\right] + \left[-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\right]$$

$$= x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z} - y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y} = (x-y)\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

exercisesson page 40 worked. (done above)

~~$[fX,hY] = (fX)(hY)-(hY)(fX) = f\underbrace{X(hY)}_{(Xh)Y+hXY}-h\underbrace{Y(fX)}_{(Yf)X+fYX}$~~

~~$= f(Xh)Y + fhXY - h(Yf)X - hfYX$~~

~~$= fh(XY-YX) + f(Xh)Y - h(Yf)X$~~

~~$= fh[X,Y] + f(Xh)Y - h(Yf)X.$~~

$$[e_r, e_\varphi] = \left[\frac{\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)}{(x^2+y^2+z^2)^{1/2}},\ -y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\right]$$

$$= \frac{1}{(x^2+y^2+z^2)^{1/2}}\underbrace{\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z},\ -y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\right]} - \underbrace{\left[\left(-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\right)(x^2+y^2+z^2)^{-1/2}\right]}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)$$

$$x\underbrace{\left(\frac{\partial}{\partial x}x\right)}_{1}\frac{\partial}{\partial y} + y\overbrace{\frac{\partial}{\partial y}(-y)}^{-1}\frac{\partial}{\partial x} \qquad -\frac{1}{2}(\ldots)^{-3/2}\underbrace{\left(-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\right)(x^2+y^2+z^2)}$$

$$-\left(-y\underbrace{\frac{\partial}{\partial x}(x)}_{1}\frac{\partial}{\partial x} + x\underbrace{\frac{\partial}{\partial y}(y)}_{1}\frac{\partial}{\partial y}\right. \qquad -y(2x)+x(2y) = 0$$

$$\underbrace{x\frac{\partial}{\partial y} \underbrace{- y\frac{\partial}{\partial x} + y}\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}} = 0$$

$= 0$ ✓

$$e_{\hat r} = \frac{\partial}{\partial r} \qquad e_{\hat\theta} = \frac{1}{r}\frac{\partial}{\partial \theta} \qquad e_{\hat\varphi} = \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}$$

$$[e_{\hat r}, e_{\hat\theta}] = \left[\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}\right] = -\frac{1}{r^2}\frac{\partial}{\partial \theta} = -\frac{1}{r}e_{\hat\theta}$$

$$[e_{\hat\theta}, e_{\hat\varphi}] = \left[\frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\right] = \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\right)\frac{\partial}{\partial \varphi} = -\frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial}{\partial \varphi} = -\frac{1}{r}\cot\theta\, e_{\hat\varphi}$$

$$[e_{\hat\varphi}, e_{\hat r}] = \left[\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial r}\right] = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \varphi} = \frac{1}{r}e_{\hat\varphi}$$

exercise on page 42 worked

$e_{\hat\rho}=\frac{\partial}{\partial\rho}$ $\quad e_{\hat\varphi}=\frac{1}{\rho}\frac{\partial}{\partial\varphi}$ $\quad e_{\hat z}=\frac{\partial}{\partial z}$

$$[e_{\hat\rho},e_{\hat\varphi}]=\left[\frac{\partial}{\partial\rho},\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right]=-\frac{1}{\rho^2}\frac{\partial}{\partial\varphi}=-\frac{1}{\rho}e_{\hat\varphi}$$

$$[e_{\hat\varphi},e_{\hat z}]=\left[\frac{1}{\rho}\frac{\partial}{\partial\varphi},\frac{\partial}{\partial z}\right]=0$$

$$[e_{\hat z},e_{\hat\rho}]=\left[\frac{\partial}{\partial z},\frac{\partial}{\partial\rho}\right]=0.$$

so $C^{\hat\varphi}{}_{\hat\rho\hat\varphi}=-\frac{1}{\rho}\left(=-C^{\hat\varphi}{}_{\hat\varphi\hat\rho}\right)$ is the single independent structure function

For the spherical coordinate case instead:

$$C^{\hat\theta}{}_{\hat r\hat\theta}=-\frac{1}{r},\quad C^{\hat\varphi}{}_{\hat\theta\hat\varphi}=-\frac{1}{r}\cot\theta,\quad C^{\hat\varphi}{}_{\hat\varphi\hat r}=\frac{1}{r}.$$

I used the cyclic order $\overset{\text{1 missing}}{23},\overset{\text{2 missing}}{31},\overset{\text{3 missing}}{12}\rightarrow\hat r\hat\theta,\hat\theta\hat\varphi,\hat\varphi\hat r$

in computing the Lie brackets, rather than $i<j$: 23, 13, 12.

exercise on page 43

substitute e_1,e_2,e_3 by dx,dy,dz (ie p-vectors → p-covectors)

$*1=dx\wedge dy\wedge dz$

$*(X_1dx+X_2dy+X_3dz)=X_1\,dy\wedge dz+X_2\,dz\wedge dx+X_3\,dx\wedge dy$

$*(X_{23}\,dy\wedge dz+X_{31}\,dz\wedge dx+X_{12}\,dx\wedge dy)=X_{23}\,dx+X_{31}\,dy+X_{12}\,dz.$

$*(X_{123}\,dx\wedge dy\wedge dz)=X_{123}$

$*\underline{1}=$ ~~$d\rho\wedge d\varphi\wedge dz$~~ $\omega^{\hat\rho\hat\varphi\hat z}=\omega^{\hat\rho}\wedge\omega^{\hat\varphi}\wedge\omega^{\hat z}=d\rho\wedge(\rho\,d\varphi)\wedge(dz)=\rho\,d\rho\wedge d\varphi\wedge dz.$

$*(\mathbb{X}_{\hat\rho}\omega^{\hat\rho}+X_{\hat\varphi}\omega^{\hat\varphi}+X_{\hat z}\omega^{\hat z})=\mathbb{X}_{\hat\rho}\,\omega^{\hat\varphi\hat z}+\mathbb{X}_{\hat\varphi}\,\omega^{\hat z\hat\rho}+\underline{\mathbb{X}}_{\hat z}\,\omega^{\hat\rho\hat\varphi}$

$*(\mathbb{X}_{\hat\varphi\hat z}\,\omega^{\hat\varphi\hat z}+\mathbb{X}_{\hat z\hat\rho}\,\omega^{\hat z\hat\rho}+X_{\hat\rho\hat\varphi}\,\omega^{\hat\rho\hat\varphi})=\mathbb{X}_{\hat\rho\hat z}\,\omega^{\hat\rho}+\mathbb{X}_{\hat z\hat\rho}\,\omega^{\hat\rho}+X_{\hat\rho\hat\varphi}\,\omega^{\hat z}$

$*(X_{\hat\rho\hat\varphi\hat z}\,\omega^{\hat\rho\hat\varphi\hat z})=\mathbb{X}_{\hat\rho\hat\varphi\hat z}$

$*$~~etc.~~ $1=\omega^{\hat r\hat\theta\hat\varphi}=\omega^{\hat r}\wedge\omega^{\hat\theta}\wedge\omega^{\hat\varphi}=dr\wedge(r\,d\theta)\wedge(r\sin\theta\,d\varphi)=r^2\sin\theta\,dr\wedge d\theta\wedge d\varphi.$

etc.

$$\mathbb{X}=\overbrace{\frac{1}{r^2\sin\theta}\mathbb{X}_{\theta\varphi}}^{\mathbb{X}_{\hat\theta\hat\varphi}}\omega^{\hat\theta\hat\varphi}+\overbrace{\frac{1}{r\sin\theta}\mathbb{X}_{\varphi r}}^{\mathbb{X}_{\hat\varphi\hat r}}\omega^{\hat\varphi\hat r}+\overbrace{\frac{1}{r}\mathbb{X}_{r\theta}}^{\mathbb{X}_{\hat r\hat\theta}}\omega^{\hat r\hat\theta}$$

$$*\mathbb{X}=\frac{1}{r^2\sin\theta}\mathbb{X}_{\theta\varphi}\,\omega^{\hat r}+\frac{\mathbb{X}_{\varphi r}}{r\sin\theta}\omega^{\hat\theta}+\frac{\mathbb{X}_{r\theta}}{r}\omega^{\hat\varphi}$$

$$*\mathbb{X}^\#=\underbrace{\frac{1}{r^2\sin\theta}\mathbb{X}_{\theta\varphi}}_{(*\mathbb{X}^\#)^{\hat r}}e_{\hat r}+\underbrace{\frac{\mathbb{X}_{\varphi r}}{r\sin\theta}}_{[*\mathbb{X}^\#]^{\hat\theta}}e_{\hat\theta}+\underbrace{\frac{\mathbb{X}_{r\theta}}{r}}_{[*\mathbb{X}^\#]^{\hat\varphi}}e_{\hat\varphi}=\underbrace{\frac{\mathbb{X}_{\theta\varphi}}{r^2\sin\theta}}_{[*\mathbb{X}^\#]^{r}}\frac{\partial}{\partial r}+\frac{\mathbb{X}_{\varphi r}}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{\mathbb{X}_{r\theta}}{r^2\sin\theta}\frac{\partial}{\partial\varphi}$$

etc.

Suppose $\Box$ is any linear derivative operator which produces a $\binom{p}{q}$-tensor field $\Box T$ from a $\binom{p}{q}$-tensor T and which obeys the product rule for tensor products and contractions thereof. Then $\Box$ is entirely determined by how it acts on functions and how it acts on the frame vector fields.

Consider $\Box e_i$ for the frame vector fields. Each such field $\Box e_i$ is again a vector field so it can be expressed as a linear combination of the frame vector fields

$$\Box e_i = \underbrace{\Box^j{}_i}_{\substack{\text{square matrix of functions}\\ \text{called " components of } \Box\\ \text{with respect to frame "}}} e_j \quad , \quad \Box^j{}_i = \underset{\substack{\uparrow\\ \text{jth component of}\\ \text{vector field } \Box e_i}}{\omega^j}(\Box e_i)$$

However by duality $\quad \omega^i(e_j) = \delta^i{}_j \quad$ (= n^2 constant functions obtained by evaluating dual frame covector fields on framevector fields)

the contraction (evaluation) of the dual frame covector fields with the frame vector fields are constant functions whose derivative must be zero no matter what $\Box$ actually is, so by the product rule

$$\underbrace{\underbrace{(\Box\omega^i)}_{\text{covectorfield}}(e_j)}_{\text{jth component of covectorfield}} + \underbrace{\omega^i(\Box e_j)}_{\Box^i{}_j} = \Box\delta^i{}_j = 0$$

$$\Box\omega^i = [\underbrace{(\Box\omega^i)(e_j)}_{-\Box^i{}_j}]\,\omega^j = -\Box^i{}_j\,\omega^j$$

Thus the same matrix determines the components of $\Box\omega^i$ but with a minus sign.

Now take any $\binom{p}{q}$-tensor field $\quad T = T^{i\dots}{}_{j\dots}\, \underbrace{e_i\otimes\dots}_{p\text{ factors}}\otimes\underbrace{\omega^j\otimes\dots}_{q\text{ factors}}$.

This is a sum of products of functions (the components of T), and tensor products of frame vector fields and dual frame covector fields, so by the product rule

$$\Box T = \underbrace{\Box(T^{i\dots}{}_{j\dots})}_{\text{functions}}\, e_i\otimes\dots\otimes\omega^j\otimes\dots + T^{i\dots}{}_{j\dots}\,(\Box e_i)\otimes\dots\otimes\omega^j + \underbrace{\dots}_{\substack{p-1\text{ other terms for}\\ \text{other frame vector fields}}}$$

$$+\, T^{i\dots}{}_{j\dots}\, e_i\otimes\dots\otimes(\Box\omega^j)\otimes\dots + \underbrace{\dots}_{\substack{q-1\text{ other terms for}\\ \text{other dual frame}\\ \text{covector fields}}}$$

$$= \Box(T^{i\dots}{}_{j\dots})\, e_i\otimes\dots\otimes\omega^j\otimes\dots + T^{i\dots}{}_{j\dots}\,(\Box^k{}_i\, e_k)\otimes\dots\otimes\omega^j\otimes\dots + \dots$$

$$+\, T^{i\dots}{}_{j\dots}\, e_i\otimes\dots\otimes(-\Box^j{}_k\,\omega^k)\otimes\dots + \dots$$

$$= \left.\begin{aligned} &\Box(T^{i\dots}{}_{j\dots})\, e_i\otimes\dots\otimes\omega^j\otimes\dots + \Box^i{}_k T^{k\dots}{}_{j\dots}\, e_i\otimes\dots\otimes\omega^j\otimes\dots + \dots \\ &- \Box^k{}_j T^{i\dots}{}_{k\dots}\, e_i\otimes\dots\otimes\omega^j\otimes\dots - \dots \end{aligned}\right\} \begin{array}{l}\text{just relable}\\ \text{dummy indices}\\ \text{so same indices}\\ \text{on basis tensors}\end{array}$$

$$= \underbrace{\left[\Box(T^{i\dots}{}_{j\dots}) + \Box^i{}_k T^{k\dots}{}_{j\dots} + \dots - \Box^k{}_j T^{i\dots}{}_{k\dots} - \dots\right]}\, e_i\otimes\dots\otimes\omega^j\otimes\dots$$

$$\boxed{[\Box T]^{i\dots}{}_{j\dots} = \Box(T^{i\dots}{}_{j\dots}) + \Box^i{}_k T^{k\dots}{}_{j\dots} + \dots - \Box^k{}_j T^{i\dots}{}_{k\dots} - \dots}$$

$$\underbrace{\qquad\qquad}_{\substack{\text{contribution}\\ \text{from }\Box\text{ derivatives}\\ \text{of component}\\ \text{functions.}}} \qquad \underbrace{\qquad\qquad\qquad\qquad\qquad}_{\substack{\text{contribution from }\Box\text{ derivatives of}\\ \text{frame and dual frame fields}}}$$

WARNING In old fashioned tensor analysis only using components and no frame vectors or dual frame covectors, one drops square brackets

$$\Box T^{i\dots}{}_{j\dots} \quad \text{means} \quad [\Box T]^{i\dots}{}_{j\dots}$$

This notation no longer distinguishes between the derivative of the components and the components of the derivative, so one has to be careful. Thus $T^{i\dots}{}_{j\dots;k}$ are the components of the covariant derivative of T, not the covariant derivative of the components (which are always the ordinary partial derivatives)

Note on class of Cartesian coord systems

on page 45 we saw that the new coordinate components $\bar{X}^i{}_{,j}$ for a nonzero constant vector field X vanish only if

$$\frac{\partial^2 \bar{x}^j}{\partial x^n \partial x^m} = 0 .$$

The general solution of this is $\bar{x}^i = A^i{}_j x^j + b^i$ where $A^i{}_j$ and b^j are constants. This corresponds to allowing new Cartesian coordinates adapted to any basis of the vector space $\mathbb{R}^n$ and with any choice of origin. The mathematical structure associated with this larger class of Cartesian coordinate systems for which no preferred origin exists is called an "affine structure". An affine space is basically a vectorspace modulo a choice of origin. Difference vectors between points in the space make sense, but no absolute position vector does since that requires first an arbitrary choice of origin.

when we think of "physical 3-space" whether doing calculus or physics, it is really this affine space (since we have to arbitrarily choose an origin for our axes) together with the Euclidean inner product for difference vectors that we work with.

Some constant tensors

The Kronecker delta tensor field $\delta = \delta^i{}_j \, e_i \otimes \omega^i = e_i \otimes \omega^i$ has constant components in every frame, not only in a Cartesian coordinate frame. The same is true of the generalized Kronecker deltas

$$\delta^{(p)} = \delta^{i_1 \ldots i_p}{}_{j_1 \ldots j_p} \, e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes \omega^{j_1} \otimes \cdots \otimes \omega^{j_p} .$$

All of these are therefore constant tensor fields on $\mathbb{R}^n$.

By the rule for covariant differentiation in a general coordinate frame $\{\bar{e}_i\}$

$$\bar\delta^i{}_{j;k} = \underbrace{\bar\delta^i{}_{j,k}}_{\delta^i{}_{j,k}=0} + \bar\Gamma^i{}_{k\ell} \underbrace{\bar\delta^\ell{}_j}_{\delta^\ell{}_j} - \bar\Gamma^k{}_{kj} \underbrace{\bar\delta^i{}_k}_{\delta^i{}_k} = \bar\Gamma^i{}_{kj} - \bar\Gamma^i{}_{kj} = 0$$

which confirms its covariant constancy, ie, $\nabla \delta = 0$.

exercise. Repeat this calculation for $\delta^{(2)} = \delta^{ij}{}_{mn} \, e_i \otimes e_j \otimes \omega^m \otimes \omega^n$, for an arbitrary coordinate frame $\{\bar{e}_i\}$, where $\bar\delta^{ij}{}_{mn} = \delta^{ij}{}_{mn}$.

[Note that by the product and sum rules, $\nabla \delta = 0$ implies that $\nabla \delta^{(p)} = 0$ for all allowed p values, but using the formula to verify this helps familiarize you with the rules of covariant differentiation.]

exercise on pages 49–50 worked $\qquad (\rho, \varphi, z) \sim (1,2,3)$

Inverting the 2x2 block of $\underline{A}$ gives $\underline{A}^{-1} = \begin{pmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Check that $\underline{A}^{-1}\underline{A} = \underline{I}$. [Also given at bottom of page 31.]

$$\underline{\bar{\omega}} = \underline{A}\, d\underline{A}^{-1} = \begin{pmatrix} c & s & 0 \\ -\rho^{-1}s & \rho^{-1}c & 0 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} -\sin\varphi\, d\varphi & -\rho\cos\varphi\, d\varphi - \sin\varphi\, d\rho & 0 \\ \cos\varphi\, d\varphi & -\rho\sin\varphi\, d\varphi + \cos\varphi\, d\rho & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$= \begin{pmatrix} -s & -\rho c & 0 \\ c & -\rho s & 0 \\ 0 & 0 & 0 \end{pmatrix} d\varphi + \begin{pmatrix} 0 & -s & 0 \\ 0 & c & 0 \\ 0 & 0 & 0 \end{pmatrix} d\rho$$

$$= \begin{pmatrix} 0 & -\rho & 0 \\ \rho^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\varphi + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rho^{-1} d\rho$$

so $\bar{\Gamma}^{\rho}{}_{\varphi\varphi} = -\rho$, $\bar{\Gamma}^{\varphi}{}_{\varphi\rho} = \rho^{-1}$, $\bar{\Gamma}^{\varphi}{}_{\rho\varphi} = \rho^{-1}$

are the only nonzero connection components, from the three nonzero entries of $\bar{\omega}$ recalling the matrix indices $(1,2,3) \sim (\rho, \varphi, z)$.

The matrix $\underline{a}$ for the associated orthonormal frame is obtained from $\underline{A}$ by setting $\rho = 1$, so one can derive the corresponding result by putting $\rho = 1$ into the above calculation:

$$\hat{\omega} \equiv (\bar{\omega}^{\hat{i}}{}_{\hat{k}}) \equiv (\bar{\Gamma}^{\hat{i}}{}_{\hat{j}\hat{k}}\, \bar{\omega}^{\hat{j}}) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\varphi$$

$\bar{\Gamma}^{\hat{\rho}}{}_{\varphi\varphi} = -1$, $\Gamma^{\varphi}{}_{\varphi\rho} = 1$ are the only nonzero connection components, due to the rotation by the angle φ of the orthonormal vector fields $e_{\hat{\rho}}$ and $e_{\hat{\varphi}}$ relative to e_x and e_y.

exercise. Since the connection has no components along z, the covariant derivative of $\partial/\partial z = e_z$ is equal to the ordinary derivative

$$\nabla_{e_i} e_z = \underbrace{\Gamma^j{}_{iz}}_{0} e_j = 0 \qquad \text{so} \quad \nabla e_z \text{ is zero,}$$

i.e., $\partial/\partial z$ is covariant constant as we already knew before from our definition of covariant differentiation.

From page 34:

$$\frac{\partial}{\partial x} = \cos\varphi \frac{\partial}{\partial \rho} - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial \varphi} \leftrightarrow \left[\frac{\partial}{\partial x}\right]^\rho = \cos\varphi, \left[\frac{\partial}{\partial x}\right]^\varphi = \frac{-\sin\varphi}{\rho}$$

$$\frac{\partial}{\partial y} = \sin\varphi \frac{\partial}{\partial \rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial \varphi} \leftrightarrow \left[\frac{\partial}{\partial y}\right]^\rho = \sin\varphi, \left[\frac{\partial}{\partial y}\right]^\varphi = \frac{\cos\varphi}{\rho}$$

Verify that $X^i{}_{;j} = X^i{}_{,j} + \Gamma^i{}_{jk} X^k = 0$ for $X = \partial/\partial x$ and $X = \partial/\partial y$.

Repeat for the 1-forms given on page 32: (or at least for dx)

$$dx = \cos\varphi \, d\rho - \rho\sin\varphi \, d\varphi \qquad [dx]_\rho = \cos\varphi \quad [dx]_\varphi = -\rho\sin\varphi.$$

$$dy = \sin\varphi \, d\rho + \rho\cos\varphi \, d\varphi$$

$$dz = dz$$

The "clever" way of evaluating the components of the covariant derivative

Suppose $g = g_{ij}\, dx^i \otimes dx^j$ is any constant metric on $\mathbb{R}^n$, i.e., the matrix $\underline{g} = (g_{ij})$ is symmetric and has nonzero determinant so that it can be inverted, and is constant. (for Cartesian coordinate frame).

Then its covariant derivative must vanish. Expressing this in a general coordinate system leads to

$$0 = \nabla g \Leftrightarrow 0 = \bar{g}_{ij;k} = \bar{g}_{ij,k} - \underbrace{\bar{g}_{\ell j} \bar{\Gamma}^{\ell}{}_{ki}}_{\bar{\Gamma}_{jki}} - \underbrace{\bar{g}_{i\ell} \bar{\Gamma}^{\ell}{}_{kj}}_{\Gamma_{ikj}}$$

by definition of index lowering:

so $$\bar{g}_{ij,k} = \bar{\Gamma}_{jki} + \bar{\Gamma}_{ikj} = \text{twice symmetric part in index pair } ij$$

This expresses the ordinary derivatives of the metric components in terms of a certain symmetric part of the index-lowered form of the components of the covariant derivative.

By definition $$\bar{\Gamma}^{i}{}_{jk} = \frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial^2 x^m}{\partial \bar{x}^j \partial \bar{x}^k} = \bar{\Gamma}^{i}{}_{kj}$$

is symmetric in its down index pair (partial derivatives commute) which have $n(n+1)/2$ independent components for each of the n values of the upper index for a total of $n^2(n+1)/2$ independent components. But the collection of partial derivatives of $\bar{g}_{ij}$ ($n(n+1)/2$ independent components times n independent derivatives) has the same number of independent components, so it is not surprising that they are equivalent, i.e., contain the same information in different packaging — in other words one can invert the above relationship to express the components of the covariant derivative in terms of the ordinary derivatives of the metric components.

This is a classic calculation of differential geometry.

$$\bar g_{ij,k} = \bar\Gamma_{ikj} + \bar\Gamma_{jki}$$

$$-\bar g_{jk,i} = -\bar\Gamma_{jik} - \Gamma_{kij} \quad \leftarrow \text{relabel indices conveniently, then add}$$

$$g_{ki,j} = \bar\Gamma_{kji} + \bar\Gamma_{ijk}$$

but by symmetry in last two indices, get simple result

$$(\ast)\quad \bar g_{ij,k} - \bar g_{jk,i} + \bar g_{ki,j} = \underbrace{(\underbrace{\bar\Gamma_{ikj}}_{\bar\Gamma_{ijk}} + \bar\Gamma_{ijk})}_{2\bar\Gamma_{ijk}} + \underbrace{(\bar\Gamma_{jki} - \underbrace{\bar\Gamma_{jik}}_{\bar\Gamma_{jki}})}_{0} + \underbrace{(\bar\Gamma_{kij} - \underbrace{\bar\Gamma_{kji}}_{\bar\Gamma_{kij}})}_{0}$$

so $\bar\Gamma_{ijk} = \frac{1}{2}(\bar g_{ij,k} - \bar g_{jk,i} + \bar g_{ki,j})$

and $\boxed{\bar\Gamma^i{}_{jk} = \bar g^{il}\bar\Gamma_{ljk} = \frac{1}{2}\bar g^{il}(\bar g_{lj,k} - \bar g_{jk,l} + \bar g_{kl,j})}$

~~by raising the first index back to its original position.~~

In other words as long as the metric is a covariant constant nondegenerate symmetric $\binom{0}{2}$-tensor field on $\mathbb{R}^n$ (corresponding to a symmetric nondegenerate inner product on the vector space $\mathbb{R}^n$), the components of the ~~connection~~ covariant derivative can be represented in terms of its components and their derivatives in a given coordinate system. No matter what such metric we start with, the result is always the same set of components $\bar\Gamma^i{}_{jk}$ and the same covariant derivative which defines constant tensor fields on $\mathbb{R}^n$ in terms of zero covariant derivative. The covariant derivative is really only connected with the translational symmetry of $\mathbb{R}^n$, not the Euclidean inner product.

For an orthogonal coordinate system (diagonal metric component matrix), index-shifting amounts to scaling by certain factors, while off diagonal metric components are zero, so the above formula is easy to evaluate in practice.

exercise. Show the preceding formula for $\bar{\Gamma}^{i}{}_{jk}$ is symmetric in (jk).

exercise Recalculate the nonzero components $\bar{\Gamma}^{i}{}_{jk}$ of the covariant derivative for cylindrical coordinates by first calculating the nonzero $\bar{\Gamma}_{ijk}$ and then raising the first index, evaluating the above formula in terms of the Euclidean metric.

If you feel ambitious, repeat for spherical coordinates.

exercise

The constant tensor

$$T = dx \otimes dx = (\cos\varphi\, d\rho - \rho \sin\varphi\, d\varphi) \otimes (\cos\varphi\, d\rho - \rho\sin\varphi\, d\varphi)$$

$$= \underbrace{\cos^2\varphi}_{T_{\rho\rho}}\ d\rho\otimes d\rho \ + \underbrace{\rho^2 \sin^2\varphi}_{T_{\varphi\varphi}}\ d\varphi \ \underbrace{-\ \rho\sin\varphi\cos\varphi}_{T_{\rho\varphi} = T_{\varphi\rho}}\ (d\rho\otimes d\varphi + d\varphi\otimes d\rho)$$

is covariant constant: $\bar{T}_{ij;k} = 0$.

Verify that all components of this covariant derivative are indeed zero.

Well, we are in good shape for computing the components of the covariant derivative in a coordinate frame entirely in terms of the metric, but what is the corresponding formula for a more general frame with nonzero structure functions?

The expression for the Lie bracket in a coordinate frame

$$[X,Y]^k = XY^k - YX^k = Y^k{}_{,i}X^i - X^k{}_{,i}Y^i$$

is the "ordinary derivative" commutator of two vector derivatives. Suppose we introduce the corresponding covariant derivative commutator

$$[\nabla_X Y - \nabla_Y X]^k = Y^k{}_{;i}X^i - X^k{}_{;i}Y^i$$

$$= Y^k{}_{,i}X^i - X^k{}_{,i}Y^i + \Gamma^k{}_{ij}X^iY^j - \underbrace{\Gamma^k{}_{ij}Y^iX^j}_{\Gamma^k_{ji}X^iY^j}$$

$$= [X,Y]^k + \underbrace{2\Gamma^k{}_{[ij]}}_{\substack{\text{twice antisymmetric part} \\ = 0 \text{ for coordinate frame by definition}}} X^iY^j$$

Thus $\nabla_X Y - \nabla_Y X = [X,Y]$.

Although obtained in a coordinate frame, this equation is valid for arbitrary vector fields X and Y since it is a frame independent formula involving tensor fields. Another way of obtaining it is to notice that in a Cartesian coordinate frame, there is no difference between ordinary and covariant differentiation so the equation which holds there must be a tensor equation valid independent of what frame we choose to use.

Thus
$$\underbrace{\nabla_{e_i} e_j}_{\Gamma^k{}_{ij}e_k} - \underbrace{\nabla_{e_j} e_i}_{\Gamma^k{}_{ji}e_c} = \underbrace{[e_i,e_j]}_{C^k{}_{ij}e}$$

$$[\Gamma^k{}_{ij} - \Gamma^k{}_{ji}]e = C^k{}_{ij}e_k$$

$$\boxed{\Gamma^k{}_{ij} - \Gamma^k{}_{ji} = C^k{}_{ij}} \qquad \text{or} \qquad \Gamma^k{}_{[ij]} = \tfrac{1}{2}C^k{}_{ij}$$

In a noncoordinate frame, the antisymmetric part of the components of the covariant derivative in the lower index pair equals the corresponding structure function. The symmetry in a coordinate frame follows from the vanishing of the structure functions.

Let us reconsider equation (4) on page 61 which is valid in any frame. We then used the symmetry of $\bar{\Gamma}^i{}_{jk}$ in a coordinate frame to go on and ~~solve~~ invert the relationship between $\bar{\Gamma}^i{}_{jk}$ and the derivatives of the metric. Let's drop the bar notation

$$g_{ij,k} - g_{jk,i} + g_{ki,j} = (\underbrace{\Gamma_{ikj}}_{\Gamma_{ijk} - C_{ijk}} + \Gamma_{ijk}) + (\underbrace{\Gamma_{jki} - \Gamma_{jik}}_{C_{jki}}) - (\underbrace{\Gamma_{kij} - \Gamma_{kji}}_{C_{kij}})$$

(where we extend index-shifting to the structure functions

$$C_{ijk} = g_{i\ell} C^\ell{}_{jk}$$

so $\Gamma_{ijk} - \Gamma_{ikj} = C_{ijk}$ or $\Gamma_{ijk} = \Gamma_{ikj} + C_{ijk}$)

Solving for Γ_{ijk}:

$$\Gamma_{ijk} = \tfrac{1}{2}[\,g_{ij,k} - g_{jk,i} + g_{ki,j} \;+\; C_{ijk} - C_{jki} + C_{kij}\,]$$

$$\Gamma^i{}_{jk} = \underbrace{\frac{1}{2}\, g^{i\ell}(\,g_{ij,k} - g_{jk,i} + g_{ki,j})}_{\equiv \left\{ {i \atop jk} \right\}} + \frac{1}{2}(C^i{}_{jk} - C_{jk}{}^i + C_k{}^i{}_j)$$

exercise The first part of this formula $\left\{ {i \atop jk} \right\}$ is called a Christoffel symbol and we saw above that it is symmetric in its lower indices. Show that the antisymmetric part of the second ~~part~~ reduces to $\frac{1}{2} C^i{}_{jk}$.

exercise Use this formula to evaluate the components of the covariant derivative for the orthonormal frame associated with cylindrical coordinates.

Clarifying remark for page 64 derivation — lower index to get corresponding formula:

$$g_{k\ell}[\,\Gamma^{\ell}{}_{ij} - \Gamma^{\ell}{}_{ji} = C^{\ell}{}_{ij}\,] \longrightarrow \Gamma_{kij} - \Gamma_{kji} = C_{kij}$$

$$\text{or} \quad \Gamma_{kij} = \Gamma_{kji} + C_{kij}$$

$$= \Gamma_{kji} - C_{kji}$$

(due to antisymmetry of last 2 indices on structure functions)

so $\Gamma_{ikj} = \Gamma_{ijk} - C_{ijk}$ from last of these, relabeling indices. which is the first substitution. The other two substitutions are the basic identity.

The formula $\Gamma^{i}{}_{jk} = \frac{1}{2} g^{i\ell}(g_{\ell j,k} - g_{jk,\ell} + g_{k\ell,j}) + \frac{1}{2}(C^{i}{}_{jk} - C_{jk}{}^{i} + C_{k}{}^{i}{}_{j})$ has two extremes.

In a coordinate frame $C^{i}{}_{jk}=0$ so $\Gamma^{i}{}_{jk} = \left\{ {}^{\;i}_{jk} \right\}$ is just the first part.

In an orthonormal frame $g_{ij} = \delta_{ij}$ so $\left\{ {}^{\;i}_{jk} \right\} = 0$ and only the second part remains

$$\Gamma^{i}{}_{jk} = \frac{1}{2}(C^{i}{}_{jk} - C_{jk}{}^{i} + C_{k}{}^{i}{}_{j})$$

$$\overset{*}{=} \frac{1}{2}(C^{i}{}_{jk} - C^{j}{}_{ki} + C^{k}{}_{ij})$$

true only in ON frame with $g_{ij} = \delta_{ij}$ (Euclidean).

In any other kind of frame both parts contribute. We havent had an example of a frame which is not coordinate or orthonormal yet.