Non-Cartesian Coordinates on $\mathbb{R}^n$

The dual basis covectors $\omega^i \equiv x^i$ to the standard ~~Cartesian coor~~ basis of $\mathbb{R}^n$ $\{e_i\}$ are the standard Cartesian coordinates on $\mathbb{R}^n$.

Any change of basis leads to a new set of Cartesian coordinates

$$x^{i'} = A^i{}_j x^j \quad , \quad (A^i{}_j) \text{ a constant matrix}$$

The Cartesian coordinates also induce a basis $\{\partial/\partial x^i|_p\}$ of ~~each~~ tangent space, with dual basis $\{dx^i|_p\}$ of the corresponding cotangent space. The set of vector fields $\{\partial/\partial x^i\}$ is a frame on $\mathbb{R}^n$ in terms of which any tensor field may be expressed

$$T = T^{i_1\cdots i_p}{}_{j_1\cdots j_q} \, \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} \otimes \omega^{j_1} \otimes \cdots \otimes \omega^{j_q}$$

$$T^{i_1\cdots i_p}{}_{j_1\cdots j_q} = T\left(dx^{i_1}, \ldots, dx^{i_p}, \frac{\partial}{\partial x^{j_1}}, \ldots \frac{\partial}{\partial x^{j_q}}\right)$$

in terms of its components, which are functions on $\mathbb{R}^n$. The "constant" tensor fields on $\mathbb{R}^n$ whose Cartesian coordinate component functions are just constants. are in a 1-1 correspondence with the tensors on the vector space $\mathbb{R}^n$. Their components are clearly constants in any Cartesian frame.

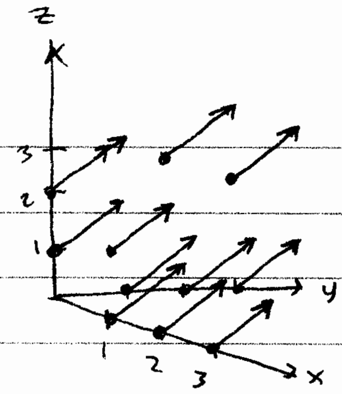
For example $X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ is a constant vector field on $\mathbb{R}^3$, while $G = dx \otimes dx + dy \otimes dy + dz \otimes dz = \delta_{ij}\, dx^i \otimes dx^j$ is a constant metric field, the Euclidean metric tensor field on $\mathbb{R}^3$. The self-inner product of X with itself

$$G(X,X) = \underbrace{dx\left(\frac{\partial}{\partial x} + \cancel{\frac{\partial}{\partial y}} + \cancel{\frac{\partial}{\partial z}}\right)}_{1} \underbrace{dx\left(\frac{\partial}{\partial x} + \cancel{\frac{\partial}{\partial y}} + \cancel{\frac{\partial}{\partial z}}\right)}_{1} + \underbrace{dy\left(\cancel{\frac{\partial}{\partial x}} + \frac{\partial}{\partial y} + \cancel{\frac{\partial}{\partial z}}\right)}_{1} \underbrace{dy \cdots}$$

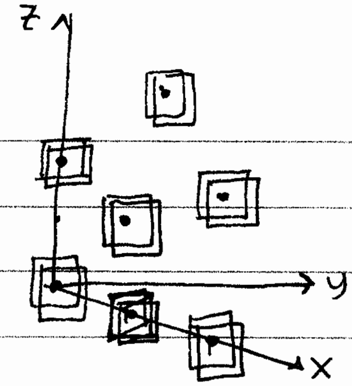
$$= 1 + 1 + 1 = 3 = (1,1,1)\cdot(1,1,1)$$

is just the self-inner product of the corresponding vector $(1,1,1) \in \mathbb{R}^3$.

The covector field $\theta = \frac{1}{4}\, dx$ is a constant 1-form on $\mathbb{R}^3$.



$$X = \frac{1}{3}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)$$

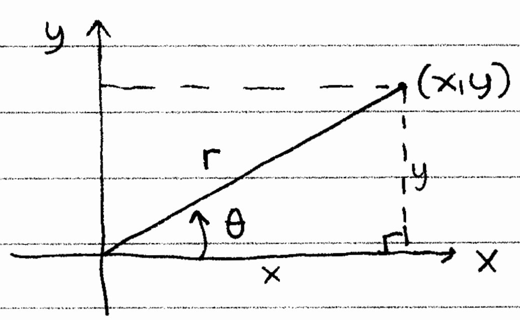


$$\sigma = \frac{1}{4}\, dx$$

We can picture the 1-form as a field of pieces of the pair of planes which represent its value at each point just like we picture a vector field as a field of arrows with initial point at the point where they represent a value. Similar pictures hold for nonconstant vector and covector fields.

Non-Cartesian coordinates on $\mathbb{R}^n$ often prove useful, especially when a problem under consideration has a symmetry associated with special families of surfaces like concentric spheres or cylinders. Polar coordinates $\{r, \theta\}$ on $\mathbb{R}^2$ are the most familiar example, followed by cylindrical coordinates $\{\rho, \varphi, z\}$ on $\mathbb{R}^3$ and spherical coordinates $\{r, \theta, \varphi\}$ on $\mathbb{R}^3$.

Consider polar coordinates on $\mathbb{R}^2$. The usual picture is



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2} \geq 0 \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

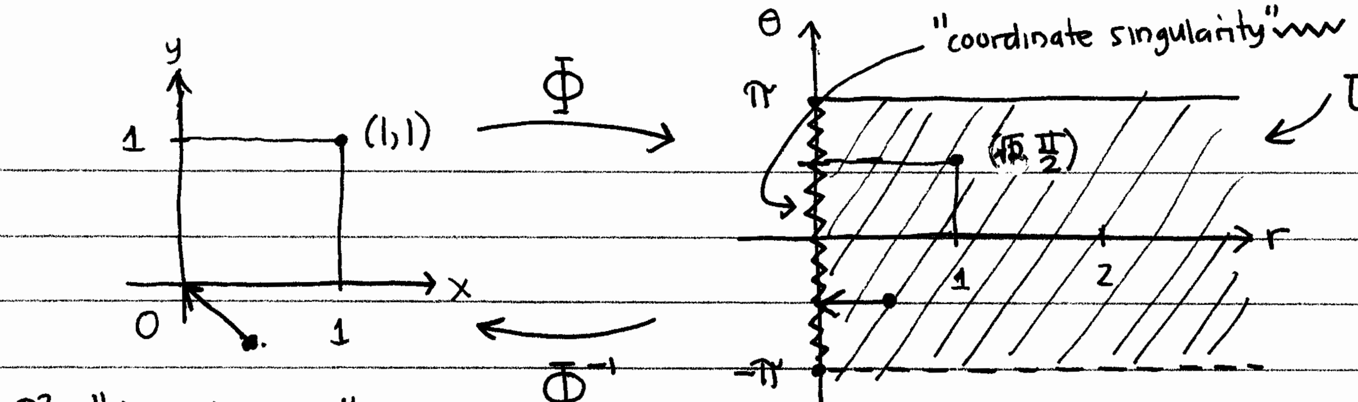
If we agree to choose $\theta \in (-\pi, \pi]$, we get a unique polar angle for every point except the origin: $\theta = \boldsymbol{\Theta} \equiv \begin{cases} \tan^{-1}\frac{y}{x} & , x>0 \\ \tan^{-1}\frac{y}{x}+\pi & , x<0, y>0 \\ \tan^{-1}\frac{y}{x}-\pi & , x<0, y>0 \\ \pi/2 & , x=0, y>0 \\ -\pi/2 & , x=0, y<0 \\ \pi & , x<0, y=0 \end{cases}$

[let $\boldsymbol{\Theta}$ denote this particular choice of polar coord]

Thus a unique pair of values of the polar coordinates characterize every point in the plane except the origin where $r=0$ but θ is undetermined and no choice of value for θ there will make the function continuous at the origin. *This is called a "coordinate singularity."*

WARNING: We use the symbols x and y or r and θ for different things! We interpret x and y as functions on the plane, ~~but~~ we also use (x, y) to represent a particular point in the plane, i.e., the pair of values of the Cartesian coordinate functions at that point. This sloppy habit means we have to be careful so that in any given instance we understand which meaning is intended. Or to be clear we can use notation which distinguishes them.

What is really going on with the above picture and relationships between the Cartesian and polar coordinates? Well, first of all we have two distinct copies of $\mathbb{R}^2$, a "physical space" of points which has a lot of mathematical structure, and a "coordinate space" on which operations involving the polar coordinates occur. The relationship between the two sets of coordinates define two maps between these spaces going in opposite directions. The "coordinate map" associates with each point in the "physical space" a point in the coordinate space which is the pair of values of the polar coordinates there. The "parametrization map" associates with each point in the coordinate space, the point in the "physical space" that they represent.



$\mathbb{R}^2$: "physical space"

$\mathbb{R}^2$: "coordinate space"

- $\Phi : \mathbb{R}^2 - \{\vec{0}\} \longrightarrow U \in \mathbb{R}^2$

  $\Phi(u^1, u^2) = \Big(\underbrace{\sqrt{(u^1)^2 + (u^2)^2}}_{r(u^1,u^2)}, \Theta(u^1, u^2)\Big)$

  $\boxed{\text{coordinate map:}}$
  maps ~~associates~~ points in physical space to its coordinate representation
  (maps all of $\mathbb{R}^2$ onto U)

- $\Phi^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

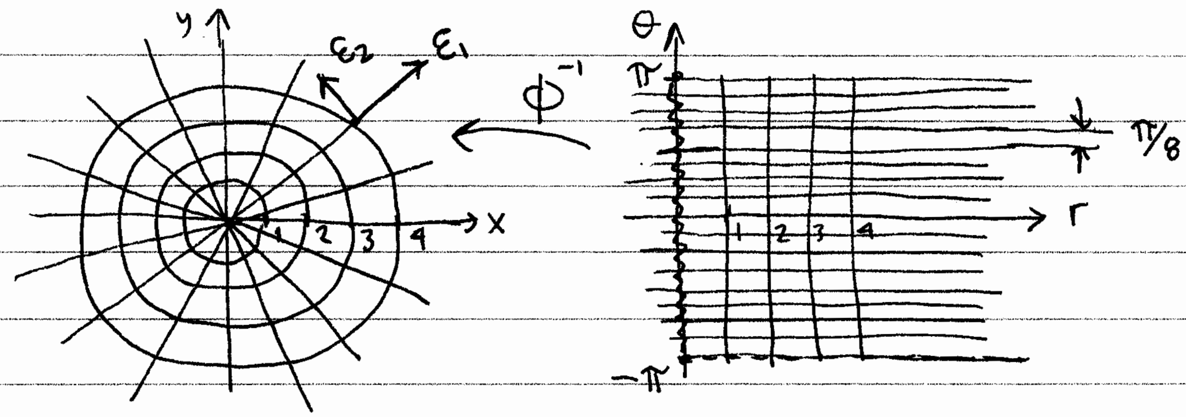
  $\Phi^{-1}(u^1, u^2) = (u^1 \cos u^2, u^1 \sin u^2)$

  $\boxed{\text{parametrization map:}}$
  maps the coordinate representation of a physical point onto that point
  (maps all of $\mathbb{R}^2$ onto all of $\mathbb{R}^2$ many times (infinite #))

Notice that $\Phi \circ \Phi^{-1}$ maps all of the coordinate space onto the set U which is the image of the coordinate map Φ. For points in U, this is the identity map. For points outside of U this associates our specific choice of polar coordinates with any other possible choices, like fixing $\theta \in [0, 2\pi)$, or allowing negative r. The vertical segment ~~The map $\Phi^{-1} \circ \Phi$~~ between $-\pi$ and π on the θ axis (not in U) corresponds to the origin in physical space in the sense that approaching it from any nearby point of U corresponds to approaching the origin in physical space in a certain direction.

The map $\Phi^{-1}\circ\Phi^{\cancel{-1}}$ maps all of physical space except for the origin onto itself, where it is the identity map. Restricting Φ^{-1} to the set U makes it the inverse of the map Φ.

The parametrization map represents the plane as a 2-parameter family of parametrized curves ("coordinate lines") which make up the polar coordinate grid.



$$(u^1, u^2) = \Phi^{-1}(r,\theta) = (r\cos\theta,\ r\sin\theta).$$

Hold θ fixed, vary r: r coordinate lines (half ray's from origin)

Hold r fixed, vary θ: θ coordinate lines (circles centered at origin)

We can easily compute the tangents to these parametrized curves:

$$\vec{\mathcal{E}}_1(r,\theta) = \left(\frac{\partial u^1}{\partial r}, \frac{\partial u^2}{\partial r}\right) = (\cos\theta, \sin\theta)$$

$$\vec{\mathcal{E}}_2(r,\theta) = \left(\frac{\partial u^1}{\partial \theta}, \frac{\partial u^2}{\partial \theta}\right) = (-r\sin\theta, r\cos\theta)$$

$$\left.\begin{aligned} \mathcal{E}_1(r,\theta) &= \cos\theta \left.\frac{\partial}{\partial x}\right|_{(r\cos\theta, r\sin\theta)} + \sin\theta \left.\frac{\partial}{\partial y}\right|_{(r\cos\theta, r\sin\theta)} \\ \mathcal{E}_2(r,\theta) &\ -r\sin\theta \left.\frac{\partial}{\partial x}\right|_{(r\cos\theta, r\sin\theta)} + r\cos\theta \left.\frac{\partial}{\partial y}\right|_{(r\cos\theta, r\sin\theta)} \end{aligned}\right\} \in T\mathbb{R}^2_{(r\cos\theta, r\sin\theta)}$$

Suppose $f(x,y) = x^2 - y^2$ is a function on $\mathbb{R}^2$. Then the tangent vector $\mathcal{E}_1(r,\theta)$ acts on it to produce the number

$$\mathcal{E}_1(r,\theta) f = \cos\theta \underbrace{\underbrace{\frac{\partial}{\partial x}\Big|_{(r\cos\theta, r\sin\theta)} (x^2-y^2)}_{2x|_{(r\cos\theta,r\sin\theta)}}}_{2r\cos\theta} + \sin\theta \underbrace{\underbrace{\frac{\partial}{\partial y}\Big|_{(r\cos\theta, r\sin\theta)} (x^2-y^2)}_{0|_{(r\cos\theta,r\sin\theta)}}}_{0}$$

$$= 2r\cos^2\theta$$

for given values of r and θ. Note that this is just

$$\frac{\partial}{\partial r} f(r\cos\theta, r\sin\theta) = \frac{\partial}{\partial r}\left(r^2\cos^2\theta - r^2\sin^2\theta\right) = 2r\cos^2\theta.$$

The inner products of $\vec{\mathcal{E}}_1$ and $\vec{\mathcal{E}}_2$ are

$$\vec{\mathcal{E}}_1(r,\theta)\cdot\vec{\mathcal{E}}_1(r,\theta) = \cos^2\theta + \sin^2\theta = 1 \quad = G|_{(r\cos\theta, r\sin\theta)}(\mathcal{E}_1(r,\theta), \mathcal{E}_1(r,\theta))$$

$$\vec{\mathcal{E}}_2(r,\theta)\cdot\vec{\mathcal{E}}_2(r,\theta) = r^2\sin^2\theta + r^2\cos^2\theta = r^2 \quad = \cdots \quad \text{(ditto)}$$

$$\vec{\mathcal{E}}_1(r,\theta)\cdot\vec{\mathcal{E}}_2(r,\theta) = -r\cos\theta\sin\theta + r\cos\theta\sin\theta = 0 \quad = \cdots$$

so $\mathcal{E}_1(r,\theta)$ and $\mathcal{E}_2(r,\theta)$ are mutually orthogonal tangent vectors of lengths 1 and r respectively.

Now going back to the sloppy notation $x = r\cos\theta$, $y = r\sin\theta$

then $\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$, $\sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$.

So define the vector fields on physical space by

$$E_1 = (x^2+y^2)^{-1/2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$

$$E_2 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

Their values at $(r\cos\theta, r\sin\theta)$ are just $\mathcal{E}_1(r,\theta)$ and $\mathcal{E}_2(r,\theta)$:

$$E_i|_{(r\cos\theta, r\sin\theta)} = \mathcal{E}_i(r,\theta) \qquad i = 1,2$$

In other words E_1 at a given point equals the tangent vector to the curve through the point corresponding to translations in the coordinate r, while E_2 is the same for θ. Their action on a function

$$E_1\big|_{(r\cos\theta, r\sin\theta)} f = \mathcal{E}_1(r,\theta) f = \frac{\partial}{\partial r} f(r\cos\theta, r\sin\theta)$$

$$= \frac{\partial}{\partial r}[f\circ\Phi^{-1}](r,\theta)$$

$$E_2\big|_{(r\cos\theta, r\sin\theta)} f = \mathcal{E}_2(r,\theta) f = \frac{\partial}{\partial\theta} f(r\cos\theta, r\sin\theta)$$

$$= \frac{\partial}{\partial\theta}[f\circ\Phi^{-1}](r,\theta).$$

The function $f\circ\Phi^{-1}$ on the coordinate space is just the function one gets by expressing f in terms of the coordinate functions r and $\Theta = \Theta$ and we write

$$E_1 = \frac{\partial}{\partial r}, \qquad E_2 = \frac{\partial}{\partial\theta}$$

In other words $\{E_i\}$ is the coordinate frame associated with the polar coordinates. The change in frame

$$E_i = A^{-1\,j}{}_i \frac{\partial}{\partial x^j} \qquad \underline{A} = \begin{bmatrix} (x^2+y^2)^{-1/2}x & -y \\ (x^2+y^2)^{-1/2}y & x \end{bmatrix}, \quad \det \underline{A} = (x^2+y^2)^{1/2}$$

is invertible everywhere except at the origin where $E_2 = 0$ and E_1 has no unique limiting value.

$$\underline{A}^{-1} = \frac{\begin{bmatrix} x & y \\ -(x^2+y^2)^{-1/2}y & (x^2+y^2)^{-1/2}x \end{bmatrix}}{(x^2+y^2)^{1/2}} =$$

If we let $(x^{1'}, x^{2'}) = (r,\theta)$,

then
$$\frac{\partial x^{i'}}{\partial x^j} = \underline{A}^{-1\,i}{}_j \qquad \frac{\partial x^i}{\partial x^{j'}} = A^i{}_j$$

so
$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y}.$$

Using polar coordinates basically means re-expressing everything in terms of them, i.e., moving over to the coordinate space, and doing calculus operations there.

The polar coordinate frame vectors

$$E_1 = (x^2+y^2)^{-1/2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial r}\,, \quad E_2 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \frac{\partial}{\partial \theta}$$

do not form a frame at the origin. E_1 is not defined and has no limit there, while E_2 vanishes. If we remove the factor $(x^2+y^2)^{-1/2}$ from E_1, it is defined but also equal to zero at the origin. This means we cannot use them to express tangent vectors at the origin. Everywhere else they are fine. We therefore need the idea of a LOCAL FRAME and a LOCAL COORDINATE PATCH to handle frames and coordinate systems which have problems at certain points of space ($\mathbb{R}^n$).

A local frame (on an open set $U \subset \mathbb{R}^n$) will be a set of n vector fields which form a basis of the tangent space at each point of U. If $U = \mathbb{R}^n$, it will be called a global frame or just frame.

A local coordinate patch will be an open set $U \subset \mathbb{R}^n$ and a set of n coordinate functions such that the associated coordinate vector fields form a local frame on U. If $\{x^i\}$ are Cartesian coordinates and

$$\frac{\partial}{\partial x^{i'}} = \frac{\partial x^j}{\partial x^{i'}}\frac{\partial}{\partial x^j} = A^{-1\,j}{}_i \frac{\partial}{\partial x^j}\,,$$

this requires that $\det \underline{A}^{-1} = \det\left(\frac{\partial x^j}{\partial x^{i'}}\right) \neq 0$ on U.
For polar coordinates $U = \mathbb{R}^2 - \{\vec{0}\}$.

In order to treat tensor fields or tangent tensors at a given point of $\mathbb{R}^n$, it must be an interior point of the open set U of the local frame we wish to use or of the local coordinate patch we wish to use.

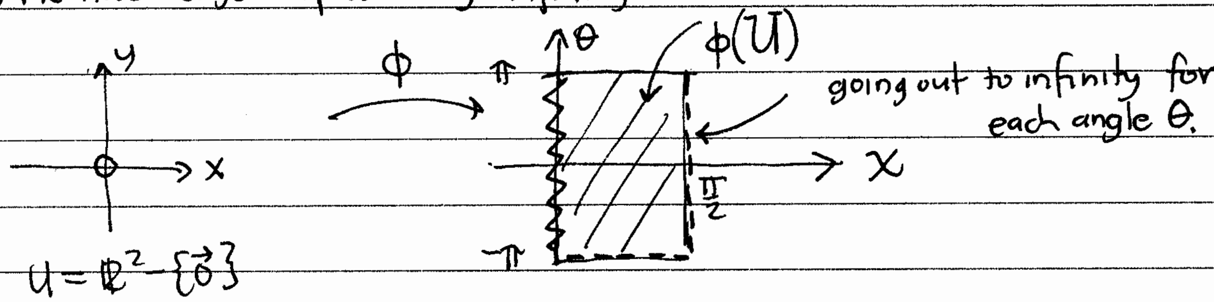
example Let U be the interior of a circle of radius $\epsilon > 0$ about the origin in the plane. The cartesian coordinates $\{x, y\}$ are local coords on U, for every value of $\epsilon > 0$. In order to use polar coordinates which fail at the origin, we must use some other local coordinate patch like one of these which contains the origin in order to handle that problem point. The polar coordinates themselves are local coordinates on the plane minus the origin. This local coordinate patch has to be supplemented by some other patch like the ones we have suggested in order to figure out what's going on at the origin. The two patches together then form a "coordinate covering" of the plane, with each point "covered" by at least one local coordinate patch.

When a coordinate patch just misses being global, the points where it fails are called coordinate singularities. The origin is a coordinate singularity for polar coordinates. The coordinate map ϕ fails to be well defined since the parametrization map is no longer 1-1, so one has many choices for the coordinates to be assigned there.

Suppose we introduce a new radial coordinate by

$$\underset{[0,\infty)}{\underset{\Uparrow}{r}} = \tan \underset{[0,\frac{\pi}{2})}{\underset{\Uparrow}{\chi}} \qquad \chi = \tan^{-1} r$$

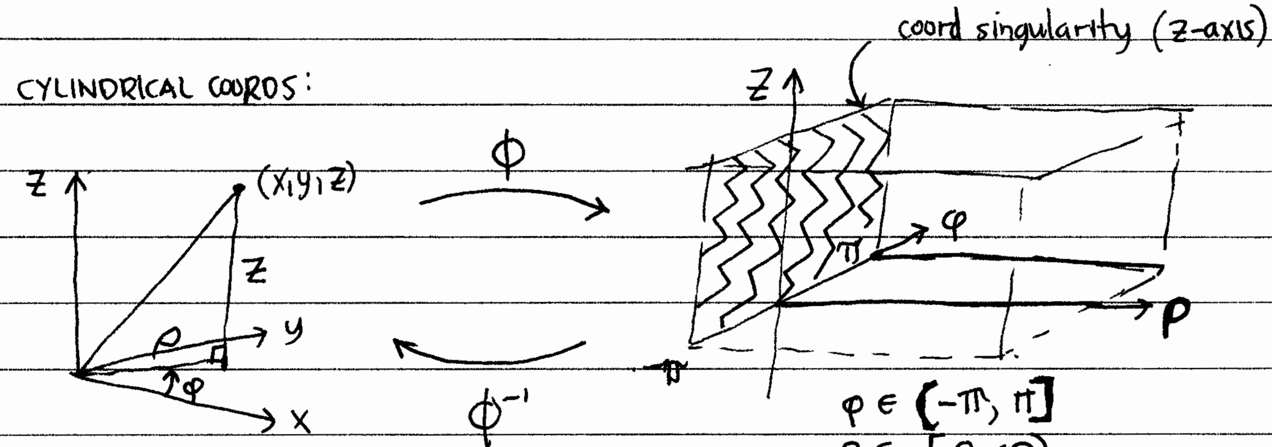
Although the coordinate lines ~~are~~ still the same, all of physical space except for the origin is mapped onto a rectangle in coordinate space with the new "edge" representing "infinity"



CYLINDRICAL AND SPHERICAL COORDINATES ON $\mathbb{R}^3$

2-dimensions is a bad example for some things since $1 = n-1$, so surfaces and lines coincide. 3-dimensions gives us a better picture of higher dimensions. Also let me change my choice of symbol U to refer to the subspace of the physical space associated with a given coordinate patch instead of its image in the coordinate space.

CYLINDRICAL COORDS:

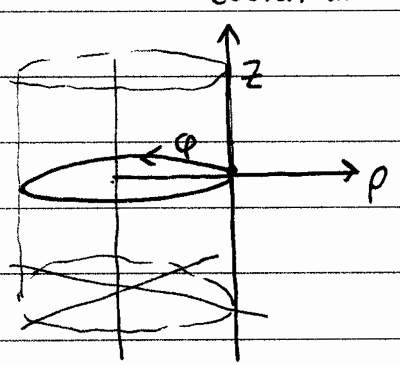


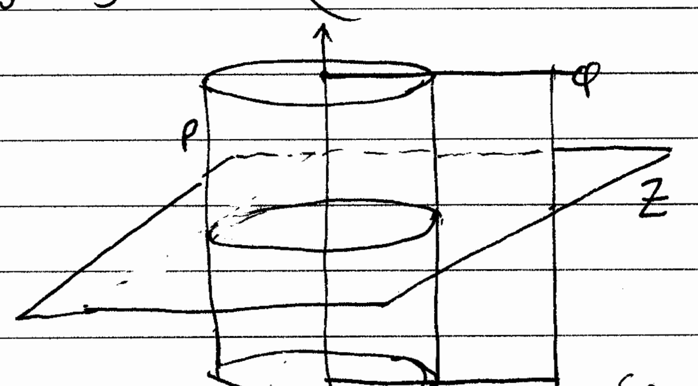
$\varphi \in (-\pi, \pi]$
$\rho \in [0, \infty)$

$$\Phi^{-1}: \begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases} \qquad \Phi: \begin{cases} \rho = \sqrt{x^2+y^2} \\ \tan\varphi = \dfrac{y}{x} \quad \text{(same soln for } \varphi \text{ as before)} \\ z = z \end{cases}$$

$U = \mathbb{R}^3 - \{(x\,y\,z) \mid y=0, x \le 0\}$
negative xz half plane where jump in φ occurs
plus z axis coordinate singularity

($\{\rho, \varphi\}$ are just the polar coordinates $\{r, \theta\}$ of the projection down to the xy plane)

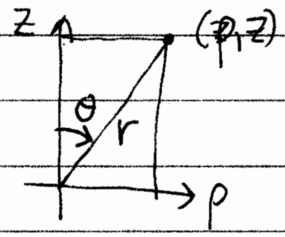




coordinate lines: $\begin{cases} \rho & \text{half lines} \\ \varphi & \text{circles} \\ z & \text{lines} \end{cases}$

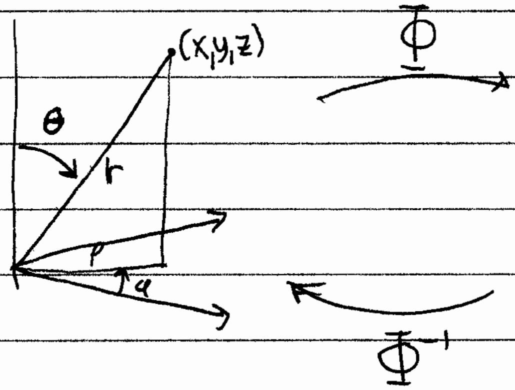
coordinate surfaces: $\begin{cases} \rho & \text{cylinders} \\ \varphi & \text{halfplanes} \\ z & \text{planes} \end{cases}$

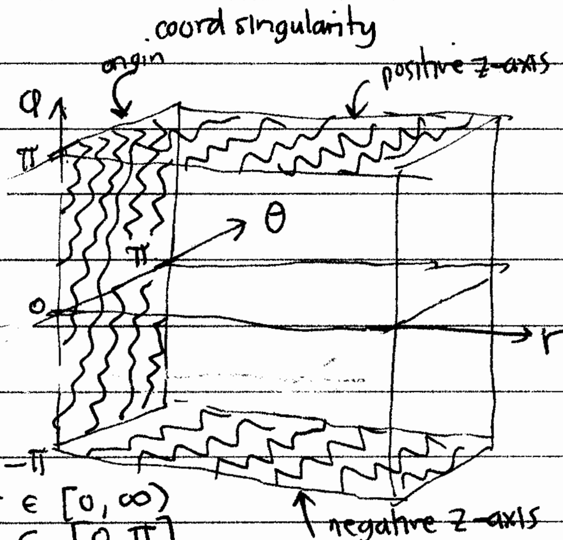
SPHERICAL COORDINATES



To go from cylindrical coordinates to spherical coordinates, one introduces polar coordinates $\{r, \theta\}$ in the ρ-z plane

$z = r\cos\theta$
$\rho = r\sin\theta$





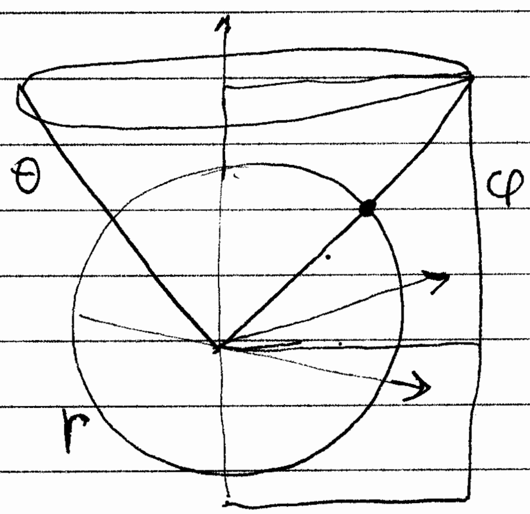
$r \in [0, \infty)$
$\theta \in [0, \pi]$
$\varphi \in (-\pi, \pi]$

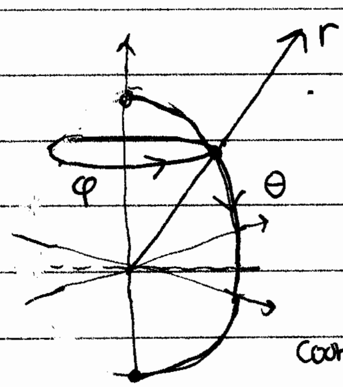
$$\Phi^{-1}: \begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$$\Phi: \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \cos^{-1} \dfrac{z}{\sqrt{x^2+y^2+z^2}} \\ \tan\varphi = \dfrac{y}{x} \quad \text{[same solution as before]} \end{cases}$$

$U = \mathbb{R}^3 - \{\underbrace{(x,y,z) \mid y=0, x\leq 0}\}$

negative xz half space where jump in φ occurs plus z-axis where coordinate singularity occurs





coord lines: $\begin{cases} r & \text{half lines} \\ \theta & \text{half circles} \\ \varphi & \text{circles} \end{cases}$

coord surfaces: $\begin{cases} r & \text{spheres} \\ \theta & \text{half cones} \\ \varphi & \text{half planes} \end{cases}$

In both of these cases the coordinate map Φ is discontinuous on the half plane $\{(x,y,z) \mid x<0, y=0\}$ where φ has a jump of 2π and undefined on the z-axis where the angular coordinate φ is not defined. A "coordinate singularity" occurs at the z-axis for this reason, while at the origin θ is also undefined, making the singularity worse. The parametrization map maps a line segment (different φ values) onto each point on the z-axis except at the origin where a rectangle (all θ and φ values) is mapped onto a single point.

The first thing to do is compute the new coordinate frame vectorfields and dual 1-forms. The next step is to evaluate their inner products are re-express the Euclidean metric.

Parentthetic remark We have some empty space here so I might as well include a short remark. I havent been consistent about the open set U of a local coordinate patch. For polar coordinates I included the negative x-axis where the jump in the angular coordinate occurs, but not the corresponding half plane for cylindrical & spherical coords. If we require Φ to be continuous then we must exclude points of discontinuity. The coordinate frame and dual frame are perfectly fine there, however, because of periodicity, so the local frame $\{E_1, E_2, E_3\}$ expressed in terms of Cartesian coordinates is valid everywhere except on the z-axis, i.e., its U includes the discontinuous points of Φ.

$$\begin{cases} d\rho = d\,(x^2+y^2)^{1/2} = \dfrac{d(x^2+y^2)}{2(x^2+y^2)^{1/2}} = \dfrac{x\,dx + y\,dy}{(x^2+y^2)^{1/2}} & \equiv W^1 \\[2ex] d\varphi = d\left(\tan^{-1}\dfrac{y}{x} + \text{const}\right) = \dfrac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = \dfrac{-\frac{y}{x^2}dx + \frac{dy}{x}}{(x^2+y^2)/x^2} = \dfrac{-y\,dx + x\,dy}{x^2+y^2} & \equiv W^2 \\[2ex] dz = dz & \equiv W^3 \end{cases}$$

$$(x^{1'}, x^{2'}, x^{3'}) \equiv (\rho, \varphi, z) \qquad (x^1, x^2, x^3) \equiv (x, y, z)$$

$$dx^{i'} = A^i{}_j\, dx^j = \frac{\partial x^{i'}}{\partial x^j} dx^j \,, \quad \frac{\partial}{\partial x^{i'}} = A^{-1\,j}{}_i \frac{\partial}{\partial x^j}$$

so $$(A^i{}_j) = \left(\frac{\partial x^{i'}}{\partial x^j}(x)\right) = \begin{pmatrix} \frac{x}{(x^2+y^2)^{1/2}} & \frac{y}{(x^2+y^2)^{1/2}} & 0 \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To invert 2x2 submatrix use $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}{ad-bc}$

and $$(A^{-1\,i}{}_j) = \left(\frac{\partial x^i}{\partial x^{j'}}(x)\right) = \begin{pmatrix} \frac{x}{(x^2+y^2)^{1/2}} & -y & 0 \\ \frac{y}{(x^2+y^2)^{1/2}} & x & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so the coordinate vector fields are

$$\frac{\partial}{\partial \rho} \equiv \frac{\partial}{\partial x^{1'}} = A^{-1\,i}{}_1 \frac{\partial}{\partial x^i} = (x^2+y^2)^{-1/2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \equiv E_1$$

$$\frac{\partial}{\partial \varphi} \equiv \frac{\partial}{\partial x^{2'}} = A^{-1\,i}{}_2 \frac{\partial}{\partial x^i} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \equiv E_2$$

$$\frac{\partial}{\partial z} \equiv \frac{\partial}{\partial x^{3'}} = A^{-1\,i}{}_3 \frac{\partial}{\partial x^i} = \frac{\partial}{\partial z} \equiv E_3$$

On the other hand, by the chain rule, the tangents to the parametrized coordinate lines are

$$\frac{\partial}{\partial \rho}\Big|_{(\rho\cos\varphi,\rho\sin\varphi,z)} = \left[\frac{\partial x}{\partial \rho}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \rho}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \rho}\frac{\partial}{\partial z}\right]\Big|_{(\rho\cos\varphi,\rho\sin\varphi,z)} = \cos\varphi\frac{\partial}{\partial x}\Big|_{(\rho\cos\varphi,\rho\sin\varphi,z)} + \sin\varphi\frac{\partial}{\partial y}\Big|_{(\rho\cos\varphi,\rho\sin\varphi,z)}$$

$$\frac{\partial}{\partial \varphi}\Big|_{(\rho\cos\varphi,\rho\sin\varphi,z)} = [\cdots] = -\rho\sin\varphi\frac{\partial}{\partial x}\Big|_{(\rho\cos\varphi,\rho\sin\varphi,z)} + \rho\cos\varphi\frac{\partial}{\partial y}\Big|_{(\rho\cos\varphi,\rho\sin\varphi,z)}$$

$$\frac{\partial}{\partial z}\Big|_{(\rho\cos\varphi,\rho\sin\varphi,z)} = \frac{\partial}{\partial z}\Big|_{(\rho\cos\varphi,\rho\sin\varphi,z)}$$

Note that the coordinate vector fields $\{E_1, E_2, E_3\}$ fail to be linearly independent on the z-axis where E_2 vanishes, while at the origin E_1 vanishes. This leads to W^1 not having a well-defined limit at the z-axis and causes W^2 to have components which become infinite there.

Now we need a change in notation for the Euclidean metric. We have been using g for functions, and G for symmetric inner product tensors. By convention one uses g for symmetric inner product tensors.

The Euclidean metric tensor field is

$$g = \delta_{ij}\, dx^i \otimes dx^j, \quad \delta_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \equiv g_{ij}$$

We can re-express it in terms of the new frame

$$g = g_{i'j'}\, dx^{i'} \otimes dx^{j'}, \quad g_{i'j'} = g\left(\frac{\partial}{\partial x^{i'}}, \frac{\partial}{\partial x^{j'}}\right) = A^{-1\,m}{}_i A^{-1\,n}{}_j\, \delta_{mn}$$
$$= [(\underline{A}^{-1})^T \underline{I}\, \underline{A}^{-1}]_{ij}$$

One can directly take the inner products of $\{E_i\}$ or use the matrix transformation law to obtain the new components as functions of the Cartesian coordinates, or one can just evaluate the differentials of x, y, z which will lead to expressions in terms of the new components.

The matrix calculation (exercise) yields

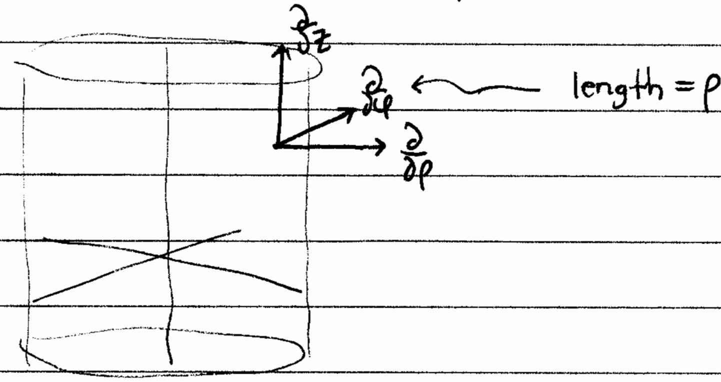
$$(g_{i'j'}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^2+y^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} g_{\rho\rho} & g_{\rho\varphi} & g_{\rho z} \\ g_{\varphi\rho} & g_{\varphi\varphi} & g_{\varphi z} \\ g_{z\rho} & g_{z\varphi} & g_{zz} \end{pmatrix}$$

while

$$\left.\begin{aligned} dx &= d(\rho\cos\varphi) = \cos\varphi\, d\rho - \rho\sin\varphi\, d\varphi \\ dy &= d(\rho\sin\varphi) = \sin\varphi\, d\rho + \rho\cos\varphi\, d\varphi \\ dz &= d(z) = dz \end{aligned}\right\} \rightarrow dx^i = \frac{\partial x^i}{\partial x^{j'}} dx^{j'} = A^{-1\,i}{}_j dx^j \rightarrow \underline{A}^{-1}(x') = \begin{pmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} g = dx\otimes dx + dy\otimes dy + dz\otimes dz &= \underbrace{(\cos\varphi\, d\rho - \rho\sin\varphi\, d\varphi)\otimes(\cos\varphi\, d\rho - \rho\sin\varphi\, d\varphi) + (\sin\varphi\, d\rho + \rho\cos\varphi\, d\varphi)\otimes(\sin\varphi\, d\rho + \rho\cos\varphi\, d\varphi)} + dz\otimes dz \\ &= \underbrace{1}_{g_{\rho\rho}} d\rho\otimes d\rho + \underbrace{\rho^2}_{g_{\rho\rho}} d\varphi\otimes d\varphi + \underbrace{1}_{g_{zz}} dz\otimes dz \end{aligned}$$

The coordinate frame is orthogonal (since mutual inner products vanish) and $\frac{\partial}{\partial\rho}$ and $\frac{\partial}{\partial z}$ are unit vector fields, while $\frac{\partial}{\partial\varphi}$ has length ρ.



Note that re-expressing $\underline{A}$ in terms of the new coordinates leads to

$$(A^i{}_j(\underline{x}')) = \left(\frac{\partial x^{i'}}{\partial x^j}(\underline{x}')\right) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\frac{\sin\varphi}{\rho} & \frac{\cos\varphi}{\rho} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The four matrices $\underline{A}$ and $\underline{A}^{-1}$ expressed in terms of the old and new coordinates may be used to transform any tensor fields from one coordinate system to the other.

For example

$$X^{i'}(x') = \underbrace{\frac{\partial x^{i'}}{\partial x^j}}_{\text{change to new components.}}\underbrace{(x(x'))}\; X^j\underbrace{(x(x'))}_{\text{old coords expressed in terms of new}}$$

The vector field $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ has components $(X^i) = \begin{pmatrix} y \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} \rho\sin\varphi \\ \rho\cos\varphi \\ 0 \end{pmatrix}$

so its new components are

$$(X^{i'}) = \begin{pmatrix} X^\rho \\ X^\varphi \\ X^z \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\frac{\sin\varphi}{\rho} & \frac{\cos\varphi}{\rho} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \rho\sin\varphi \\ \rho\cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} 2\rho\sin\varphi\cos\varphi \\ \cos^2\varphi - \sin^2\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} \rho\sin 2\varphi \\ \cos 2\varphi \\ 0 \end{pmatrix}$$

so $\quad X = \rho\sin 2\varphi\frac{\partial}{\partial\rho} + \cos 2\varphi\frac{\partial}{\partial\varphi}.$

Similarly $\underline{X}^\flat = y\,dx + x\,dy$ can be transformed:

$$(\underline{X}_{i'}) = (\underline{X}_j A^{-1j}{}_i) = (\underset{\rho\sin\varphi}{y}, \underset{\rho\cos\varphi}{x}, 0)\begin{pmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} = (2\rho\cos\varphi\sin\varphi, (\rho^2\cos^2\varphi - \sin^2\varphi), 0)$$
$$= (\rho\sin 2\varphi, \rho^2\cos 2\varphi, 0)$$

so $\underline{X}^\flat = \underbrace{\rho\sin 2\varphi}_{\underline{X}_\rho = g_{\rho\rho}\underline{X}^\rho} d\rho + \underbrace{\rho^2\cos 2\varphi}_{\underline{X}_\varphi = g_{\varphi\varphi}\underline{X}^\varphi} d\varphi$

$$\underline{X}_\rho = \underbrace{g_{\rho\rho}}_{1}\underbrace{\underline{X}^\rho}_{\rho\sin 2\varphi} \qquad \underline{X}_\varphi = \underbrace{g_{\varphi\varphi}}_{\rho^2}\underbrace{\underline{X}^\varphi}_{\cos 2\varphi}$$

The same result could have been obtained using the re-expressed metric.

Since the frame is orthogonal, index shifting reduces to multiplication by the corresponding diagonal metric component.

We can also just re-expression the old coordinate ~~derivative~~ frame vector fields in terms of the new

$$\frac{\partial}{\partial x^i} = A^j{}_i \frac{\partial}{\partial x^{j'}} \rightarrow \begin{cases} \dfrac{\partial}{\partial x} = \cos\varphi\dfrac{\partial}{\partial\rho} - \dfrac{\sin\varphi}{\rho}\dfrac{\partial}{\partial\varphi} \\ \dfrac{\partial}{\partial y} = \sin\varphi\dfrac{\partial}{\partial\rho} + \dfrac{\cos\varphi}{\rho}\dfrac{\partial}{\partial\varphi} \\ \dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial z} \end{cases}$$

<u>Exercise</u>: Evaluate the 2nd order linear differential operator

$$\nabla^2 = \delta^{ij}\frac{\partial^2}{\partial x^i \partial x^j} = \left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}\right)$$

by substituting the above expressions for the Cartesian coordinate vector fields.

exercise worked

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial x}\right)^2 = \left(\cos\varphi \frac{\partial}{\partial\rho} - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}\right)\left(\cos\varphi\frac{\partial}{\partial\rho} - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}\right)$$

$$= \cos^2\varphi \frac{\partial^2}{\partial\rho^2} - \frac{\cos\varphi\sin\varphi}{\rho}\frac{\partial^2}{\partial\rho\partial\phi} - \frac{\sin\varphi}{\rho}(-\sin\varphi)\frac{\partial}{\partial\rho} + \frac{\sin^2\varphi}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \left[\cos\varphi\left(\frac{\sin\varphi}{\rho^2}\right) + \frac{\sin\varphi\cos\varphi}{\rho^2}\right]\frac{\partial}{\partial\varphi}$$

$$= \cos^2\varphi\frac{\partial^2}{\partial\rho^2} + \frac{\sin^2\varphi}{\rho}\frac{\partial}{\partial\rho} + \frac{\sin^2\varphi}{\rho^2}\frac{\partial^2}{\partial\varphi^2} - \frac{\cos\varphi\sin\varphi}{\rho}\frac{\partial^2}{\partial\rho\partial\varphi} + \frac{2\cos\varphi\sin\varphi}{\rho^2}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial y}\right)^2 = \left(\sin\varphi\frac{\partial}{\partial\rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi}\right)\left(\sin\varphi\frac{\partial}{\partial\rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi}\right)$$

$$= \sin^2\varphi\frac{\partial^2}{\partial\rho^2} + \frac{\cos\varphi(\cos\varphi)}{\rho}\frac{\partial}{\partial\rho} + \frac{\sin\varphi\cos\varphi}{\rho}\frac{\partial^2}{\partial\rho\partial\phi} + \frac{\cos^2\varphi}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \left[\frac{\sin\varphi\cos\varphi}{-\rho^2} + \frac{\cos\varphi}{\rho}\left(\frac{-\sin\varphi}{\rho}\right)\right]\frac{\partial}{\partial\varphi}$$

$$= \sin^2\varphi\frac{\partial^2}{\partial\rho^2} + \frac{\cos^2\varphi}{\rho}\frac{\partial}{\partial\rho} + \frac{\cos^2\varphi}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\sin\varphi\cos\varphi}{\rho}\frac{\partial^2}{\partial\rho\partial\phi} - \frac{2\cos\varphi\sin\varphi}{\rho^2}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = (\cos^2\varphi + \sin^2\varphi)\frac{\partial^2}{\partial\rho^2} + \frac{\sin^2\varphi + \cos^2\varphi}{\rho}\frac{\partial}{\partial\rho} + \frac{\sin^2\varphi + \cos^2\varphi}{\rho^2}\frac{\partial^2}{\partial\varphi^2}$$

$$+ \left[-\frac{\cos\varphi\sin\varphi}{\rho} + \frac{\cos\varphi\sin\varphi}{\rho}\right]\frac{\partial^2}{\partial\rho\partial\phi} + \left[\frac{2\cos\varphi\sin\varphi}{\rho^2} - \frac{2\cos\varphi\sin\varphi}{\rho^2}\right]\frac{\partial}{\partial\varphi}$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}$$

[If $(\rho, \varphi) \to (r, \theta)$ this gives ∇^2 on $\mathbb{R}^2$ in polar coords]

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}}$$

Note that $\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) = \frac{\partial^2 f}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial\rho}$, so the first two terms can be rewritten as: $\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)$.

exercise from page 17b worked

$$[X,Y] = [A^i{}_j x^j \tfrac{\partial}{\partial x^i},\ B^m{}_n x^n \tfrac{\partial}{\partial x^n}] = A^i{}_j x^j B^m{}_n \underbrace{\frac{\partial x^n}{\partial x^i}}_{\delta^m{}_i} \frac{\partial}{\partial x^m} - B^m{}_n x^n A^i{}_j \underbrace{\frac{\partial x^j}{\partial x^m}}_{\delta^j{}_m} \frac{\partial}{\partial x^i}$$

$$= B^m{}_i A^i{}_j x^j \frac{\partial}{\partial x^m} - A^i{}_j B^j{}_m x^n \frac{\partial}{\partial x^i} = B^m{}_i A^i{}_j x^j \frac{\partial}{\partial x^m} - A^m{}_i B^i{}_j x^j \frac{\partial}{\partial x^m}$$

$$= [\underline{B},\underline{A}]^m{}_j x^j \frac{\partial}{\partial x^m} = -[\underline{A},\underline{B}]^m{}_j x^j \frac{\partial}{\partial x^m}$$

$$[X,Z] = [A^i{}_j x^j \tfrac{\partial}{\partial x^j},\ b^e \tfrac{\partial}{\partial x^e}] = -b^e A^i{}_j \underbrace{\frac{\partial x^j}{\partial x^e}}_{\delta^j{}_e} \frac{\partial}{\partial x^i} = -A^i{}_j b^j \frac{\partial}{\partial x^i}$$

$$= -[\underline{A}\underline{b}]^i \frac{\partial}{\partial x^i}$$

$$[Z,W] = [b^e \tfrac{\partial}{\partial x^e},\ c^k \tfrac{\partial}{\partial x^k}] = 0 \quad (\text{constant components})$$

We can make these results look mathematically pretty by defining $\sigma(\underline{A}) = A^i{}_j x^j \frac{\partial}{\partial x^i}$, $\mathcal{J}(\underline{b}) = b^i \frac{\partial}{\partial x^i}$

then

(1) $[\sigma(\underline{A}), \sigma(\underline{B})] = \sigma([\underline{A},\underline{B}])$

(2) $[\sigma(\underline{A}), \mathcal{J}(\underline{b})] = -\mathcal{J}(\underline{A}\underline{b})$

(3) $[\mathcal{J}(\underline{b}), \mathcal{J}(\underline{c})] = 0$

$\sigma : \{n\times n \text{ matrices}\} \rightarrow \{\text{vector fields on } \mathbb{R}^n\}$ is a linear map (check: $\sigma(a\underline{A}+b\underline{B}) = a\sigma(A)+b\sigma(B)$) from a vector space with a commutator into a vector space with a commutator (Lie bracket). Relation (1) says you can do the commutator before or after the map and still get the same result. [A vectorspace with a commutator is called a Lie algebra and such a map between Lie algebras mapping one commutator into the next is called a Lie algebra homomorphism.]

exercise at bottom of page 17b worked

$$[\nabla^2, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}] = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right) - \left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}x\frac{\partial}{\partial y}\right) - y\underset{②}{\frac{\partial^3}{\partial x^3}} + x\underset{③}{\frac{\partial^3}{\partial y^3}} - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}y\frac{\partial}{\partial x}\right) \qquad -\left(x\underset{①}{\frac{\partial^3}{\partial y\partial x^2}} - y\underset{②}{\frac{\partial^3}{\partial x^3}} + x\underset{③}{\frac{\partial^3}{\partial y^3}} - y\underset{④}{\frac{\partial^3}{\partial x\partial y^2}}\right)$$

$$\frac{\partial}{\partial x}\left(x\frac{\partial^2}{\partial x\partial y}+\frac{\partial}{\partial y}\right) \qquad -\frac{\partial}{\partial y}\left(y\frac{\partial^2}{\partial y\partial x}+\frac{\partial}{\partial x}\right)$$

$$x\underset{①}{\frac{\partial^3}{\partial x^2\partial y}} + \underset{⑤}{\frac{\partial^2}{\partial x\partial y}} + \underset{⑤}{\frac{\partial^2}{\partial x\partial y}} + -y\underset{④}{\frac{\partial^3}{\partial y^2\partial x}} - \underset{⑤}{\frac{\partial^2}{\partial y\partial x}} - \underset{⑤}{\frac{\partial^2}{\partial y\partial x}}$$

$= 0$ (everything cancels)

The same calculation applies to

$$\left[\delta^{ij}\frac{\partial^2}{\partial x^i\partial x^j},\ x^m\frac{\partial}{\partial x^n} - x^n\frac{\partial}{\partial x^m}\right] \qquad m\neq n$$

define $x^m = x$
$x^n = y$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\underbrace{\sum_{i\neq m,n}\frac{\partial^2}{\partial (x^i)^2}},\ x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right]$$

this term commutes with $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$

since x, y constants with respect to remaining coordinates.

exercise on bottom of page 17c worked

$$\mathfrak{X}f = \left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\left(\frac{y}{x}\right) = xy\frac{\partial}{\partial x}\left(\frac{1}{x}\right) + y\frac{1}{x}\frac{\partial}{\partial y}(y) = -\frac{y}{x}+\frac{y}{x} = 0$$

$$\mathfrak{X}g = \left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\left(\frac{z}{\sqrt{x^2+y^2}}\right) = \underbrace{-\frac{z}{2(x^2+y^2)^{3/2}}\underbrace{\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)(x^2+y^2)}_{2x^2+2y^2}}_{\cancel{z\sqrt{x^2+y^2}}\quad -\frac{z}{\sqrt{x^2+y^2}}} + \frac{z}{\sqrt{x^2+y^2}} = 0$$

$$df = d\left(\frac{y}{x}\right) = -\frac{y}{x^2}dx + \frac{1}{x}dy = \frac{-y\,dx+x\,dy}{x^2}$$

$$dg = d\left(\frac{z}{\sqrt{x^2+y^2}}\right) = -\frac{1}{2(x^2+y^2)^{3/2}}\underbrace{d(x^2+y^2)}_{2x\,dx+2y\,dy} + \frac{dz}{\sqrt{x^2+y^2}}$$

$$= -\frac{z(x\,dx+y\,dy)}{(x^2+y^2)^{3/2}} + \frac{dz}{\sqrt{x^2+y^2}}$$

$$dg(\mathfrak{X}) = -\frac{z(x\cdot\mathfrak{X}^1+y\mathfrak{X}^2)}{(x^2+y^2)^{3/2}} + \frac{\mathfrak{X}^3}{\sqrt{x^2+y^2}} = \underbrace{-\frac{z(x^2+y^2)}{(x^2+y^2)^{3/2}}}_{-\frac{z}{\sqrt{x^2+y^2}}} + \frac{z}{\sqrt{x^2+y^2}} = 0$$

$$df(\mathfrak{X}) = \frac{-y\mathfrak{X}^1+x\mathfrak{X}^2}{x^2} = \frac{-yx+xy}{x^2} = 0.$$

Summary of what we did for cylindrical coordinates

$$(1)\quad dx^{i'} = \frac{\partial x^{i'}}{\partial x^j}(x)\, dx^j$$

$$(2)\quad \frac{\partial}{\partial x^{i'}} = \frac{\partial x^j}{\partial x^{i'}}(x'(x))\frac{\partial}{\partial x^j}$$

defines new coordinate frame entirely in terms of Cartesian coordinates

$$(3)\quad dx^i \underset{(5)}{=} \frac{\partial x^i}{\partial x^{j'}}(x')\, dx^{j'}$$

$$(4)\quad \frac{\partial}{\partial x^i} = \frac{\partial x^{j'}}{\partial x^i}(x(x'))\frac{\partial}{\partial x^{j'}}$$

represents Cartesian frame in terms of new coordinates

In words:

(1) Taking the differential of the coordinate map $\phi = (x^{1'}, x^{2'}, x^{3'})$ yields the new coordinate dual frame in terms of the Cartesian coordinates and the matrix $A^i{}_j(x)$ which can be inverted to give $A^{-1\,i}{}_j(x)$ and Cartesian coordinate expressions (2) for the new coordinate vector fields.

Substitution of the parametrization map into these two matrices re-expresses them in terms of the new coordinates which may then be used to represent the coordinate frame and dual frame in terms of the new coordinates.

(5) Alternatively one can take the differential of the parametrization map to directly yield $A^{-1\,i}{}_j(x(x'))$ which can be inverted to get $A^i{}_j(x(x'))$.

(6) The we re-expressed the Euclidean metric $g = \delta_{ij}\, dx^i \otimes dx^j$

$$= \delta_{ij} \frac{\partial x^i}{\partial x^{m'}} \frac{\partial x^j}{\partial x^{n'}}\, dx^{m'} \otimes dx^{n'} \equiv g_{m'n'}\, dx^{m'} \otimes dx^{n'}$$

either by substituting (3) into g or by using the matrix transformation

$$\underline{g}' = \underline{A}^{-1}(x')^T \underline{I}\; \underline{A}^{-1}(x') \equiv (g_{m'n'})$$

(7) Then we evaluated $\nabla^2 = \delta^{ij} \frac{\partial^2}{\partial x^i \partial x^j} = \ldots$

EXERCISE. Repeat for spherical coords. Re-express $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ and $X^\flat = y\,dx + x\,dy$.

exercise on page 35 worked (SPHERICAL COORDINATES)

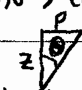
$x^i = x^i(\bar{x})$ $\qquad (A^{-1\,i}{}_j(\bar{x})) = \left(\frac{\partial x^i}{\partial \bar{x}^j}(\bar{x})\right)$

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

$$\text{"}\begin{pmatrix} \partial x/\partial\bar{x}^j \\ \partial y/\partial\bar{x}^j \\ \partial z/\partial\bar{x}^j \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

$$(A^{-1\,i}{}_j(x)) = \left(\frac{\partial x^i}{\partial \bar{x}^j}(x)\right) = \begin{pmatrix} \frac{x}{(x^2+y^2+z^2)^{1/2}} & \frac{xz}{(x^2+y^2)^{1/2}} & -y \\ \frac{y}{(x^2+y^2+z^2)^{1/2}} & \frac{yz}{(x^2+y^2)^{1/2}} & x \\ \frac{z}{(x^2+y^2+z^2)^{1/2}} & -(x^2+y^2)^{1/2} & 0 \end{pmatrix}$$

first column just $\frac{1}{r}\begin{pmatrix} x \\ y \\ z\end{pmatrix}$.
second column; first two entries $\sin\theta \to \cos\theta = \sin\theta\,\cot\theta$



$\to \quad \frac{z}{\rho} = \frac{z}{(x^2+y^2)^{1/2}}$.
last entry obvious, just $-\rho$.
last column: interchange x, y plus minus sign.

$\bar{x}^i = \bar{x}^i(x)$

$r = (x^2+y^2+z^2)^{1/2}$

$\theta = \cos^{-1}\frac{z}{(x^2+y^2+z^2)^{1/2}}$ — $\frac{d}{du}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}$ $\qquad \frac{\partial\theta}{\partial x} = -\frac{1}{\sqrt{1-\frac{z^2}{(x^2+y^2+z^2)}}}\left[-\frac{1}{2}\frac{z(2x)}{(x^2+y^2+z^2)^{3/2}}\right] =$ simplify, same for $\partial\theta/\partial y$

$\frac{\partial\theta}{\partial z} = -\frac{1}{\sqrt{1-\frac{z^2}{(x^2+y^2+z^2)}}}\left[-\frac{1}{2}\frac{z(2z)}{(x^2+y^2+z^2)^{3/2}} + \frac{1}{(x^2+y^2+z^2)^{1/2}}\right]$ simplify

$\varphi = \tan^{-1}\frac{y}{x} + \text{"}c\text{"}$ — $\frac{d}{du}\tan^{-1}u = \frac{1}{1+u^2}$

$$\left(\frac{\partial\bar{x}^i}{\partial x^j}(x)\right) = (A^i{}_j(x)) = \begin{pmatrix} \partial r/\partial x^j \\ \partial\theta/\partial x^j \\ \partial\varphi/\partial x^j \end{pmatrix} = \begin{pmatrix} \frac{x}{(x^2+y^2+z^2)^{1/2}} & \frac{y}{(x^2+y^2+z^2)^{1/2}} & \frac{z}{(x^2+y^2+z^2)^{1/2}} \\ \frac{xz}{(x^2+y^2+z^2)\,(x^2+y^2)^{1/2}} & \frac{yz}{(x^2+y^2+z^2)(x^2+y^2)^{1/2}} & \frac{-(x^2+y^2)}{(x^2+y^2+z^2)(x^2+y^2)^{1/2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix}$$

$$\left(\frac{\partial\bar{x}^i}{\partial x^j}(\bar{x})\right) = (A^i{}_j(\bar{x})) = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \frac{1}{r}\cos\theta\cos\varphi & \frac{1}{r}\cos\theta\sin\varphi & -\frac{1}{r}\sin\theta \\ -\frac{\sin\varphi}{r\sin\theta} & \frac{\cos\varphi}{r\sin\theta} & 0 \end{pmatrix}$$

first row just $\frac{(x,y,z)}{r}$.
second row $\frac{x(r\cos\theta)}{r^2(r\sin\theta)}\ \ldots$
third row $\frac{-y}{r^2\sin^2\theta}\ \ldots$

exercise to this solution: check that $\underline{A}(\bar{x})\,\underline{A}^{-1}(\bar{x}) = \underline{I}$.

The matrix $A^{-1}(\bar{x}) = \left(\frac{\partial x^i}{\partial r} \; \frac{\partial x^i}{\partial \theta} \; \frac{\partial x^i}{\partial \varphi} \right)$

has as its columns the Cartesian coordinate components of the tangents to the new coordinate lines parametrized by those coordinates. The first column are the old fashioned components of the tangent vector of the curve which results from holding θ and φ fixed and varying r, for example.

Since the new coordinate system is orthogonal, these three tangent vectors are orthogonal as one can verify by taking dot products of the corresponding vectors in $\mathbb{R}^3$. In fact

$$[\underline{A}^{-1}(\bar{x})]^T \underline{A}^{-1}(\bar{x})$$

is the matrix of all possible inner products of these vectors. By orthogonality it will be diagonal. The diagonal elements will be the self-dot products of the three tangent vectors, i.e., the lengths of the three column matrices thought of as vectors in $\mathbb{R}^3$.

If we divide each column by its lengths, the new columns will be orthonormal. This defines an orthogonal matrix.

The rows of $A^{-1}(\bar{x})$ are the new components of the differentials of the old coordinates

$$dx^i = \frac{\partial x^i}{\partial \bar{x}^j}(\bar{x})\, d\bar{x}^j \leftrightarrow \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = A^{-1}(\bar{x}) \begin{pmatrix} dr \\ d\theta \\ d\varphi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi\, dr + r\cos\theta\cos\varphi\, d\theta - r\sin\theta\sin\varphi\, d\varphi \\ \sin\theta\sin\varphi\, dr + r\cos\theta\sin\varphi\, d\theta + r\sin\theta\cos\varphi\, d\varphi \\ \cos\theta\, dr - r\sin\theta\, d\theta \end{pmatrix}$$

column matrix

The columns of $A(\bar{x})$ are the new components of the old coordinate frame vectors

$$\frac{\partial}{\partial x^i} = \frac{\partial \bar{x}^j}{\partial x^i}(\bar{x}) \frac{\partial}{\partial \bar{x}^j} \leftrightarrow \left(\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z} \right) = \left(\frac{\partial}{\partial r}\ \frac{\partial}{\partial \theta}\ \frac{\partial}{\partial \varphi} \right) A(\bar{x})$$

row matrix

no derivative of A, just do matrix product and put derivatives to right again.

$$= \left(\sin\theta\cos\varphi \frac{\partial}{\partial r} + \frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi} \;\Big|\; \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta} - \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi} \;\Big|\; \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \right)$$

These, together with the parametrization map $x^i = x^i(\bar{x})$ are needed to transform the components of tensor fields.

For a vector field $X = X^i \frac{\partial}{\partial x^i} = \bar{X}^i \frac{\partial}{\partial \bar{x}^i}$: $\bar{X}^i(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^j}(\bar{x})\, X^j(x(\bar{x})) \leftrightarrow \begin{pmatrix} X^r \\ X^\theta \\ X^\varphi \end{pmatrix} = A(\bar{x}) \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix}$

example: $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \leftrightarrow \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix} = \begin{pmatrix} y \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} r\sin\theta\sin\varphi \\ r\sin\theta\cos\varphi \\ 0 \end{pmatrix}$

$$\begin{pmatrix} X^r \\ X^\theta \\ X^\varphi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \frac{1}{r}\cos\theta\cos\varphi & \frac{1}{r}\cos\theta\sin\varphi & -\frac{1}{r}\sin\theta \\ \frac{-\sin\varphi}{r\sin\theta} & \frac{\cos\varphi}{r\sin\theta} & 0 \end{pmatrix} \begin{pmatrix} r\sin\theta\sin\varphi \\ r\sin\theta\cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} r\sin^2\theta\cos\varphi\sin\varphi + r\sin^2\theta\sin\varphi\cos\varphi \\ \sin\theta\cos\theta\sin\varphi\cos\varphi + \sin\theta\cos\theta\sin\varphi\cos\varphi \\ -\sin^2\varphi + \cos^2\varphi \end{pmatrix}$$

$$= \begin{pmatrix} r\sin^2\theta\sin 2\varphi \\ \sin\theta\cos\theta\sin 2\varphi \\ \cos 2\varphi \end{pmatrix} \quad \text{so} \quad X = \sin\theta\sin 2\varphi \left(r\sin\theta \frac{\partial}{\partial r} + \cos\theta \frac{\partial}{\partial\theta} \right) + \cos 2\varphi \frac{\partial}{\partial\varphi}.$$

For a covector field $X^\flat = X_i dx^i = \bar{X}_i d\bar{x}^i$: $\bar{X}_i(\bar{x}) = \frac{\partial x^j}{\partial \bar{x}^i}(\bar{x})\, X_j(x(\bar{x})) \leftrightarrow (X_r\ X_\theta\ X_\varphi) = (X_1 X_2 X_3) A^{-1}(\bar{x})$

example $X^\flat = y\,dx + x\,dy \leftrightarrow (X_1 X_2 X_3) = (y, x\ 0) = \cdots$

$$(X_r X_\theta X_\varphi) = (r\sin\theta\sin\varphi\ \ r\sin\theta\cos\varphi\ \ 0) \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

$$= \left(r\sin^2\theta\sin\varphi\cos\varphi + r\sin^2\theta\sin\varphi\cos\varphi \;\Big|\; r^2\sin\theta\cos\theta\sin\varphi\cos\varphi + r^2\sin\theta\cos\theta\sin\varphi\cos\varphi \;\Big|\; -r^2\sin^2\theta\sin^2\varphi + r^2\sin^2\theta\cos^2\varphi \right)$$

$$= \left(r\sin^2\theta\sin 2\varphi \quad r^2\sin\theta\cos\theta\sin 2\varphi \quad r^2\sin^2\theta\cos 2\varphi \right)$$

so $\underline{X}^\flat = r\sin\theta\left(\sin 2\varphi\left[\sin\theta\frac{\partial}{\partial r} + r\cos\theta\frac{\partial}{\partial\theta}\right] + r\cos 2\varphi\sin\theta\frac{\partial}{\partial\varphi}\right)$

Similarly to transform the metric $g = \delta_{ij}\, dx^i \otimes dx^j = (\sin\theta\cos\varphi\, dr + \dots)\otimes(\sin\theta\cos\varphi\, dr + \dots) + \dots$ etc, but matrix methods are more efficient

$$(\bar g_{ij}(\bar x)) = \left(\frac{\partial x^m}{\partial \bar x^i}(\bar x)\,\delta_{mn}\,\frac{\partial x^n}{\partial \bar x^j}(\bar x)\right) = A^{-1}(\bar x)^T A^{-1}(\bar x)$$

$$= \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ r\cos\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\theta \\ -r\sin\theta\sin\varphi & r\sin\theta\cos\varphi & 0 \end{pmatrix}\begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

columns: unit vector $\quad$ $r\times$(unit vector) $\quad$ $r\sin\theta \times$ (unit vector)

columns are mutually orthogonal as vectors in $\mathbb{R}^3$

$$= \dots = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

so $g = \underbrace{1}_{g_{rr}}\, dr\otimes dr + \underbrace{r^2}_{g_{\theta\theta}}\, d\theta\otimes d\theta + \underbrace{r^2\sin^2\theta}_{g_{\varphi\varphi}}\, d\varphi\otimes d\varphi$

and $\eta = \underbrace{|\det \bar g|^{1/2}}_{(g_{rr}g_{\theta\theta}g_{\varphi\varphi})^{1/2}}\, dr\wedge d\theta\wedge d\varphi = r^2\sin\theta\; dr\wedge d\theta\wedge d\varphi$

and $g^{-1} = \frac{\partial}{\partial r}\otimes\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\otimes\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}\otimes\frac{\partial}{\partial\varphi}$.

We can check $\underline{X}^\flat$ is related to $\underline{X}$ by index lowering: $\quad (r^2\sin^2\theta)(\cos 2\varphi)$ ✓

$X_r = g_{rr}X^r = X^r$ ✓, $X_\theta = g_{\theta\theta}X^\theta = r^2(\sin\theta\cos\theta\sin 2\varphi)$ ✓, $X_\varphi = g_{\varphi\varphi}X^\varphi =$

The matrices $A(x)$ and $A^{-1}(x)$ are necessary for transforming in the opposite direction, from spherical to cartesian components.

The rows of $A(x)$ are the old components of the differentials of the new coordinates

$$\begin{pmatrix} dr \\ d\theta \\ d\varphi \end{pmatrix} = A(x)\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \dots = \begin{pmatrix} \dfrac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{1/2}} \\[2ex] \dfrac{z(x\,dx + y\,dy) - (x^2+y^2)\,dz}{(x^2+y^2+z^2)(x^2+y^2)^{1/2}} \\[2ex] \dfrac{-y\,dx + x\,dy}{x^2+y^2} \end{pmatrix} \equiv \begin{pmatrix} \omega^r \\ \omega^\theta \\ \omega^\varphi \end{pmatrix}$$

The columns of $A^{-1}(x)$ are old components of the new coordinate frame vector fields

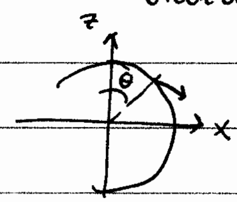
$$\left(\frac{\partial}{\partial r}\ \frac{\partial}{\partial\theta}\ \frac{\partial}{\partial\varphi}\right) = \left(\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\right) A^{-1}(x) = \dots$$

$$= \left(\underbrace{\frac{\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)}{(x^2+y^2+z^2)^{1/2}}} \quad \underbrace{\frac{z\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right] - [x^2+y^2]\frac{\partial}{\partial z}}{(x^2+y^2)^{1/2}}} \quad \underbrace{-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}} \right) \equiv (e_r\ e_\theta\ e_\varphi)$$

$\frac{\vec r}{\|\vec r\|} = \hat r$ physics notation for position vector

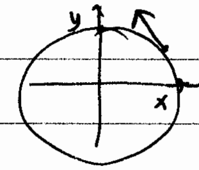
unit vector in radial direction

check out $y=0$ case, $x>0$ becomes



$z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$

opposite direction to xy plane discussion but tangent to circle, length r



already saw this to be tangent to circles, length $\rho = \sqrt{x^2+y^2} = r\sin\theta$

exercise: express $Y = r\frac{\partial}{\partial r} + \frac{\partial}{\partial\varphi}$ in cartesian coords.

If we take the expressions $\{e_r, e_\theta, e_\varphi\}$ for the spherical coordinate frame vector fields in Cartesian coordinates and their dual frame covector fields $\{\omega^r, \omega^\theta, \omega^\varphi\}$, we can use them as an orthogonal frame on $\mathbb{R}^3$, forgetting about their representation in terms of spherical coordinates.

Clearly e_r is undefined at the origin and e_θ along the z-axis, while e_φ is zero on the z-axis, so the frame is only valid off the z-axis.

We can normalize this frame to an orthonormal frame by dividing by lengths " $e_{\hat{\bar x}^i} \equiv (g_{ii})^{-1/2} e_{\bar x^i}$, $\omega^{\hat{\bar x}^i} = (g_{ii})^{1/2}\omega^{\bar x^i}$ "

$$\{e_{\hat r}, e_{\hat\theta}, e_{\hat\varphi}\} = \{e_r, \underbrace{\frac{1}{(x^2+y^2+z^2)^{1/2}}}_{\frac{1}{r}} e_\theta, \underbrace{\frac{1}{(x^2+y^2)^{1/2}}}_{\frac{1}{r\sin\theta}} e_\varphi\} = \left\{\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right\}$$

Components in an orthonormal frame are called "physical components" since given the orientation of the frame at a point we can visualize a vector in terms of its orthonormal components. For example, for our vector field X we have:

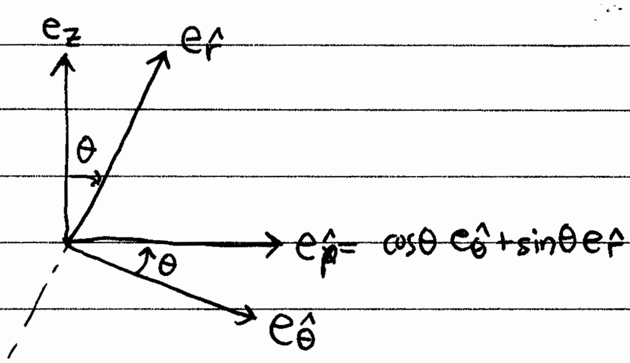
$X^{\hat r} = r\sin^2\theta\sin 2\varphi$ $\quad X^{\hat\theta} = r\sin\theta\cos\theta\sin 2\varphi$ $\quad X^{\hat\varphi} = r\sin\theta\cos 2\varphi$

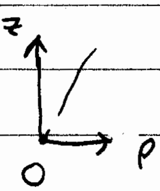
$= X_{\hat r}$ $\quad = X_{\hat\theta}$ $\quad = X_{\hat\varphi}$

and $X = r\sin\theta\left[\sin 2\varphi\,(\sin\theta\, e_{\hat r} + \cos\theta\, e_{\hat\theta}) + \cos 2\varphi\, e_{\hat\varphi}\right]$

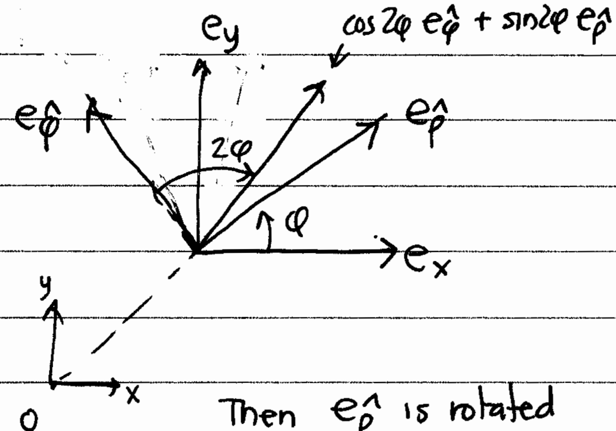
Note that $\mathbb{X}\cdot\mathbb{X} = \delta_{ij}\mathbb{X}^i\mathbb{X}^j = y^2+x^2 = r^2\sin^2\theta$, so $\|\mathbb{X}\| = r\sin\theta$, which is the factor outside the square brackets.

We can also visualize the contents of the square brackets in terms of two successive rotations of $e_{\hat\theta}$ yielding a unit vector:





First $e_{\hat\theta}$ is rotated to $e_{\hat\rho}$ in the z-ρ plane (fixed φ)
(horizontal, ie, no z-component)



Then $e_{\hat\rho}$ is rotated in the horizontal plane an angle 2φ in the opposite direction.

$\mathbb{X}$ is this vector times $r\sin\theta$.

---

exercise for bob: verify that $\left[\frac{\partial}{\partial r}, \frac{\partial}{\partial\theta}\right] = [e_r, e_\theta] = 0$ entirely in cartesian coords:

$$[e_r, e_\theta] = \left[\frac{x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}}{(x^2+y^2+z^2)^{1/2}},\ \frac{z\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)-(x^2+y^2)\frac{\partial}{\partial z}}{(x^2+y^2)^{1/2}}\right]$$

$\boxed{\text{Note } [f\mathbb{X}, h\mathbb{Y}] = fh[\mathbb{X},\mathbb{Y}] + f(\mathbb{X}h)\mathbb{Y} - h(\mathbb{Y}f)\mathbb{X} \text{ why? (exercise) } (f,h = \text{functions})}$

$$= \frac{1}{(x^2+y^2+z^2)^{1/2}(x^2+y^2)^{1/2}}\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z},\ z\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)-(x^2+y^2)\frac{\partial}{\partial z}\right] \quad ①$$

$$+\frac{1}{(x^2+y^2+z^2)^{1/2}}\left[\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)(x^2+y^2)^{-1/2}\right]\left[z\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)-(x^2+y^2)\frac{\partial}{\partial z}\right] ②$$

$$-\frac{1}{(x^2+y^2)^{1/2}}\left\{\left[z\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)-(x^2+y^2)\frac{\partial}{\partial z}\right]\left[(x^2+y^2+z^2)^{-1/2}\right]\right\}\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right] ③$$

change sign

$$① = \frac{x\left(z\frac{\partial}{\partial x}-2x\frac{\partial}{\partial z}\right)+y\left(z\frac{\partial}{\partial y}-2y\frac{\partial}{\partial z}\right)+z\left(\underset{0}{x\frac{\partial}{\partial x}}+\underset{0}{y\frac{\partial}{\partial y}}\right) - zx\frac{\partial}{\partial x}-zy\frac{\partial}{\partial y}+(x^2+y^2)\frac{\partial}{\partial z}}{(x^2+y^2+z^2)^{1/2}(x^2+y^2)^{1/2}} \rightarrow z\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)-(x^2+y^2)\frac{\partial}{\partial z}$$

$$② \quad -\frac{1}{2}\frac{(2x^2+2y^2)}{(x^2+y^2)^{3/2}(x^2+y^2+z^2)^{1/2}}\left[z\left(x\frac{\partial}{\partial x}\ldots\right)-(x^2+y^2)\frac{\partial}{\partial z}\right] = -\frac{1}{(x^2+y^2)^{1/2}(x^2+y^2+z^2)^{1/2}}\left[z\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)-(x^2+y^2)\frac{\partial}{\partial z}\right]$$

cancel

$$③ \quad -\frac{1}{(x^2+y^2)^{1/2}}\left(-\frac{1}{2(x^2+y^2+z^2)^{3/2}}\right)\left[zx(2x)+zy(2y)-(x^2+y^2)(2z)\right]\Big\} = 0$$

$= 0.$ exercise. Try easier calc.: $[e_r, e_\varphi] = 0$.

exercise on page 40 worked

these parentheses, though not necessary, reinforce the instruction to let Y act on a function, multiply by h to get a new function, then let X act on that new function and multiply it by f

$$[fX, hY] = \underbrace{fX(hY)}_{(Xh)Y + hXY} - hY\underbrace{(fX)}_{(Yf)X + fYX}$$

these parentheses confine action of X only to h which gives a function which then multiplies the operator Y.

$$= f(Xh)Y + fhXY - h(Yf)X - hfYX$$

$$= fh\underbrace{(XY - YX)}_{\equiv [X,Y]} + f(Xh)Y - h(Yf)X$$

$$= fh[X,Y] + f(Xh)Y - h(Yf)X.$$

an example of this would be the following, involving sums of such terms

$$[\underbrace{X^i}_{f}\underbrace{\tfrac{\partial}{\partial x^i}}_{X}, \underbrace{Y^j}_{h}\underbrace{\tfrac{\partial}{\partial x^j}}_{Y}] = [\sum_{i=1}^{n} X^i\frac{\partial}{\partial x^i}, \sum_{j=1}^{n} Y^j\frac{\partial}{\partial x^j}] = \sum_{i=1}^{n}\sum_{j=1}^{n}\underbrace{[X^i\frac{\partial}{\partial x^i}, Y^j\frac{\partial}{\partial x^j}]}$$

above — make summations explicit

each of these ~~individual~~ terms is as above

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(X^iY^j\overbrace{[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}]}^{0} + X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j} - Y^j\frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial x^i}\right)$$

$$= X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j} - Y^j\frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial x^i}$$ now suppress summation

$$= (XY^j)\frac{\partial}{\partial x^j} - (YX^i)\frac{\partial}{\partial x^i} = (XY^i - YX^i)\frac{\partial}{\partial x^i}$$

$$= (Y^i{}_{,j}X^j - X^i{}_{,j}Y^j)\frac{\partial}{\partial x^i}$$

Thus the coordinate formula

$$[X,Y]^i = XY^i - YX^i = Y^i{}_{,j}X^j - X^i{}_{,j}Y^j$$

is a consequence of this basic rule for how commutators work when you stick functions in them.

another example is the frame formula for Lie brackets.

$$[X^i e_i, Y^j e_j] = \left[\sum_{i=1}^{n} X^i e_i, \sum_{j=1}^{n} Y^j e_j\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} [\underbrace{X^i}_{f}\underbrace{e_i}_{X}, \underbrace{Y^j}_{h}\underbrace{e_j}_{Y}] \quad \text{in original formula}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(X^i Y^j \underbrace{[e_i,e_j]}_{C^k_{ij}\,e_k} + (\underbrace{X^i e_i Y^j}_{XY^j}) e_j - (\underbrace{Y^j e_j X^i}_{YX^i}) e_i \right)$$

$$= (XY^j)e_j - (YX^i)e_i + C^k_{ij}X^iY^j e_k$$

$$= (XY^i)e_i - (YX^i)e_i + C^i_{jk}X^jY^k e_i$$

$$= [XY^i - YX^i + C^i_{jk}X^jY^k]\, e_i$$

so the frame formula for the Lie bracket in terms of components with respect to the given frame is

$$[X,Y]^i = XY^i - YX^i + C^i{}_{jk}X^jY^k$$
$$= Y^i{}_{,j}X^j - X^i{}_{,j}Y^j + C^i{}_{jk}X^jY^k$$

if we agree that $f_{,j} \equiv e_j f$ is the derivative of a function f along the jth frame vector as with coordinate frames.

new exercise

(page 34)

$$\begin{cases} X = y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y} = \cdots = \rho\sin 2\varphi\dfrac{\partial}{\partial\rho} + \cos 2\varphi\dfrac{\partial}{\partial\varphi} = \underbrace{\rho\sin 2\varphi}_{X^{\hat\rho}}\, e_{\hat\rho} + \underbrace{\rho\cos 2\varphi}_{X^{\hat\varphi}}\, e_{\hat\varphi} \\ Y = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} = \cdots = \rho\dfrac{\partial}{\partial\rho} = \underbrace{\rho}_{Y^{\hat\rho}}\, e_{\hat\rho} \end{cases}$$

why?

Compute the components $[X,Y]^{\hat\rho}$, $[X,Y]^{\hat\varphi}$, $[X,Y]^{\hat z}$ from the above formulas using the fact that $C^{\hat\varphi}{}_{\rho\hat\varphi} = -\frac{1}{\rho} = -C^{\hat\varphi}{}_{\hat\varphi\hat\rho}$ are the only nonzero structure functions.

Compare with the result $[X,Y]$ done entirely in terms of cartesian coordinates.

Repeat for $Y = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \cdots = \frac{\partial}{\partial\varphi} = \rho\, e_{\hat\varphi}$ and the same X.

exercise. Compute all of the Lie brackets of $\{e_{\hat r}, e_{\hat\theta}, e_{\hat\varphi}\}$ working in <u>spherical coordinates</u> (this is easy).

<u>Laplacian calculation</u>:

$$
\begin{aligned}
\nabla^2 = \left(\frac{\partial}{\partial x}\right)^2 &= \left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\varphi} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\\
+\left(\frac{\partial}{\partial y}\right)^2 &+ \left(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta} - \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\left(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta} - \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\\
+\left(\frac{\partial}{\partial z}\right)^2 &+ \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)
\end{aligned}
$$

$$
= \dots = \underbrace{\begin{pmatrix}\sin^2\theta\cos^2\varphi\\ +\sin^2\theta\sin^2\varphi\\ +\cos^2\theta\end{pmatrix}}_{1}\frac{\partial^2}{\partial r^2} + \underbrace{\begin{pmatrix}\frac{\cos^2\theta\cos^2\varphi}{r^2}\\ \frac{\cos^2\theta\sin^2\varphi}{r^2}\\ +\frac{\sin^2\theta}{r^2}\end{pmatrix}}_{\frac{1}{r^2}}\frac{\partial^2}{\partial\theta^2} + \underbrace{\begin{pmatrix}\frac{\sin^2\varphi}{r^2\sin^2\theta}\\ +\frac{\cos^2\varphi}{r^2\sin^2\theta}\end{pmatrix}}_{\frac{1}{r^2\sin^2\theta}}\frac{\partial^2}{\partial\varphi^2}
$$

$$+\ 0\ \frac{\partial^2}{\partial r\partial\theta} + 0\ \frac{\partial^2}{\partial\theta\partial\varphi} + 0\ \frac{\partial^2}{\partial r\partial\varphi}$$ (due to orthogonality of coefficient vectors)

$$
+ \underbrace{\begin{bmatrix}\sin\theta\cos\varphi\ \cos\theta\cos\varphi\left(-\frac{1}{r^2}\right)\\ \sin\theta\sin\varphi\ \cos\theta\sin\varphi\left(-\frac{1}{r^2}\right)\\ \cos\theta(-\sin\theta)\left(-\frac{1}{r^2}\right)\end{bmatrix}}_{0}\frac{\partial}{\partial\theta} + \dots
$$

$$
+ \underbrace{\begin{bmatrix}\frac{\cos\theta\cos\varphi}{r}(\cos\theta\cos\varphi) + \frac{\cos\theta\sin\varphi}{r}(\cos\theta\sin\varphi) - \frac{\sin\theta}{r}(-\sin\theta)\\ -\frac{\sin\varphi}{r\sin\theta}(-\sin\theta\sin\varphi) + \frac{\cos\varphi}{r\sin\theta}(\sin\theta\cos\varphi)\end{bmatrix}}_{2/r}\frac{\partial}{\partial r} + \dots
$$

there is a better way to do this, but it doesn't hurt to check some of the coefficients by this brute force method.

The result can be expressed as

$$
\begin{aligned}
\nabla^2 &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\\
&= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}
\end{aligned}
$$

2 first order terms, rest is just $g^{ij}\frac{\partial^2}{\partial x^i\partial x^j}$. Back to this later.

exercise. Normalize the cylindrical coordinate frame to obtain the orthonormal frame $e_{\hat\rho} = \frac{\partial}{\partial \rho}$, $e_{\hat\phi} = \frac{1}{\rho}\frac{\partial}{\partial\phi}$, $e_{\hat z} = \frac{\partial}{\partial z}$.

Evaluate the nonzero Lie brackets of these vector fields using their ~~cylindrical coordinate expressions (almost trivial).~~

Given any frame $\{e_i\}$, define a set of "structure functions" for the frame in the following way. Each of the commutators $[e_i, e_j]$ is itself a vector field which may be expressed as a linear combination of $\{e_i\}$ so:

$$[e_i, e_j] = \underbrace{C^k{}_{ij}}_{\omega^k([e_i,e_j])} e_k$$

$\omega^k([e_i,e_j])$ = kth component of Lie bracket vector field $[e_i, e_j]$

Clearly $C^k{}_{ij} = 0$ for $i=j$ and $C^k{}_{ij} = -C^k{}_{ji}$ for $i \neq j$, ie this object is antisymmetric in its lower indices.

If we wish we can define a tensor having these components

$$C(e) = C^i{}_{jk}\, e_i \otimes \omega^j \otimes \omega^k = \tfrac{1}{2} C^i{}_{jk}\, e_i \otimes \omega^{jk},$$

but it changes if we change the frame, i.e., does not define the same tensor in every frame like $\delta^i{}_j$ does.

It is enough to list $\{C^k{}_{ij}\}_{i<j}$ to specify all the independent structure functions. Do this for the normalized cylindrical and spherical coordinate frames, i.e., $\{e_{\hat\rho}, e_{\hat\phi}, e_{\hat z}\}$ and $\{e_{\hat r}, e_{\hat\theta}, e_{\hat\phi}\}$.

exercise

On page 41 (part I) we gave the duality map for the basis p-vectors and p-covectors in an orthonormal basis $\{e_i\}$. These are valid pointwise for the orthonormal frames $\{e_x, e_y, e_z\}$, $\{e_{\hat\rho}, e_{\hat\varphi}, e_{\hat z}\}$, $\{e_{\hat r}, e_{\hat\theta}, e_{\hat\varphi}\}$.

Evaluate the following duals:

x, y, z:	$\hat\rho, \hat\varphi, \hat z$
$*1$ (3-form)	$*1$ (3-form)
$*(X_1 dx + X_2 dy + X_3 dz)$	$*(X_{\hat\rho}\,\omega^{\hat\rho} + X_{\hat\varphi}\,\omega^{\hat\varphi} + X_{\hat z}\,\omega^{\hat z})$
$*(X_{23}\, dy\wedge dz + X_{31}\, dz\wedge dx + X_{12}\, dx\wedge dy)$	$*(X_{\hat\varphi\hat z}\,\omega^{\hat\varphi\hat z} + X_{\hat z\hat\rho}\,\omega^{\hat z\hat\rho} + X_{\hat\rho\hat\varphi}\,\omega^{\hat\rho\hat\varphi})$
$*(X_{123}\, dx\wedge dy\wedge dz)$	

$\hat r, \hat\theta, \hat\varphi$:

$*1$ (3-form)

$*(X_{\hat r}\,\omega^{\hat r} + X_{\hat\theta}\,\omega^{\hat\theta} + X_{\hat\varphi}\,\omega^{\hat\varphi})$

$*(X_{\hat\theta\hat\varphi}\,\omega^{\hat\theta\hat\varphi} + X_{\hat\varphi\hat r}\,\omega^{\hat\varphi\hat r} + X_{\hat r\hat\theta}\,\omega^{\hat r\hat\theta})$

$*(X_{\hat r\hat\theta\hat\varphi}\,\omega^{\hat r\hat\theta\hat\varphi})$

---

exercise: If $X = \frac{1}{2}\bar X_{jk}\, d\bar x^j \wedge d\bar x^k = X_{\theta\varphi}\, d\theta\wedge d\varphi + X_{\varphi r}\, d\varphi\wedge dr + X_{r\theta}\, dr\wedge d\theta$

use the formula $(*X)^i = \frac{1}{2}\eta^{ijk} X_{jk}$ to evaluate the vector field

$$*X^\# = (*X^\#)^r \frac{\partial}{\partial r} + (*X^\#)^\theta \frac{\partial}{\partial\theta} + (*X^\#)^\varphi \frac{\partial}{\partial\varphi}.$$

What are its physical components?