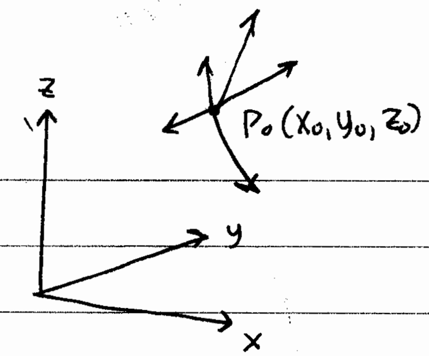
# DIFFERENTIAL GEOMETRY

SPRING 91

bob jantzen

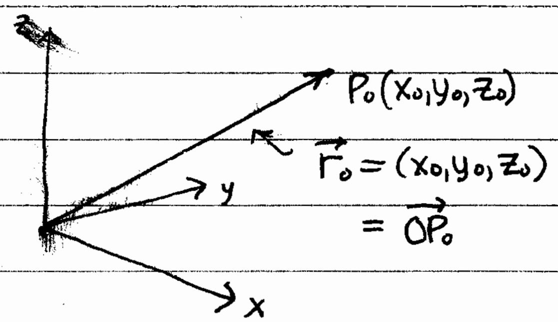
# PART II: CALCULUS

# The Tangent Space in Multivariable Calculus



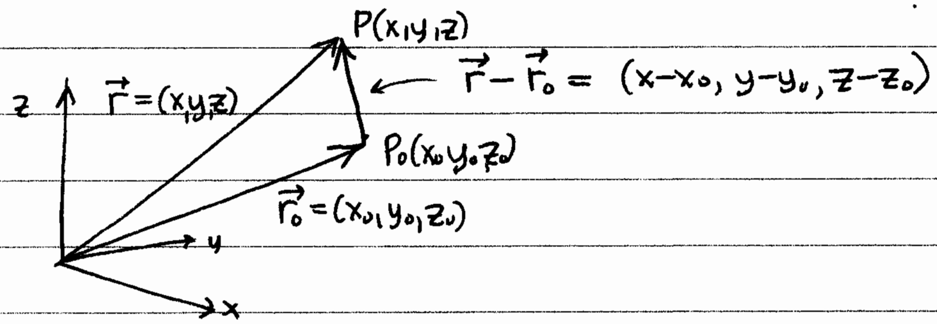

tangent space at $P_0 \in \mathbb{R}^3$

The space $\mathbb{R}^3$ has many different mathematical structures. It can be thought of as a space of points with no additive structure, with the values of the three cartesian coordinates $\{x, y, z\}$ at a point serving to locate that point relative to the standard orthogonal axes on the space. Alternatively one can think of $\mathbb{R}^3$ as a space of vectors, i.e., as a vector space with vector addition and scalar multiplication. In this case the points $P$ of $\mathbb{R}^3$ are reinterpreted as directed line segments or "arrows" with initial point at the origin $O$ and terminal point at the point $P$. The notation $\vec{r} = (x, y, z)$ for the position vector of the point $P(x,y,z)$ emphasizes this vector interpretation of the point $(x, y, z) \in \mathbb{R}^3$. In this case the cartesian coordinates of a point are reinterpreted as the components of the corresponding vector with respect to the standard basis $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ of $\mathbb{R}^3$ as a vector space. The cartesian coordinates $\{x, y, z\}$ are real valued linear functions on $\mathbb{R}^3$ which pick out the associated component of a vector with respect to the standard basis. In other words they are just the basis dual to the standard basis of $\mathbb{R}^3$.




point-vector duality

The closest one gets to the terminology "tangent space" that one gets in multivariable calculus is the tangent plane to the graph of a function of two variables or to the level surface of a function of three variables. The idea of the tangent space to a point $P_0$ is not formally introduced but it is nontheless used and understood. It is just the space of all difference vectors $\vec{r}-\vec{r}_0 = (x-x_0, y-y_0, z-z_0)$ for all points $P(x,y,z)$ of $\mathbb{R}^3$. These difference vectors are pictured as arrows with initial point at $P_0$ and terminal point at $P$, i.e., as the directed line segments $\overrightarrow{P_0P}$. They are called tangent vectors at $P_0$.




A basis for this tangent space is the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ thought of as directed line segments with their initial points at $P_0$. To recall this interpretation, the symbols $\{e_1|_{P_0}, e_2|_{P_0}, e_3|_{P_0}\}$ can be used. Each such tangent space is a real vector space isomorphic to $\mathbb{R}^3$ itself and usually no distinction is made between them in multivariable calculus. However, tangent vectors are discussed in relation to curves and surfaces in $\mathbb{R}^3$. The tangent vector to a parametrized curve is always thought of as attached to the point on the curve at which it is defined, while a normal vector determining the orientation of the tangent plane to a surface at a point is always thought of as attached to that point. Each of these are examples of tangent vectors.

The cartesian coordinate differentials $\{dx, dy, dz\}$ at the point $P_0$ are sometimes introduced as new cartesian coordinates translated from the origin to $P_0$, but notationally the point $P_0$ is suppressed.

$$(dx, dy, dz)|_{P_0} = (x-x_0, y-y_0, z-z_0) = \vec{r} - \vec{r}_0 .$$

The value of these new coordinates at a point $P_1(x_1, y_1, z_1)$

$$dx|_{P_0}(P_1) = x_1 - x_0, \dots,$$

are just the components of the difference vector with respect to the basis $\{e_1|_{P_0}, e_2|_{P_0}, e_3|_{P_0}\}$ of the tangent space at $P_0$. In other words the cartesian coordinate differentials $\{dx|_{P_0}, dy|_{P_0}, dz|_{P_0}\}$ form the dual basis to this basis of the tangent space at $P_0$. However, to interpret the differentials as the dual basis, we must agree to evaluate them on the difference vectors rather than their terminal points relative to $P_0$. If

$$\vec{X} = X_1 e_1|_{P_0} + X_2 e_2|_{P_0} + X_3 e_3|_{P_0}$$

is a tangent vector at $P_0$, then

$$dx|_{P_0}(\vec{X}) = X_1, \dots .$$

The differential of an arbitrary (differentiable) function $f$ at $P_0$ is defined in terms of the partial derivatives of $f$ at $P_0$

$$df|_{P_0} = f_x(x_0,y_0,z_0)\, dx|_{P_0} + f_y(x_0,y_0,z_0)\, dy|_{P_0} + f_z(x_0,y_0,z_0)\, dz|_{P_0} .$$

It is a real valued linear function on the tangent space at $P_0$, i.e., a covector or 1-form. When evaluated on a tangent vector $\vec{X}$ as above, it produces the result

$$\begin{aligned} df|_{P_0}(\vec{X}) &= f_x(x_0,y_0,z_0) X_1 + f_y(x_0,y_0,z_0) X_2 + f_z(x_0,y_0,z_0) X_3 \\ &= \vec{X} \cdot \vec{\nabla} f\,(x_0,y_0,z_0), \end{aligned}$$

where $\vec{\nabla} f(x_0,y_0,z_0) = f_x(x_0,y_0,z_0)\, e_1|_{P_0} + f_y(x_0,y_0,z_0)\, e_2|_{P_0} + f_z(x_0,y_0,z_0)|_{P_0}$ is the gradient of $f$ at $P_0$. The differential $df|_{P_0}$ represents to linear approximation to the function $f - f(x_0,y_0,z_0)$ at $P_0$, as a function of the difference vectors relative to $P_0$.

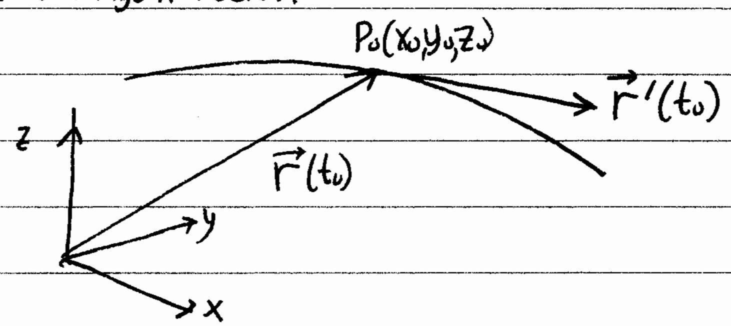
Two important uses of the tangent space in multivariable calculus occur in the discussion of tangent vectors to parametrized curves and in directional derivatives of functions, and they come together in the chain rule. Given a parametrized curve

$$\vec{r} = \vec{r}(t) = (x(t), y(t), z(t))$$

which passes through a point $P_0(x_0,y_0,z_0)$ at $t = t_0$, one produces a tangent vector at $P_0$ by differentiating to produce the tangent vector to the curve at $P_0$

$$\vec{r}\,'(t_0) = (x'(t_0), y'(t_0), z'(t_0))$$
$$= x'(t_0)\, e_1|_{P_0} + y'(t_0)\, e_2|_{P_0} + z'(t_0)\, e_3|_{P_0},$$

where the last equality reminds us notationally of the connection of the tangent vector to the point $P_0$. This distinction is never made but it is an integral part of the intuitive picture one has of the tangent vector.




One can think of the tangent space at $P_0$ as the space of tangent vectors at $P_0$ to all possible parametrized curves passing through $P_0$. This is in fact a useful idea which generalizes to more complicated settings.

The chain rule evaluates the derivative of a function $f$ on $\mathbb{R}^3$ along the parametrized curve as a function of the parameter.

$$\frac{d}{dt} f(\vec{r}(t))\Big|_{t=t_0} = f_x(x_0,y_0,z_0)\,x'(t_0) + f_y(x_0,y_0,z_0)\,y'(t_0) + f_z(x_0,y_0,z_0)\,z'(t_0)$$
$$= \vec{r}\,'(t_0)\cdot\vec{\nabla} f(x_0,y_0,z_0).$$

However, this is just the value of the differential of $f$ at $P_0$ on the tangent vector $\vec{r}\,'(t_0)$

$$\frac{d}{dt} f(\vec{r}(t))\Big|_{t=t_0} = df|_{P_0}(\vec{r}\,'(t_0)).$$

One can also differentiate a function along a given direction at $P_0$ without having an explicit parametrized curve. For this one introduces the directional derivative which generalizes the partial derivatives to an arbitrary direction specified by a unit vector $\hat{u} = (u_1, u_2, u_3)$ , $\hat{u}\cdot\hat{u} = 1$. Taking the arclength parametrized straight line in the direction $\hat{u}$ at $P_0$

$$x = x_0 + su_1\,,\quad y = y_0 + su_2\,,\quad z = z_0 + su_3\,,$$

one defines the directional derivative of a function $f$ at $P_0$ in the direction $\hat{u}$ by an application of the chain rule

$$\frac{d}{dt} f(\vec{r}(s))\Big|_{s=0} = \hat{u}\cdot\vec{\nabla} f(x_0,y_0,z_0) \equiv D_{\hat{u}} f(x_0,y_0,z_0)$$
$$= \left(u_1 \frac{\partial}{\partial x}\Big|_{P_0} + u_2 \frac{\partial}{\partial y}\Big|_{P_0} + u_3 \frac{\partial}{\partial z}\Big|_{P_0}\right) f$$
$$= df|_{P_0}(\hat{u}).$$

Along the coordinate directions this reduces to the ordinary partial derivatives, using the $\hat{\imath}, \hat{\jmath}, \hat{k}$ notation

$$D_{\hat{\imath}} f = f_x\,,\quad D_{\hat{\jmath}} f = f_y\,,\quad D_{\hat{k}} f = f_z\,.$$

Note that the directional derivative $D_{\hat{u}} f(x_0, y_0, z_0)$ may be interpreted either as the result of allowing the first order differential operator

$$u_1 \frac{\partial}{\partial x}\Big|_{P_0} + u_2 \frac{\partial}{\partial y}\Big|_{P_0} + u_3 \frac{\partial}{\partial z}\Big|_{P_0} = \hat{u} \cdot \vec{\nabla}\Big|_{P_0}$$

to act on the function $f$ or by evaluating the differential of the function $f$ at $P_0$ on the unit tangent vector $\hat{u}$. However, the condition that $\hat{u}$ be a unit vector requires the use of the Euclidean metric, so if we want to generalize the directional derivative to a setting where no metric is required, this restriction must be dropped.

So introduce the derivative of $f$ at $P_0$ along the tangent vector $\vec{X}$ by

$$\begin{aligned} \nabla_X f(x_0, y_0, z_0) &= X_1 f_x(x_0, y_0, z_0) + X_2 f_y(x_0, y_0, z_0) + X_3 f_z(x_0, y_0, z_0) \\ &= \left(X_1 \frac{\partial}{\partial x}\Big|_{P_0} + X_2 \frac{\partial}{\partial y}\Big|_{P_0} + X_3 \frac{\partial}{\partial z}\Big|_{P_0}\right) f \\ &= df\big|_{P_0}(\vec{X}). \end{aligned}$$

In this way the chain rule links the derivative of a function along the parametrized curve to the derivative along its tangent vector.

$$\begin{aligned} \frac{d}{dt} f(\vec{r}(t))\Big|_{t=t_0} &= \nabla_{\vec{r}\,'(t_0)} f(x_0, y_0, z_0). \\ &= \left(x'(t_0) \frac{\partial}{\partial x}\Big|_{P_0} + y'(t_0) \frac{\partial}{\partial y}\Big|_{P_0} + z'(t_0) \frac{\partial}{\partial z}\Big|_{P_0}\right) f \end{aligned}$$

This in turn may be interpreted as the result of a uniquely associated first order differential operator at $P_0$ acting on the function.

Still we haven't gone far enough with the tangent space idea. The notion of a tangent vector as a difference vector requires an underlying vector space. If we want to generalize this idea to a setting without vector space structure, the difference vector interpretation must be abandoned. The answer lies with tangent vectors to parametrized curves and derivatives of functions along them.

One can always differentiate functions along parametrized curves and the chain rule shows that this is equivalent to the derivative of those functions along the corresponding tangent vectors, regardless of how we try to interpret those tangent vectors. In fact with every tangent vector at a point $P_0$ there is a uniquely associated first order linear differential operator which accomplishes the derivatives of functions along that tangent vector. It is just the linear combination of the partial derivatives at $P_0$ whose coefficients are the corresponding components of the tangent vector. Why not simply define the differential operator to be the tangent vector? This makes the partial derivative operators a basis of the tangent space at each point. The components of a tangent vector with respect to this basis are exactly what we've been calling the components all along. So this definition can be looked at as a bookkeeping trick. It turns out to be extremely useful.

So our previous expansion of a tangent vector at $P_0$

$$\vec{X} = X_1 e_1|_{P_0} + X_2 e_2|_{P_0} + X_3 e_3|_{P_0}$$

can still be used if we re-interpret the symbols $e_i|_{P_0}$ to mean the corresponding partial derivatives at $P_0$. The familiar notation $\{x, y, z\}$ instead of $\{x_1, x_2, x_3\}$ must also go so indexed equations using the summation convention can make formula writing simple. Finally index positioning must be respected. We can remind ourselves of the differential operator interpretation by dropping the arrow notation & just let $X$ denote the above tangent vector.

Also, since we have changed our definition of tangent vectors, and differentials were defined to be dual to tangent vectors, i.e., real valued linear functions on tangent vectors, their definition must be changed if we insist on maintaining duality.
The differential of a function $f$ at $P_0$ will be defined by an equation already used above

$$df|_{P_0}(X) = Xf \equiv \left(X_1 \frac{\partial}{\partial x^1}\Big|_{P_0} + X_2 \frac{\partial}{\partial x^2}\Big|_{P_0} + X_3 \frac{\partial}{\partial x^3}\Big|_{P_0}\right) f .$$

The right hand side is a real valued linear function of the tangent vector $X$ and so defines a covector or 1-form at $P_0$. The coordinate differential $dx$ is no longer a new cartesian coordinate $x - x_0$, but the real valued linear map obtained by letting a tangent vector $X$ act on the function $x$

$$dx(X) = Xx = X_1, \ldots$$

and so on. Thus the coordinate differentials merely pick out the components of tangent vectors in the coordinate derivative basis. This identification of the tangent space and its dual enables us to extend the concept to spaces which are locally like $\mathbb{R}^n$, called manifolds.

Some Problems on 3-D Calculus. $\left[ \vec{X} = X^i e_i \leftrightarrow X = X^i \frac{\partial}{\partial x^i}\Big|_P \text{ for tangent space at point } P \right]$

① Suppose $x = t$, $y = t^2+1$, $z = 2-t$, or $\vec{r}(t) = (t, t^2+1, 2-t)$.

Evaluate $\vec{r}\,'(t)$. What is the tangent vector at $t = 1$?

Express it as a first order linear differential operator, call it $r'(1)$ with no overarrow. (and $r'(t)$ in general).

Consider the function $f(x,y,z) = x^2+y^2-3z^2$.

What is $df(x,y,z)$? $df(1,2,1)$?

What is $df(1,2,1)\,(r'(1))$?

What is $r'(1)\,f$? [action of derivative operator on function $f$].

Find expressions for $x, y, z$ as functions of $t$ for some other parametrized curve which has the same tangent at $t=0$ as the previous curve (such that at $t=0$, $x=0$, $y=1$, $z=2$ as with the previous curve). (This is easy!)

If $X = 2 \frac{\partial}{\partial x}\Big|_{(1,2,1)} - \frac{\partial}{\partial y}\Big|_{(1,2,1)} + 4 \frac{\partial}{\partial z}\Big|_{(1,2,1)}$, what is $df(1,2,1)\,(X)$?

If $\theta = dx\big|_{(1,2,1)} + 2\,dy\big|_{(1,2,1)} - dz\big|_{(1,2,1)}$

what is $\theta(r'(1))$? $\theta(X)$?

## More Motivation for the Re-interpretation of the Tangent Space

The standard Cartesian coordinates on $\mathbb{R}^n$ are those functions which pick out the individual components of vectors — these are just the dual basis ~~co~~covectors:

$$x^i \equiv \omega^i$$

$$x^i((v^1, \ldots, v^n)) = v^i.$$

However, since we are going to emphasize different mathematical structure on $\mathbb{R}^n$, we will use a different notation. If $P = (v^1, \ldots, v^n)$ is a point in $\mathbb{R}^n$, de-emphasizing its vector nature using a capital letter as we conventionally do for points, then

$$x^i\big|_P \equiv x^i(P) = v^i$$

will indicate the value of $x^i$ at $P$.

We can now re-interpret a change of basis on $\mathbb{R}^n$ as a change of Cartesian coordinates:

$$\left.\begin{aligned} \omega^{i'} &= A^i{}_j \omega^j \\ e_{i'} &= A^{-1\,j}{}_i \, e_j \end{aligned}\right\} \quad \text{where the column's of } \underline{A^{-1}} \text{ are the old components of the new basis vectors.}$$

~~becomes~~ $\quad x^{i'} = A^i{}_j x^j \quad$ or $\quad x^i = A^{-1\,i}{}_j x^{j'}$.

By definition of partial differentiation for a given coordinate system

$$\frac{\partial x^j}{\partial x^i} = \delta^j{}_i \,, \qquad \frac{\partial x^{j'}}{\partial x^{i'}} = \delta^j{}_i$$

so
$$\frac{\partial x^{i'}}{\partial x^j} = \frac{\partial}{\partial x^j}\left(A^i{}_k x^k\right) = A^i{}_k \frac{\partial x^k}{\partial x^j} = A^i{}_k \delta^k{}_j = A^i{}_j$$

and
$$\frac{\partial x^i}{\partial x^{j'}} = \ldots \text{(same calculation)} \ldots = A^{-1\,i}{}_j$$

Suppose $(u^i)$ are $n$ (real-valued) functions on $\mathbb{R}^n$ and we introduce the partial derivative operator $\quad u = u^i \dfrac{\partial}{\partial x^i} \quad$ on real-valued differentiable functions on $\mathbb{R}^n$, acting in the obvious way to produce new functions

$$f \mapsto uf = \left(u^i \frac{\partial}{\partial x^i}\right) f = u^i \frac{\partial f}{\partial x^i} .$$

By the chain rule

$$uf = u^i \frac{\partial f}{\partial x^i} = \underbrace{u^i \frac{\partial x^{j'}}{\partial x^i}}_{u^{j'}} \frac{\partial f}{\partial x^{j'}} = u^{j'} \frac{\partial f}{\partial x^{j'}} = \left(u^{j'} \frac{\partial}{\partial x^{j'}}\right) f$$

$$u^{j'} \text{ by definition: } \quad u^{j'} = \frac{\partial x^{j'}}{\partial x^i} u^i = A^j{}_i u^i$$

one obtains the "transformation law" for the coefficient functions in the linear differential operator, which is said to be the transformation law for a contravariant vector (field). The corresponding differential operator is by definition independent of coordinates.

Conversely if one has a set of components which transform in this way the combination

$$\underbrace{u^{i'} \frac{\partial}{\partial x^{i'}}}_{} = A^i{}_k u^k A^{-1j}{}_i \frac{\partial}{\partial x^j} = (A^{-1j}{}_i A^i{}_k) u^k \frac{\partial}{\partial x^j}$$

$$\frac{\partial x^j}{\partial x^{i'}} \frac{\partial}{\partial x^j} = A^{-1j}{}_i \frac{\partial}{\partial x^j} \qquad = \delta^j{}_k u^k \frac{\partial}{\partial x^j} = u^j \frac{\partial}{\partial x^j}$$

is invariant. Using this same chain rule calculation one has

$$\frac{\partial f}{\partial x^{i'}} = \frac{\partial x^j}{\partial x^{i'}} \frac{\partial f}{\partial x^j} = A^{-1j}{}_i \frac{\partial f}{\partial x^j}$$

which is said to define the transformation law of a covariant vector (field). In fact these are just the coefficients of the differential $df$ expressed in terms of the coordinate differentials

$$df = \frac{\partial f}{\partial x^i} dx^i = \frac{\partial f}{\partial x^{i'}} dx^{i'}$$

which of course doesn't depend on which coordinates are used to express it.

Thus expressing the coordinate independent operations $uf$ and $df$ in particular Cartesian coordinate systems leads to the usual

transformation of the components of vectors and covectors point by point on $\mathbb{R}^n$.
By calling $u = u^i \frac{\partial}{\partial x^i}$ the vector field instead of the collection of components $(u^i)$, it enjoys the same invariant status as the differential of a function or even of an ordinary vector

$$\vec{V} = (V^1, \ldots, V^n) = V^i e_i \in \mathbb{R}^n$$

which is a quantity $\vec{V}$ independent of the choice of basis, whose components merely change with a change in basis.

On the other hand, although everybody knows the rules for evaluating differentials, the meaning of the differentials of the coordinates themselves is often lost on students or poorly presented in textbooks. We all remember that we plug in increments in the coordinates for them when we use the differential approximation, but the mathematical interpretation of the differentials themselves we quickly forget. It should therefore cause no great objection if we redefine what they mean mathematically, although the rules for taking differentials will remain the same.

Linearity of differentiation means

$$(au+bv)\,f = \left(a u^i \frac{\partial}{\partial x^i} + b v^i \frac{\partial}{\partial x^i}\right) f = a u^i \frac{\partial f}{\partial x^i} + b v^i \frac{\partial f}{\partial x^i} = a u f + b v f.$$

so $\quad (au+bv)|_P f = a u|_P f + b v|_P f \qquad$ (where $a, b$ are constants).

This means associating the value of the derivative of a function by the derivative operator at a certain point of $\mathbb{R}^n$ with the derivative operator

$$u|_P \longmapsto u|_P f \in \mathbb{R}$$

is a real-valued linear function of the derivative operator. The space of all such operators at a given point $P$ is clearly an $n$-dimensional vector space isomorphic to $\mathbb{R}^n$:

$$u^i e_i \in \mathbb{R}^n \leftrightarrow u^i \frac{\partial}{\partial x^i}\Big|_P \in \underbrace{??}_{\text{need a name for it.}}$$

(space of these operators at $P$.)

so this defines a covector on that vector space, $\{\frac{\partial}{\partial x^i}|_P\}$ is a basis of the space of these operators at $P$, and with respect to this basis, the components of the covector are $\partial f/\partial x^i$ since

$$u|_P f = \left(\frac{\partial f}{\partial x^i}\right) u^i .$$

By making the simple definition $\boxed{df|_P(u) = u|_P f}$ for this covector, thereby defining the differential of the function $f$ at $P$, we get a meaning for the differentials of the coordinates themselves

$$dx^i|_P(u) = u^j \frac{\partial}{\partial x^j} x^i = u^j \delta^i{}_j = u^i$$

as the covectors which pick out the components of these linear operators with respect to the basis $\{\partial/\partial x^i|_P\}$, i.e. the dual basis

$$dx^i|_P\left(\frac{\partial}{\partial x^j}\Big|_P\right) = \frac{\partial}{\partial x^j}\Big|_P x^i = \frac{\partial x^i}{\partial x^j} = \delta^i{}_j .$$

Let's use the notation $T\mathbb{R}^n_P$ for the tangent space to $\mathbb{R}^n$ at the point $P$ (T for tangent!) identified with the space of linear differential operators there. Then the coordinate differentials by this new definition of differential form a basis for the dual space $(T\mathbb{R}^n_P)^*$, called the cotangent space at $P$.

Thus at each point $P$ of $\mathbb{R}^n$, we have the tangent space $V = T\mathbb{R}^n_P$ with basis $\{\partial/\partial x^i|_P\}$ and its dual space the cotangent space $V^* = (T\mathbb{R}^n_P)^*$ with dual basis $\{dx^i|_P\}$, and we are free to consider all the spaces of $\binom{p}{q}$-tensors over each such $V$ and independent changes of basis at different points.

Objects defined at each point of a space are called "fields".

Picking out (smoothly) a tangent vector at each point $P$ leads

to the already familiar concept of a vector field (at least for $n=2$ and $n=3$).
The differential of a (smooth = differentiable) function leads to a covector field or "1-form field" ( p-covectors are often called p-forms) on $\mathbb{R}^n$. These are special 1-forms since the components come from the derivatives of a function. Given $n$ functions $\theta_i$ on $\mathbb{R}^n$, $\theta = \theta_i \, dx^i$ defines a general 1-form field (or just 1-form).

Similarly the Euclidean metric tensor field

$$G = \delta_{ij} \, dx^i \otimes dx^j \,, \qquad \delta_{ij} = G\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

tells us how to take the lengths of vector fields by defining the basis $\{\partial/\partial x^i|_e\}$ to be orthonormal. This just reproduces the usual inner product

$$G(u,v) = \delta_{ij} u^i v^j \qquad \text{for two vector fields } u,v$$

when expressed in terms of components.

A p-covector field or p-form field or just p-form is of the form $S = \frac{1}{p!} S_{i_1 \dots i_p} \, dx^{i_1} \wedge \dots \wedge dx^{i_p} \equiv \frac{1}{p!} S_{i_1 \dots i_p} \, dx^{i_1 \dots i_p}$
while a p-vector field is of the form $T = \frac{1}{p!} T^{i_1 \dots i_p} \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_p}}$,
where the components are now functions on $\mathbb{R}^n$.

FRAMES AND DUAL FRAMES

A smooth choice of basis for the tangent spaces to $\mathbb{R}^n$ is called a frame, and consists of $n$ vector fields whose values are $n$ linearly independent ~~vector fields~~ tangent vectors at each point of $\mathbb{R}^n$. The corresponding choice of dual basis is called the dual frame. $\{\partial/\partial x^i\}$ is such a frame, usually called a coordinate frame since the individual frame vector fields are just partial derivatives with respect to the coordinates, and $\{dx^i\}$ is its dual frame.

All the linear algebra we developed for a single vector space we can apply to each tangent space to $\mathbb{R}^n$ independently, although we must assume that what we do at different tangent spaces is a continuous or even differentiable function of position.

For example, we can change the frame, i.e., perform a change of basis on each tangent space in a continuous or differentiable fashion.

$$E_i = A^{-1\,j}{}_i \frac{\partial}{\partial x^j}, \qquad W^i = A^i{}_j \, dx^j$$

~~where now $A$ is a matrix-valued function on $\mathbb{R}^n$, with nonzero~~ determinant of course. The components of tensor fields will change according to the same formulas as before except that now both the components of the tensors and the matrix of the transformation are functions on $\mathbb{R}^n$. The special case of constant $A$ describes the change to a new frame which is the coordinate frame associated with the new Cartesian coordinates $x^{i'} = A^i{}_j x^j$ so that $E_i = \partial/\partial x^{i'}$. A more general special case corresponds to the change to a frame associated with a non-Cartesian coordinate system.

Suppose $\{x^{i'}\}$ are $n$ functions on $\mathbb{R}^n$ such that the matrix $A^{i'}{}_j = \frac{\partial x^{i'}}{\partial x^j}$ of partial derivatives has nonzero determinant at each point of $\mathbb{R}^n$.
Then the chain rule says that $E_{i'} = \partial/\partial x^{i'}$ are partial derivatives with respect to the new coordinates.

Thus we have Cartesian coordinate frames, non-Cartesian coordinate frames, and "non-coordinate" frames, namely frames for which no system of coordinates can be found so that the frame vector fields can be represented as coordinate derivatives.
There is a simple way to tell whether a frame is noncoordinate or not.
We all know that partial derivatives commute, i.e., as long as a function $f$ is well behaved

$$\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f = \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i} f$$

or $$\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j}\frac{\partial}{\partial y^i}\right) f = 0 \quad \text{for all such } f$$

or $$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] \equiv \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i} = 0$$

when acting on such well behaved functions. For any operators $B$ and $C$, their commutator is defined by

$$[B,C] = BC - CB$$

and when it vanishes, their order doesn't matter.

Define the commutator of any two vector fields by the same formula $\quad [u,v] = uv - vu$.
This is ~~an~~ a differential operator on functions. What is it?
We can express it in components, when acting on a function.
[Note that $[u,u] = uu - uu = 0$
and $[v,u] = vu - uv = -(uv - vu) = -[u,v]$ ]

$$[u,v]f = (uv-vu)f = uvf - vuf$$
$$= u^i\frac{\partial}{\partial x^i}\left(v^j\frac{\partial f}{\partial x^j}\right) - v^i\frac{\partial}{\partial x^i}\left(u^j\frac{\partial f}{\partial x^j}\right)$$

relable dummy indices

$$= u^i\frac{\partial v^j}{\partial x^i}\frac{\partial f}{\partial x^j} + u^i v^j\frac{\partial^2 f}{\partial x^i\partial x^j} - v^i\frac{\partial u^j}{\partial x^i}\frac{\partial f}{\partial x^j} - v^i u^j\frac{\partial^2 f}{\partial x^i \partial x^j}$$
$$= \underbrace{\left(u^i\frac{\partial v^j}{\partial x^i} - v^i\frac{\partial u^j}{\partial x^i}\right)\frac{\partial f}{\partial x^j}}_{} + u^i v^j\underbrace{\left(\frac{\partial^2 f}{\partial x^i\partial x^j} - \frac{\partial^2 f}{\partial x^j\partial x^i}\right)}_{0 \text{ for well behaved } f}$$

this defines a new vector field $[u,v]$ acting on $f$ whose components are

$$\boxed{\begin{aligned}[u,v]^j &= u^i\frac{\partial v^j}{\partial x^i} - v^i\frac{\partial u^j}{\partial x^i}\\ &= uv^j - vu^j\end{aligned}}$$

called the Lie Bracket of $u$ and $v$.

Thus if $E_{i'} = A^j{}_i\frac{\partial}{\partial x^j}$ can be represented as coordinate derivatives for some coordinate system : $E_{i'} = \partial/\partial x^{i'}$

then $[E_{i'}, E_{j'}] = [\frac{\partial}{\partial x^{i'}}, \frac{\partial}{\partial x^{j'}}] = 0$ since partial derivatives commute.

A necessary condition for this is therefore the vanishing of the Lie brackets of all pairs of distinct frame vector fields.

exercise. Compute the nonzero Lie brackets among the vector fields

$$u = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},\quad v = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\quad w = (x^2+y^2)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) + \frac{\partial}{\partial z}$$

namely, $[u,v]$, $[u,w]$ and $[v,w]$. on $\mathbb{R}^3$.

On $\mathbb{R}^2$ do the same for $u = (x^2+y^2)^{-1/2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$ and $v = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.
If we introduce a new frame by $E_1 = u$, $E_2 = v$ then $\underline{A}^{-1} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & -y \\ \frac{y}{\sqrt{x^2+y^2}} & x\end{pmatrix}$
What is $\det \underline{A}$? Can this vanish? What does this mean?
What does your result for $[u,v]$ tell you?

<u>More on Lie Brackets</u>: A Product rule: $\frac{\partial}{\partial x^i} f \frac{\partial}{\partial x^j} = \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j} + f\frac{\partial^2}{\partial x^i \partial x^j}$

$$\left[ xy\frac{\partial}{\partial x}, \sin(x+y)\frac{\partial}{\partial y}\right] = xy\frac{\partial}{\partial x}\sin(x+y)\frac{\partial}{\partial y} - \sin(x+y)\frac{\partial}{\partial y} xy \frac{\partial}{\partial x}$$

$$= xy\cos(x+y)\frac{\partial}{\partial y} + xy\sin(x+y)\frac{\partial^2}{\partial x\partial y} - \sin(x+y)\, x\frac{\partial}{\partial x} - \sin(x+y)\, xy\frac{\partial^2}{\partial y \partial x} = xy\cos(x+y)\frac{\partial}{\partial y} - x\sin(x+y)\frac{\partial}{\partial x}$$

$$\to 0$$

So the commutator of the vector fields on $\mathbb{R}^2$ with Cartesian coordinate components $(xy, 0)$ and $(0, \sin(x+y))$ has components $(-x\sin(x+y), xy\cos(x+y))$.

<u>Exercise</u>: If $A = (A^i{}_j)$ and $B = (B^i{}_j)$ are constant matrices and $\vec{b} = (b^i)$ and $\vec{c} = (c^i)$ are constant vectors in $\mathbb{R}^n$, define the ~~three~~ four vector fields

$$X = A^i{}_j x^j \frac{\partial}{\partial x^i}, \quad Y = B^m{}_n x^n \frac{\partial}{\partial x^m}, \quad Z = b^\ell \frac{\partial}{\partial x^\ell}, \quad W = c^k \frac{\partial}{\partial x^k}$$

Evaluate $[X, Y]$ and $[X, Z]$. (Using $\frac{\partial}{\partial x^i} x^j = \delta^j{}_i$)
and $[Z, W]$.

These results allow us to ~~consider~~ evaluate the Lie brackets of any vector fields whose components are linear functions of the coordinates.

<u>Exercise</u>: $2^{nd}$ order linear differential operators are also useful.

Define $\nabla^2 = \delta^{ij}\frac{\partial^2}{\partial x^i \partial x^j} = \sum_{i=1}^{n} \frac{\partial^2}{\partial (x^i)^2}$. For $n=2$: $\nabla^2 = \left[\frac{\partial^2}{\partial (x^1)^2} + \frac{\partial^2}{\partial (x^2)^2}\right]$

Evaluate $\left[\nabla^2, x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}\right]$ for $n=2$.

ie. $\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right]$

$\left[\text{Hint: } \frac{\partial^2}{\partial x^2}\left(x\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x}\Bigg(\underbrace{\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}\right)}_{\left(1\frac{\partial}{\partial y} + x\frac{\partial^2}{\partial x\partial y}\right)}\Bigg) \text{ etc}\right]$

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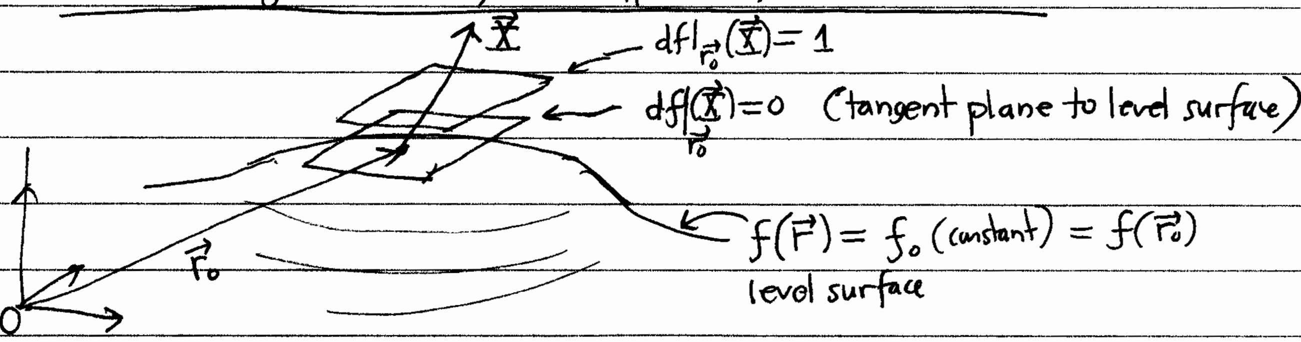
## More on tangent covectors, the differential, and vector fields




$\mathbb{R}^3$

The differential $df|_{\vec{r}_0}$ of $f$ at $\vec{r}_0$ can be represented by the pair of planes in $\mathbb{R}^3$ shown in the diagram. The value $df|_{\vec{r}_0}(\vec{X})$ using the calculus meanings of differential and tangent vector (as a difference vector) is the number of integer spaced planes of this family containing these two planes which are pierced by $\vec{X}$. Tangent vectors in the tangent plane give zero. In the old fashioned language

$$df|_{\vec{r}_0}(\vec{X}) = \vec{X} \cdot \vec{\nabla} f|_{\vec{r}_0}$$

so if this is zero, $\vec{X}$ is perpendicular to the gradient of $f$ which itself is orthogonal to the tangent plane to the level surface, making $\vec{X}$ belong to this tangent plane. [For small enough $\epsilon$, the plane $df|_{\vec{r}_0}(X) = 0$ corresponds to tangent plane to the level surface $f(\vec{r}_0 + \vec{X}) = f_0 + \epsilon$.]

The new meaning of the differential and tangent vector

$$df|_{\vec{r}_0}(X) = Xf \qquad X \in T\mathbb{R}^3_{\vec{r}_0}$$

tells us if $Xf = 0$, then $X$ belongs to the tangent plane to the level surface of $f$ at $\vec{r}_0$. Suppose we have instead a vector field $X$ such that $Xf = 0$. This means $X|_{\vec{r}_0}$ is tangent to the level surface of $f$ through $\vec{r}_0$ at every point $\vec{r}_0$.

exercise. Show that $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ is tangent to the level surfaces of the functions $f(x,y,z) = \frac{y}{x}$ and $g(x,y,z) = \frac{z}{\sqrt{x^2+y^2}}$. Compute $df$ and $dg$ and $df(X)$, $dg(X)$ as well as $Xf$ and $Xg$.