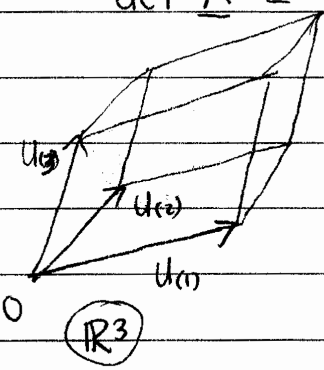
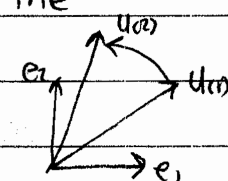
MEASURE MOTIVATION

The determinant of a matrix $\underline{A} = (\underline{u}_{(1)} \cdots \underline{u}_{(n)})$ whose columns are the column matrices corresponding to vectors $u_{(i)} \in \mathbb{R}^n$ may be thought of as an antisymmetric multilinear real-valued function of n vector arguments, i.e., a $\binom{0}{n}$-tensor on $\mathbb{R}^n$

$$\det \underline{A} = \det(\underline{u}_{(1)} \cdots \underline{u}_{(n)}) \equiv \det(u_{(1)}, \dots, u_{(n)}).$$



Recall that $\mathrm{Vol}(u_{(1)}, \dots, u_{(n)}) = |\det(u_{(1)}, \dots, u_{(n)})|$ has the interpretation as the volume of the n-parallellopiped formed with these vectors as the edges from the corner at the origin, while $\operatorname{sgn}\det(u_{(1)}, \dots, u_{(n)})$ indicates whether or not the vectors have the same "orientation" as the standard basis.



[In the plane moving from the tip of $u_{(1)}$ to the tip of $u_{(2)}$ in the counterclockwise (clockwise) direction means that the ordered pair $(u_{(1)}, u_{(2)})$ has the same (opposite) orientation as (e_1, e_2). In $\mathbb{R}^3$ we speak of righthanded (lefthanded) triplets of ordered vectors $(u_{(1)}, u_{(2)}, u_{(3)})$ if $u_{(3)}$ is on the same (opposite) side of the plane of $(u_{(1)}, u_{(2)})$ as the cross-product $u_{(1)} \times u_{(2)}$. The standard basis is righthanded.]

We'll return to the orientation later.

We want to see the way in which the following three properties characterize volume, modulo signs.

(1) $\det(u_{(1)}, \dots, \underbrace{u_{(i)} + a u_{(j)}}_{i\text{th argument}}, \dots, u_{(j)}, \dots, u_{(n)}) = \det(u_{(1)}, \dots, u_{(i)}, \dots, u_{(j)}, \dots, u_{(n)})$

[Iteration leads to adding any linear combination of the other vectors to a given vector]

(2) $\det(u_{(1)}, \dots, a u_{(i)}, \dots, u_{(n)}) = a \det(u_{(1)}, \dots, u_{(i)}, \dots, u_{(n)})$

(3) $\det(e_1, \dots, e_n) = 1$, true for any orthonormal basis of $\mathbb{R}^n$ with the same orientation as the standard basis

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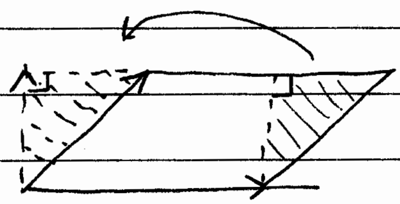
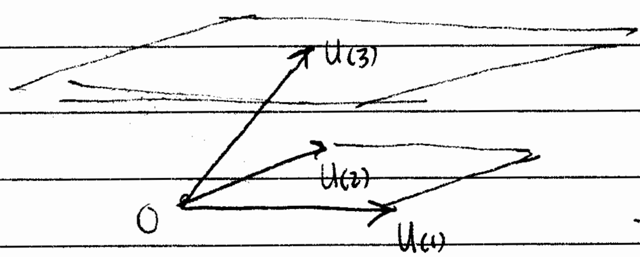
Properties (1) and (2) are independent of the Euclidean inner product and remain valid in defining volume with respect to any inner product. Property (3) basically fixes scale of the volume function in terms of the inner product.

Recall that we define volume of a rectangular solid with perpendicular edges to be the product of the lengths of its orthogonal edges from any corner. This is then extended to n-parallelopipeds by noticing one can always chop it up and re-assemble into a rectangular solid with the same volume. In the plane for example





This property of volume is exactly equivalent to property (1). In $\mathbb{R}^3$ for example, we can move $u_{(3)}$ around anywhere in the plane through its tip parallel to the plane of $u_{(1)}$ and $u_{(2)}$ without changing the "height" relative to that plane and therefore not changing the volume of the parallellopiped. In particular we can always move $u_{(3)}$ so that it is perpendicular to the plane of $u_{(1)}$ and $u_{(2)}$. Then by adding multiples of $u_{(1)}$ to $u_{(2)}$ we can make $u_{(2)}$ perpendicular to $u_{(1)}$ resulting in a rectangular solid. Such an iterative process can be used in $\mathbb{R}^n$ to reduce any n-tuple of vectors to an orthogonal n-tuple with the same volume. Property (2) allows us to pull out the factors of the lengths of the orthogonal edges, leaving the scale of the volume to be set by condition (3), that an orthonormal set of vectors has unit volume.

Property (1) is a direct result of the antisymmetry of the

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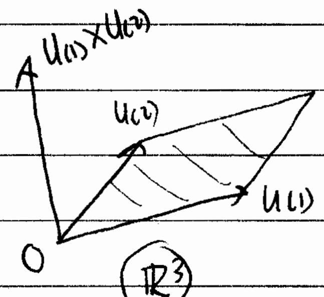
determinant, since antisymmetrization in a pair of identical or proportional objects always gives zero. It determines an equivalence relation on the ordered n-tuples of vectors which corresponds to having the same volume. Any orthogonal representative of such an equivalence class then sets the value of the volume through properties (2) and (3) together.

We will see that antisymmetrization of vectors is somehow equivalent to establishing a volume (or more generally measure) equivalence relation, while an inner product merely sets the scale.

We are also interested in the measure of p-dimensional objects in $\mathbb{R}^n$, like parallelograms in $\mathbb{R}^3$. These turn out to be connected to subdeterminants.



For example, an ordered pair of vectors $(u_{(1)}, u_{(2)})$ in $\mathbb{R}^n$ determines a parallelogram with a certain orientation in space. Consider the 3×2 matrix $(\underline{u}_{(1)}\ \underline{u}_{(2)})$. It has three 2×2 subdeterminants obtained by eliminating the first, second, and third rows respectively, and then alternating the sign. These define the components of the cross product of $u_{(1)}$ and $u_{(2)}$ whose magnitude gives the desired area information, and whose direction specifies the orientation of the plane of $u_{(1)}$ and $u_{(2)}$

$$u_{(1)} \times u_{(2)} = \left(\begin{vmatrix} u_{(1)}{}^2 & u_{(2)}{}^2 \\ u_{(1)}{}^3 & u_{(2)}{}^3 \end{vmatrix}, -\begin{vmatrix} u_{(1)}{}^1 & u_{(2)}{}^1 \\ u_{(1)}{}^3 & u_{(2)}{}^3 \end{vmatrix}, \begin{vmatrix} u_{(1)}{}^1 & u_{(2)}{}^1 \\ u_{(1)}{}^2 & u_{(2)}{}^2 \end{vmatrix} \right)$$

(explanation later)

You also know that

$$\det(u_{(1)}, u_{(2)}, u_{(3)}) = u_{(3)} \cdot (u_{(1)} \times u_{(2)}),$$

i.e., the vector $u_{(1)} \times u_{(2)}$ is basically the partial evaluation of the determinant tensor leaving one vector argument free, i.e., a covector, which is then identified with a vector by the Euclidean inner product

$$\det(u_{(1)}, u_{(2)},\ \ ) = [u_{(1)} \times u_{(2)}]^\flat .$$

The properties of the determinant function and of "antisymmetrization" in general. thus characterize p-measure in $\mathbb{R}^n$ up to a setting of scale which is accomplished via an inner product. We need to develop a notation that can more easily handle this kind of problem.

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SYMMETRY PROPERTIES & VOLUME (MEASURE)

We have already used a symmetry condition for the class of inner products we have been considering, namely $G(Y,X) = G(X,Y)$, and have mentioned antisymmetric inner products. Symmetry properties of tensors turn out to be extremely important so we need to develop a notation to handle them.

Symmetry properties involve the behavior of a tensor under interchange of two arguments or more generally under an arbitrary permutation of a certain number of arguments. Of course to even consider the value of a tensor after the permutation of some of its arguments, the arguments must be of the same type, i.e., covectors have to go in covector arguments and vectors in vector arguments and no other combinations are allowed.

The simplest case to consider are tensors with only 2 arguments of the same type. For vector arguments we have $\binom{0}{2}$-tensors. Define for such a tensor T:

$T(Y,X) = T(X,Y)$ — T is symmetric in X and Y

$T(Y,X) = -T(X,Y)$ — T is antisymmetric or alternating in X and Y.

Letting $(X,Y) = (e_i, e_j)$ and using the definition of components, we get a corresponding condition on the components

$T_{ji} = T_{ij}$ — T is symmetric in the index pair (i,j)

$T_{ji} = -T_{ij}$ — T is antisymmetric (alternating) in the index pair (i,j).

Any such tensor can be decomposed into its symmetric and antisymmetric parts by defining

$[\mathrm{SYM}\, T](X,Y) = \frac{1}{2}[T(X,Y) + T(Y,X)]$ "the sym. part of T"

$[\mathrm{ALT}\, T](X,Y) = \frac{1}{2}[T(X,Y) - T(Y,X)]$ "the antisym part of T"

Clearly $T = \mathrm{SYM}\, T + \mathrm{ALT}\, T$ since evaluating this equation on the pair (X,Y) immediately leads to an identity. [Check.]

(How to get these results?)

Again letting $(X,Y) = (e_i, e_j)$ leads to corresponding component formulas

$$[\mathrm{SYM}\, T]_{ij} = \tfrac{1}{2}[T_{ij} + T_{ji}] \equiv T_{(ij)} \qquad \text{" } n(n+1)/2 \text{ independent components "}$$
$$[\mathrm{ALT}\, T]_{ij} = \tfrac{1}{2}[T_{ij} - T_{ji}] \equiv T_{[ij]} \qquad \text{" } n(n-1)/2 \text{ independent components "}$$
$$T_{ij} = T_{(ij)} + T_{[ij]}.$$

Round brackets around a pair of indices denotes the symmetrization operation, while square brackets denotes the antisymmetrization. This is a very convenient shorthand. All of this can be repeated for $\binom{2}{0}$-tensors and just reflects what we already know about the symmetric and antisymmetric parts of matrices.

In order to consider more than a pair of indices we need to discuss the so called "symmetric group" S_n of permutations of the integers from 1 to n:

$$(1,2,\dots,n) \mapsto (\sigma(1), \sigma(2), \dots, \sigma(n)) \qquad \begin{array}{l}\text{re-arrangement of integers} \\ \text{re-ordering} = \text{permutation.}\end{array}$$

This is also written $\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$

which indicates to integer $\sigma(i)$ replacing the integer i in a two row matrix, the ordering of the columns clearly doesnt matter:

$$\sigma \sim \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{all mean} \quad (1,2,3) \mapsto (\sigma(1), \sigma(2), \sigma(3)) = (2,3,1)$$

Composition of two permutations $(1,2,\dots,n) \mapsto (\pi \circ \sigma)(1,2,3,n)$
$$= \pi(\sigma(1), \dots, \sigma(n)) = (\pi(\sigma(1)), \dots \pi(\sigma(n)))$$

is easily performed using the matrix algorithm

$$\sigma \sim \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \pi \sim \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix}$$
$$\pi \sim \begin{pmatrix} 2 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix}$$
$$\pi \circ \sigma \sim \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

line up upper row of π with lower row of σ, then erase these two rows to get two rows of "product" of two permutations.

FACT. Every permutation can be represented as a certain number N of transpositions (only two integers interchanged, others fixed), the evenness or oddness of which is an invariant. One can therefore define

the sign of a permutation to be $(-1)^N = \begin{cases} 1, & \text{even } N \\ -1, & \text{odd } N \end{cases}$
and this divides the $n!$ possible permutations of n integers equally into even and odd permutations respectively. For example

$$n=1: \begin{pmatrix}1\\1\end{pmatrix} \quad n=2: \begin{pmatrix}12\\12\end{pmatrix};\begin{pmatrix}12\\21\end{pmatrix} \quad n=3: \begin{pmatrix}123\\123\end{pmatrix}\begin{pmatrix}123\\231\end{pmatrix}\begin{pmatrix}123\\312\end{pmatrix};\begin{pmatrix}123\\132\end{pmatrix}\begin{pmatrix}123\\213\end{pmatrix}\begin{pmatrix}123\\321\end{pmatrix}$$
$$\quad + \qquad\quad + \quad - \qquad\quad + \quad + \quad + \qquad - \quad - \quad -$$

This group itself has a rich structure which also has extremely important physical applications but we have enough information for our present purposes.

Suppose we have a $\binom{0}{3}$-tensor T. For $\binom{0}{2}$-tensors we defined the symmetric and antisymmetric parts by summing over all permutations of their arguments/indices, including the sign for the antisymmetric part, and dividing by the total number (2) of such permutations. For a $\binom{0}{3}$-tensor the analogous definitions are

$$\left[\begin{pmatrix}SYM\\ALT\end{pmatrix}T\right](X,Y,Z) = \frac{1}{3!}\left[T(X,Y,Z)+T(Y,Z,X)+T(Z,X,Y) \pm T(X,Z,Y) \pm T(Y,X,Z) \pm T(Z,Y,X)\right]$$

where the upper sign (lower sign) applies for the sym. (antisym.) part.

Clearly under any permutation of the arguments, SYM T is unchanged while ALT T changes by the sign of the permutation. Letting $(X,Y,Z) = (e_i, e_j, e_k)$ leads to the component form

$$\left[\begin{pmatrix}SYM\\ALT\end{pmatrix}T\right]_{ijk} = \frac{1}{3!}\left[T_{ijk}+T_{jki}+T_{kij} \pm T_{ikj} \pm T_{jik} \pm T_{kji}\right] \equiv \begin{pmatrix}T_{(ijk)}\\T_{[ijk]}\end{pmatrix}$$

However, in addition to the completely symmetric and completely antisymmetric parts of T, there are also mixed symmetries which were not possible with only two arguments:

$$T = SYM\,T + ALT\,T + \cdots \quad / \quad T_{ijk} = T_{(ijk)} + T_{[ijk]} + \cdots$$

This question has to do with the representations of the symmetric group and belongs in a course on group theory.

[Use $\{X_{(i)}\}_{i=1,\dots,n}$ to list a set of n vectors to distinguish from the symbol X_i for the components of a single covector.]

In general for a $\binom{0}{p}$-tensor we define:

$$[\mathrm{SYM}\,T](X_{(1)},\dots,X_{(p)}) = \frac{1}{p!}\sum_{\sigma\in S_p} T(X_{(\sigma(1))},\dots,X_{(\sigma(p))})$$

$$[\mathrm{ALT}\,T](X_{(1)},\dots,X_{(p)}) = \frac{1}{p!}\sum_{\sigma\in S_p} (\operatorname{sgn}\sigma)\, T(X_{(\sigma(1))},\dots,X_{(\sigma(p))})$$

and letting $(X_{(1)},\dots,X_{(p)}) = (e_{i_1},\dots,e_{i_p})$ gives the component version

$$[\mathrm{SYM}\,T]_{i_1\dots i_p} \equiv T_{(i_1\cdots i_p)} = \frac{1}{p!}\sum_{\sigma\in S_p} T_{i_{\sigma(1)}\cdots i_{\sigma(p)}}$$

$$[\mathrm{ALT}\,T]_{i_1\cdots i_p} \equiv T_{[i_1\cdots i_p]} = \frac{1}{p!}\sum_{\sigma\in S_p} (\operatorname{sgn}\sigma)\, T_{i_{\sigma(1)}\cdots i_{\sigma(p)}}\,.$$

$\boxed{\text{WARNING! FASTEN SEAT BELTS!}}$

However, Σ-notation is bad news — we introduced the summation convention to eliminate it so far and we can do the same now by making some convenient definitions, at least for the antisymmetric case which is more manageable.

GENERALIZED KRONECKER DELTAS:

$1\le p\le n:$ $\qquad \delta^{i_1\cdots i_p}{}_{j_1\cdots j_p} \equiv p!\,\delta^{i_1}{}_{[j_1}\delta^{i_2}{}_{j_2}\cdots\delta^{i_p}{}_{j_p]} \equiv p!\,\delta^{[i_1}{}_{j_1}\delta^{i_2}{}_{j_2}\cdots\delta^{i_p]}{}_{j_p}$

notice why equivalent for $p=2$

$p=1 \qquad \delta^{i_1}{}_{j_1}$

$p=2 \qquad \delta^{i_1 i_2}{}_{j_1 j_2} = \delta^{i_1}{}_{j_1}\delta^{i_2}{}_{j_2} - \delta^{i_1}{}_{j_2}\delta^{i_2}{}_{j_1} \qquad \left(= \delta^{i_1}{}_{j_1}\delta^{i_2}{}_{j_2} - \delta^{i_2}{}_{j_1}\delta^{i_1}{}_{j_2}\right)$

$p=n \qquad \delta^{i_1\cdots i_n}{}_{j_1\cdots j_n} = \begin{cases} \operatorname{sgn}\begin{pmatrix} i_1\cdots i_n \\ j_1\cdots j_n\end{pmatrix} & \text{if no repeated indices at either level} \\ 0 & \text{otherwise.}\end{cases}$

[In fact each generalized Kronecker delta is a determinant of a matrix whose entries are Kronecker deltas

$$\delta^{i_1\cdots i_p}{}_{j_1\cdots j_p} = \begin{vmatrix} \delta^{i_1}{}_{j_1} & \cdots & \delta^{i_1}{}_{j_p} \\ \vdots & & \\ \delta^{i_p}{}_{j_1} & \cdots & \delta^{i_p}{}_{j_p}\end{vmatrix}$$

$$= \sum_{\sigma\in S_p} (\operatorname{sgn}\sigma)\,\delta^{i_1}{}_{\sigma(j_1)}\delta^{i_2}{}_{\sigma(j_2)}\cdots\delta^{i_p}{}_{\sigma(j_p)}\ ]$$

Then $\quad \frac{1}{p!}\delta^{j_1\cdots j_p}{}_{i_1\cdots i_p} T_{j_1\cdots j_p} = \delta^{j_1}{}_{[i_1}\delta^{j_2}{}_{i_2}\cdots\delta^{j_p}{}_{i_p]} T_{j_1\cdots j_p}$

$$= T_{[i_1\cdots i_p]} = [\mathrm{ALT}\,T]_{i_1\cdots i_p}$$

since each Kronecker delta contraction replaces a j-index by an i-index.

Note that if a tensor is already antisymmetric, antisymmetrization does not change it (it is equal to its antisymmetric part), or

$$\mathrm{ALT}\,(\mathrm{ALT}\ T) = \mathrm{ALT}\ T$$

equivalent to $\frac{1}{p!}\,\delta^{j_1\cdots j_p}_{i_1\cdots i_p}\left(\frac{1}{p!}\,\delta^{k_1\cdots k_p}_{j_1\cdots j_p}\,T_{k_1\cdots k_p}\right) = \frac{1}{p!}\,\delta^{k_1\cdots k_p}_{i_1\cdots i_p}\,T_{k_1\cdots k_p}$

or $\delta^{j_1\cdots j_p}_{i_1\cdots i_p}\,\delta^{k_1\cdots k_p}_{j_1\cdots j_p} = p!\,\delta^{k_1\cdots k_p}_{i_1\cdots i_p}$.

What exactly are these generalized Kronecker deltas?

Each term in the expansion of a generalized Kronecker delta is a product of Kronecker deltas, which represent the components of certain tensor products of the Kronecker delta tensor (or unit tensor) with itself and so are themselves tensors, as is the entire sum of such terms.

Thus $\delta^{i_1\cdots i_p}_{j_1\cdots j_p}$ are the components of a $\binom{p}{p}$-tensor $\delta^{(p)}$ on our vector space V which has the same numerical values of its components in any basis

$$\delta^{(p)} = \delta^{i_1\cdots i_p}_{j_1\cdots j_p}\, e_{i_1}\otimes\cdots\otimes e_{i_p}\otimes\omega^{i_1}\otimes\cdots\otimes\omega^{i_p}$$

Its value on p-vector arguments and p-covector arguments is just the determinant of the matrix of all possible evaluations of a covector on a vector

$$\delta^{(p)}(f^{(1)},\dots,f^{(p)},u_{(1)},\dots,u_{(p)}) = \underbrace{|\,f^{(i)}(u_{(j)})\,|}_{\text{determinant symbol}}$$

Finally define the (alternating, permutation, Levi-Civita) symbols by

$$\epsilon_{i_1\cdots i_n} = \delta^{1\cdots n}_{i_1\cdots i_n}\,,\qquad \epsilon^{i_1\cdots i_n} = \delta^{i_1\cdots i_n}_{1\cdots n}.$$

These do not define the components of a tensor any more than $\delta^1{}_i$ defines the components of a covector, in the sense that a single tensor exists which has these numerical component values in every basis. [$\delta^1{}_i$ are the components of $\omega^1 = \delta^1{}_i\,\omega^i$, which changes as you change the basis. Similarly $\epsilon_{i_1\cdots i_n}$ are the components of a certain tensor but a different one for each choice of basis.]

These alternating symbols are useful in representing determinants.

$$\det \underline{A} = \sum_{\sigma\in S_p} (\operatorname{sgn}\sigma)\ \underbrace{A^{\sigma(1)}{}_1 \cdots A^{\sigma(n)}{}_n}_{\text{re-arrange factors so top order is } 1,2,\dots,n}$$

$$\overset{\checkmark}{=} \sum_{\sigma\in S_p} \operatorname{sgn}\sigma\ A^1{}_{\sigma^{-1}(1)} \cdots A^n{}_{\sigma^{-1}(n)}$$

but $\operatorname{sgn}\sigma^{-1} = \operatorname{sgn}\sigma$ and a some over σ^{-1} for all $\sigma\in S_p$ is a sum over every permutation since every permutation can be represented as the inverse of another permutation (group property)

$$\sigma \sim \begin{pmatrix} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n)\end{pmatrix} \sim \begin{pmatrix} \sigma^{-1}(1) & \cdots & \sigma^{-1}(n) \\ 1 & \cdots & n\end{pmatrix}$$

so the inverse appears when you re-order the lower row of the permutation.

But by definition then

$$\boxed{\det \underline{A} = \epsilon_{i_1\cdots i_n} A^{i_1}{}_1 \cdots A^{i_n}{}_n = \epsilon^{i_1\cdots i_n} A^1{}_{i_1} \cdots A^n{}_{i_n}}$$

Also

$$(\det A)\,\epsilon_{j_1\cdots j_n} = \epsilon_{i_1\cdots i_n} A^{i_1}{}_{j_1} \cdots A^{i_n}{}_{j_n}$$ (since a permutation of the columns of A changes $\det A$ by its sign)

$$(\det A)\,\epsilon^{j_1\cdots j_n} = \epsilon^{i_1\cdots i_n} A^{j_1}{}_{i_1} \cdots A^{j_1}{}_{i_n}$$ (ditto for permutation of rows)

or replacing $\underline{A}$ by $\underline{A}^{-1}$ in the first and not in the second (recall $\det A^{-1} = (\det A)^{-1}$)

$$\epsilon_{j_1\cdots j_n} = (\det A^{-1})^{-1}\,\epsilon_{i_1\cdots i_n} A^{-1\,i_1}{}_{j_1} \cdots A^{-1\,i_n}{}_{j_n}$$

$$\epsilon^{j_1\cdots j_n} = \underbrace{(\det A^{-1})}_{\substack{\text{additional weight } (\det A^{-1})^W \text{ which rescales tensor to} \\ \text{new tensor which has same numerical values as} \\ \text{the components of the old "tensor".}}}\ \underbrace{\epsilon^{i_1\cdots i_n} A^{j_1}{}_{i_1} \cdots A^{j_n}{}_{i_n}}_{\text{tensor transformation law if } e'_i = A^{-1\,j}{}_i\, e_j}$$

So in fact these alternating symbols define the components of a 1-parameter family of proportional $\binom{0}{p}$-tensors and $\binom{p}{0}$-tensors respectively which together are referred to as a "tensor density of weight $W = -1$ and $W = 1$" respectively ($\det A \neq 0$ for a change of basis, but it can assume all nonzero values).

In fact these are not so unfamiliar. For any basis we can identify the components of vectors with column matrices

$$u = u^i e_i \rightarrow \underline{u} = \begin{bmatrix} u^1 \\ \vdots \\ u^n \end{bmatrix}$$

and the value of the tensor $\epsilon_{i_1 \cdots i_n} \omega^{i_1} \otimes \cdots \otimes \omega^{i_n}$ on n vector arguments is just

$$[\epsilon_{i_1 \cdots i_n} \omega^{i_1} \otimes \cdots \otimes \omega^{i_n}](u_{(1)}, \ldots, u_{(n)})$$

$$= \epsilon_{i_1 \cdots i_n} u^{i_1}{}_{(1)} \cdots u^{i_n}{}_{(n)} = \underbrace{\det(\underline{u}_{(1)}, \ldots, \underline{u}_{(n)})}_{\substack{\text{determinant as multilinear} \\ \text{function on } n \text{ column matrices}}} = \det \underbrace{(\underline{u}_{(1)} \cdots \underline{u}_{(n)})}_{\substack{\text{matrix whose} \\ \text{columns are} \\ \text{these column} \\ \text{matrices}}}$$

But under a change of basis.

$$u^{i\prime} = A^i{}_j u^j \quad \text{so} \quad \underline{u}' = A\underline{u} \quad \text{so}$$

$$\det(\underline{u}'_{(1)} \cdots \underline{u}'_{(n)}) = \det(A\underline{u}_{(1)} \cdots A\underline{u}_{(n)})$$

$$= \det[\underbrace{A(\underline{u}_{(1)} \cdots \underline{u}_{(n)})}_{\substack{\text{left matrix product of} \\ A \text{ against columns of right factor} \\ \text{has columns of right factor multiplied by } A}}] = \underbrace{(\det A) \det(\underline{u}_{(1)} \cdots \underline{u}_{(n)})}_{\substack{\text{product rule} \\ \text{for determinants}}}$$

the determinant of the new column matrices differs by the determinant of the transformation itself, explaining why one gets a different (but proportional) tensor for different choices of basis.

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Now I have to admit I have a sick love for lots of indices, but I wouldnt drag you through this index jungle if it werent true that the algebra of antisymmetric tensors did not play a fundamental role in differential geometry.* Questions of measure for p-dimensional surfaces · necessary for generalizing line integrals and surface integrals of vector fields and volume integrals of functions all involve this algebra in a way that will turn out to be very beautiful. Trust me.

* ~~maybe I would~~

What happened to symmetric tensors? Except for inner products they turn out not to be as important, so we dont need to develop a machine for them, which anyway involves the symmetric group in a much more nontrivial way.

But were not finished. First an easy formula:

$$\underbrace{\delta^{i_1\cdots i_n}_{j_1\cdots j_n}}_{\substack{\text{sign of permutation from}\\ \text{upper indices to lower indices}}} = \epsilon^{i_1\cdots i_n}\,\epsilon_{j_1\cdots j_n}$$

sign of permutation from upper indices to lower indices = product of signs of permutation from upper ~~to $(1\cdots n)$ and then from $(1\cdots n)$ to lower~~.

Next a hard formula (just accept it for now)

$$\delta^{i_1\cdots i_p}_{j_1\cdots j_p} = \frac{1}{(n-p)!}\,\delta^{i_1\cdots i_p\,k_{p+1}\cdots k_n}_{j_1\cdots j_p\,\underbrace{k_{p+1}\cdots k_n}_{(n-p)\text{ contractions}}} = \cancel{\frac{1}{(n-p)!}\,\epsilon^{i_1\cdots i_p k_{p+1}\cdots k_n}\,\epsilon}$$

no room

$$= \frac{1}{(n-p)!}\,\epsilon^{i_1\cdots i_p k_{p+1}\cdots k_n}\,\epsilon_{j_1\cdots j_p k_{p+1}\cdots k_n}$$

(easy from previous formula)

~~that finishes the foundation. Next we build the house.~~

[Remark. By iteration of this ugly formula you can get

$$\delta^{i_1\cdots i_p}_{j_1\cdots j_p} = \frac{(n-q)!}{(n-p)!}\,\delta^{i_1\cdots i_p\,k_{p+1}\cdots k_q}_{j_1\cdots j_p\,k_{p+1}\cdots k_q}$$

Had to sneak that in.]

It helps to look at the case $n=3$ to have a more concrete idea of what all this means:

$$\epsilon_{123} = \epsilon_{213} = \epsilon_{312} = 1$$

$$\epsilon_{132} = \epsilon_{231} = \epsilon_{321} = -1$$

$$\delta^{ij}_{kl} = \begin{vmatrix} \delta^i_k & \delta^i_l \\ \delta^j_k & \delta^j_l \end{vmatrix} = \delta^i_k \delta^j_l - \delta^i_l \delta^j_k = 2\,\delta^i_{[k}\delta^j_{l]} = 2\,\delta^{[i}_k \delta^{j]}_l$$

$$\delta^{ijk}_{mnl} = \begin{vmatrix} \delta^i_m & \delta^i_n & \delta^i_l \\ \delta^j_m & \delta^j_n & \delta^j_l \\ \delta^k_m & \delta^k_n & \delta^k_l \end{vmatrix} = \delta^i_m \delta^j_n \delta^k_l + \delta^i_n \delta^j_l \delta^k_m + \delta^i_l \delta^j_m \delta^k_n - \delta^i_m \delta^j_l \delta^k_n - \delta^i_n \delta^j_m \delta^k_l - \delta^i_l \delta^j_n \delta^k_m$$

$$\delta^{ijk}_{mnk} = \delta^i_m \delta^j_n \underbrace{\delta^k_k}_{3} + \delta^i_n \underbrace{\delta^j_k \delta^k_m}_{\delta^j_m} + \underbrace{\delta^i_k \delta^j_m \delta^k_n}_{\delta^i_n} - \delta^i_m \underbrace{\delta^j_k \delta^k_n}_{\delta^j_n} - \delta^i_n \delta^j_m \underbrace{\delta^k_k}_{3} - \underbrace{\delta^i_k \delta^j_n \delta^k_m}_{\delta^i_m} = (3-1-1)\,\delta^i_m \delta^j_n - (3-1-1)\,\delta^i_n \delta^j_m$$

$$= \delta^i_m \delta^j_n - \delta^i_n \delta^j_m = \delta^{ij}_{mn}.$$

$$\delta^{ijk}_{mjk} = \delta^{ij}_{mj} = \delta^i_m \delta^j_j - \delta^i_j \delta^j_m = (3-1)\,\delta^i_m = 2\,\delta^i_m$$

$$\delta^{ijk}_{ijk} = 2\cdot\delta^i_i = 2\cdot 3 = 3!$$

Whew!

This Kronecker delta business is just a shorthand for giving compact expressions for 3X3 determinants and subdeterminants (minors, minors of minors, ..)

$$\delta^{123}_{mnp} X^m Y^n Z^p = X^1 Y^2 Z^3 + X^3 Y^1 Z^2 + X^2 Y^1 Z^3 - \dots$$

$$= \cancel{6}\cdot \begin{vmatrix} X^1 & Y^1 & Z^1 \\ X^2 & Y^2 & Z^2 \\ X^3 & Y^3 & Z^3 \end{vmatrix}.$$

$$\delta^{12}_{mn} X^m Y^n = X^1 Y^2 - X^2 Y^1 = \begin{vmatrix} X^1 & Y^1 \\ X^2 & Y^2 \end{vmatrix}$$ = 33 minor of previous determinant.

For the case in which $A = (u_{(1)} \dots u_{(n)})$ we get on $\mathbb{R}^n$

$$\det(u_{(1)},\dots,u_{(n)}) = \epsilon_{i_1\dots i_n} A^{i_1}{}_1 \cdots A^{i_n}{}_n$$
$$= \delta^{1\dots n}_{i_1\dots i_n} u^{i_1}_{(1)} \cdots u^{i_n}_{(n)}$$
$$= p!\; u^{[1}_{(1)} \cdots u^{n]}_{(n)}$$

This is the single independent component of the antisymmetrized tensor product of the n vectors

$$p![\mathrm{ALT}(u_{(1)} \otimes \dots \otimes u_{(n)})]^{i_1\dots i_n} = \delta^{i_1\dots i_n}_{j_1\dots j_n} u^{j_1}_{(1)} \cdots u^{j_n}_{(n)}$$
$$= \epsilon^{i_1\dots i_n} \det(u_{(1)},\dots,u_{(n)}).$$

The generalized Kronecker delta $\delta^{(n)}$ areses in a very simple way

$$u^{i_1}_{(1)} \cdots u^{i_n}_{(n)} = \delta^{i_1}{}_{j_1} \cdots \delta^{i_n}{}_{j_n} u^{j_1}_{(1)} \cdots u^{j_n}_{(n)}$$

$$p!\, u^{[i_1}_{(1)} \cdots u^{i_n]}_{(n)} = \underbrace{p!\,\delta^{[i_1}{}_{j_1} \cdots \delta^{i_n]}{}_{j_n}}_{\equiv\, \delta^{i_1\dots i_n}_{j_1\dots j_n}} u^{j_1}_{(1)} \cdots u^{j_n}_{(n)}$$

as the antisymmetrizer operator modulo the factorial factor.

The antisymmetrized tensor product of p vectors in an n-dimensional vector space contains both information about the p-measure of the p-parallelopiped they form as well as its orientation within the space just like the cross product does in $\mathbb{R}^3$ (almost). An inner product merely sets the scale of the p-measure.

TOPS FORM 7525

Litho in U.S.A.

The algebra of antisymmetric tensors

We need names for the vectorspaces of tensors of given types over a vector space V.

Let $T^{(p,q)}(V)$ be the vector space of $\binom{p}{q}$- tensors over V

$$S = S^{i_1\cdots i_p}{}_{j_1\cdots j_q} \underbrace{e_{i_1}\otimes\cdots\otimes e_{i_p}\otimes\omega^{j_1}\otimes\cdots\otimes\omega^{j_q}}$$

basis: $p+q$ indices, each taking n values $\rightarrow$

$$\dim T^{(p,q)}(V) = n^{p+q}.$$

Let $\begin{cases} \Lambda^{(p)}(V) = \text{ALT } T^{(p,0)}(V) \\ \Lambda^{(p)}(V)^* = \text{ALT } T^{(0,p)}(V) \end{cases} \equiv$ linear subspaces of ~~ant~~ antisymmetric

$\binom{p}{0}$, $\binom{0}{p}$ - tensors called p-vectors, p-covectors or p-forms respectively.

$$S = S^{i_1\cdots i_p} e_{i_1}\otimes\cdots\otimes e_{i_p} \qquad S^{i_1\cdots i_p} = S^{[i_1\cdots i_p]}$$

$$S = S_{i_1\cdots i_p}\omega^{i_1}\otimes\cdots\otimes\omega^{i_p} \qquad S_{i_1\cdots i_p} = S_{[i_1\cdots i_p]}.$$

Antisymmetric tensors cannot have nonzero components with any repeated indices, since interchanging any pair of indices must change the sign of the result, but an interchange of an identical pair does change the component so it can only be zero:

$$S_{ijk} = -S_{jik} \rightarrow S_{112} = -S_{112} \rightarrow S_{112} = 0.$$

Thus an antisymmetric tensor can have at most n indices without being identically zero. The norepeat condition tells us the dimension of the space of antisymmetric tensors of a given "rank" p, equivalently the number of "independent components" of such a tensor. The number of p-tuples of distinct integers chosen from the set of integers $(1,\ldots,n)$ is by definition the number of combinations of n things take r at a time

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$$\dim \Lambda^{(p)}(V) = \dim \Lambda^{(p)}(V)^* = \binom{n}{p} = \frac{n!}{p!(n-p)!}.$$

If we define $\Lambda^{(0)}(V) = \Lambda^{(0)}(V)^* = \mathbb{R}$, i.e. the $\binom{0}{0}$-tensors or scalars are identified with antisymmetric tensors with no indices (1 index tensors are antisymmetric by default), then we have $(n+1)$ such spaces for the contravariant and covariant cases which pair off by dimension since

$$\binom{n}{p} = \binom{n}{n-p}$$

So the case $p=0$ and $p=n$ are both 1-dimensional

$p=1$ and $p=n-1$ n-dimensional

$p=2$ and $p=n-2$ $\frac{n(n-1)}{2}$ dimensional etc.

<u>EX.</u> $n=3$.

A 2-vector $S^{ij} e_i \otimes e_j$ has 3 independent components (S^{23}, S^{31}, S^{12})

A 3-vector $S^{ijk} e_i \otimes e_j \otimes e_k$ has a single independent component S^{123}.

Any time we get our hands on a vector space, we try to find a convenient basis. We can do the same here. Consider the p-vector case

Since $S^{i_1 \cdots i_p} = S^{[i_1 \cdots i_p]} = \frac{1}{p!} \delta^{i_1 \cdots i_p}{}_{j_1 \cdots j_p} S^{j_1 \cdots j_p}$, then

$$S = S^{i_1 \cdots i_p} e_{i_1} \otimes \cdots \otimes e_{i_p} = \frac{1}{p!} S^{j_1 \cdots j_p} \underbrace{\delta^{i_1 \cdots i_p}{}_{j_1 \cdots j_p} e_{i_1} \otimes \cdots \otimes e_{i_p}}$$

make the definition: $\equiv e_{i_1 \cdots i_p} \equiv p!\, e_{[i_1} \otimes \cdots \otimes e_{i_p]}$

$$S = \frac{1}{p!} \underbrace{S^{j_1 \cdots j_p}}_{\text{same}} \underbrace{e_{j_1 \cdots j_p}}_{\substack{p! \text{ orderings} \\ \text{only differ by sign}}} = \underbrace{\sum_{j_1 < \cdots < j_p}}_{\substack{\text{sum only over} \\ \text{ordered p-tuplets, leave out } p! \text{ factor}}} S^{j_1 \cdots j_p} e_{j_1 \cdots j_p}$$

The set $\{e_{i_1\cdots i_p}\}_{i_1<\cdots<i_p}$ is a basis for p-vectors.

[Exercise. We have shown that any p-vector is a linear combination of them. How do we show linear independence, ie:

$$\sum_{i_1<\cdots<i_p} S^{i_1\cdots i_p} e_{i_1\cdots i_p} = 0 \rightarrow \quad S^{j_1\cdots j_p} = 0 \text{ for all possible index values ?}$$
]

<u>EX</u> $n=3$, $p=2$

$$S = S^{ij} e_i \otimes e_j = S^{12} e_1\otimes e_2 + S^{31} e_3 \otimes e_1 + S^{12} e_1 \otimes e_2$$
$$+ S^{21} e_2 \otimes e_1 + S^{13} e_1 \otimes e_3 + S^{21} e_2 \otimes e_1$$

$$= S^{12} \underbrace{(e_1\otimes e_2 - e_2 \otimes e_1)}_{e_{12}} + S^{13} \underbrace{(e_1 \otimes e_3 - e_3 \otimes e_1)}_{e_{13}} + S^{12} \underbrace{(e_1\otimes e_2 - e_2\otimes e_1)}_{e_{12}}$$

For this case it turns out that the basis $\{e_{12}, e_{31}, e_{12}\}$ is more useful, as we will see later.

Well, rather than write $\sum_{i_1<\cdots<i_p}$ (Σ-notation is bad remember) we just sum over all orderings and divide by $p!$ when we represent a p-vector abstractly, or we introduce more notation

$$S = \frac{1}{p!} S^{i_1\cdots i_p} e_{i_1\cdots i_n} = \sum_{i_1<\cdots<i_p} S^{i_1\cdots i_p} e_{i_1\cdots i_p}$$
$$\equiv S^{i_1\cdots i_p} e_{|i_1\cdots i_p|}$$

Vertical bars enclosing a p-tuple of antisymmetric indices mean sum only over ordered p-tuple values.

WHOA! (western movie expression for "stop")

Okay, before going on, let's see what we've done so far to reassure ourselves that we have a general idea what we have done. Just let $V=\mathbb{R}^n$.
We have been expanding on the following structure

- $(\mathbb{R}^n, \cdot, \det)$

$\{e_i\}$ standard basis

$e_i \cdot e_j = \delta_{ij}$ standard Euclidean inner product

"det" = $\binom{0}{n}$-tensor
- sgn det: standard orientation
- abs.val. det: Euclidean volume function for parallellopipeds

- dual space $[\mathbb{R}^n]^*$

dual basis $\{\omega^i\}$: just "cartesian coordinates $\{x^i\}$"

duality $\omega^i(e_j) = \delta^i{}_j$ (definition of standard basis components)

- dual of dual identification: $u(f) \equiv f(u) = f_i u^i$

$$e_j(\omega^i) \equiv \omega^i(e_j) = \delta^i{}_j$$

So we know how to evaluate vectors on forms and viceversa, and the index pair contraction just symbolizes evaluation of a real-valued linear function of a vector or a covector when expressed in terms of its components

$$u = u^i e_i \qquad u^i = \omega^i(u)$$
$$f = f_i \omega^i \qquad f_i = e_i(f) = f(e_i)$$

We then generalize to multilinear real-valued functions accepting p covector arguments and q vector arguments

- $T^{(p,q)}(\mathbb{R}^n)$: $\quad T = T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}\, e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes \omega^{j_1} \otimes \cdots \otimes \omega^{j_q}$

where the tensor product just holds the vectors and covectors apart in order until they accept their arguments, at which they become real numbers value of T: $T(f,g,\dots,u,v,\dots) =$

$$= T^{i_1\cdots i_p}{}_{j_1\cdots j_q}\, e_{i_1}(f)\, e_{i_2}(g) \cdots \omega^{j_1}(u)\,\omega^{j_2}(v)\cdots$$

$$\equiv T^{i_1\cdots i_p}{}_{j_1\cdots j_p}\, f_{i_1} g_{i_2} \cdots u^{j_1} v^{j_2} \cdots$$

value of T on basis vectors, covectors:

$$T(\omega^{i_1},\omega^{i_2},\dots,e_{j_1},e_{j_2},\dots) = T^{m_1\cdots m_p}{}_{n_1\cdots n_q} \underbrace{[\omega^{i_1}]_{m_1}}_{\delta^{i_1}{}_{m_1}} \underbrace{[\omega^{i_2}]_{m_2}}_{\delta^{i_2}{}_{m_2}} \cdots \underbrace{[e_{j_1}]^{n_1}}_{\delta^{n_1}{}_{j_1}} \cdots$$

$$= T^{i_1\cdots i_p}{}_{j_1\cdots j_q}$$

~~defines the components as particular values of the tensor~~

■ $"\cdot" = G = G_{ij}\,\omega^i \otimes \omega^j$, $G_{ij} = G(e_i,e_j) = e_i \cdot e_j = \delta_{ij}$

$$= \delta_{ij}\,\omega^i\otimes\omega^j$$

$$= \sum_{i=1}^{n} \omega^i \otimes \omega^i \qquad \text{so self inner product}$$

$$u\cdot u = G(u,u) = \sum_{i=1}^{n} [\omega^i(u)]^2 = \sum_{i=1}^{n} (u^i)^2$$

$$u\cdot v = G(u,v) = G_{ij}\,\omega^i(u)\, v^j(v) = G_{ij}\, u^i v^j$$

We reinterpret the dot product as a $\binom{0}{2}$-tensor with components in the standard basis equal to the unit matrix, ie., the standard basis is orthonormal.

■ $\det(u,v,\dots,w) = \underbrace{\det(e_{i_1},e_{i_2},\dots,e_{i_n})}_{\equiv\, \epsilon_{i_1 i_2\cdots i_n}}\, u^{i_1} v^{i_2} \cdots w^{i_n}$ (just a convenient def)

$$= \epsilon_{i_1 i_2\cdots i_n}\, u^{i_1} v^{i_2}\cdots w^{i_n} \equiv \delta^{1\cdots n}{}_{i_1\cdots i_n}\, u^{i_1} v^{i_2} \cdots w^{i_n}$$

$$= \sum_{\sigma\in S_n} (\operatorname{sgn}\sigma)\, u^{\sigma(1)} v^{\sigma(2)} \cdots w^{\sigma(n)} \equiv n!\, u^{[1} v^2 \cdots w^{n]}$$

So $\det = \epsilon_{i_1\cdots i_n}\,\omega^{i_1}\otimes\cdots\otimes\omega^{i_n}$

$$= n!\,\omega^{[1}\otimes\cdots\otimes\omega^{n]}$$

Any antisymmetric $\binom{0}{n}$-tensor is completely determined by a single nonzero component (for $\mathbb{R}^n$!)

$T_{i_1\cdots i_n} = t\,\epsilon_{i_1\cdots i_n}$, $t = T_{12\cdots n}$ If T antisymmetric (which just means it changes sign under exchange of any two arguments).

Positively (negatively) oriented n-forms have $t>0$ $(t<0)$, equivalent to $\det(e_{1'}, e_{2'}, \dots, e_{n'}) > 0$ (<0) for a positively (negatively) oriented basis $\{e_{i'}\}$.

In fact $\underbrace{u^{i_1} v^{i_2} \cdots w^{i_n}}_{n \text{ factors}} \equiv T^{i_1 i_2 \cdots i_n}$ is an $\binom{n}{0}$-tensor

with antisymmetric part $[ALT\,T]^{i_1\cdots i_n} = u^{[i_1} v^{i_2} \cdots w^{i_n]}$

which has a single independent component

$$[ALT\ T]^{1\cdots n} = u^{[1} v^2 \cdots w^{n]} = \frac{1}{n!}\det(u, v, \dots, w)$$

The antisymmetric tensor product turns out to be very useful. We'll get to it next.

MATRIX NOTATION.

You did all this stuff first in matrix notation. Let's go back to it to remind ourselves

$u \in \mathbb{R}^n \mapsto \underline{u} = (u^i)$ column matrix

$f \in [\mathbb{R}^n]^* \mapsto \underline{f} = (f_i)$ row matrix

$f(u) = f_i u^i = \underline{f}\,\underline{u}$ matrix product gives evaluation

$u_i = \delta_{ij} u^j$ = components of $u^\flat \in [\mathbb{R}^n]^* \mapsto \underline{u}^T$ row matrix

$u^i = \delta^{ij} u_j$ $\quad \underline{u}^T \mapsto (\underline{u}^T)^T = \underline{u}$

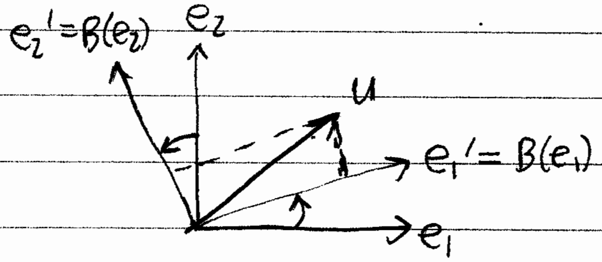
The transpose corresponds to raising and lowering indices in this correspondence. The dot product is then

$$G(u,v) \equiv u \cdot v = \delta_{ij} u^i v^j = \underline{u^i \delta_{ij} v^j} = \underline{u}^T \underline{I}\,\underline{v}$$

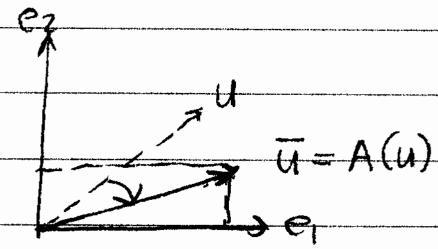
and the determinant is

$$\det(u, v, \dots, w) = \det\underbrace{(\underline{u}\ \underline{v} \cdots \underline{w})}_{\text{matrix}}$$ in the original matrix determinant sense

■ Change of basis. Let $A = B^{-1}$ be a linear transformation of $\mathbb{R}^n$



passive transformation:
u fixed but express in new basis $\{e_{i'}\}$ obtained by active linear transformation of standard basis —
new components $u^{i'}$



active transformation:
basis fixed but vector u changes to new vector $\bar{u}$ with new components $\bar{u}^i$ wrt old basis

Components of u wrt $\{e_{i'}\}$ equal the components of $\bar{u}$ with respect to $\{e_i\}$ as in picture: $u^{i'} = \bar{u}^i$

$$e_{i'} = B^j{}_i e_j = A^{-1j}{}_i e_j \quad \text{or} \quad e_i = B^{-1j}{}_i e_{j'} = A^j{}_i e_{j'}$$
$$\omega^{i'} = B^{-1i}{}_j \omega^j = A^i{}_j \omega^j \qquad \omega^i = B^i{}_j \omega^{j'} = A^{-1i}{}_j \omega^{j'}$$

evaluate on u to get new components

$$\underbrace{\omega^{i'}(u)}_{u^{i'}} = A^i{}_j \underbrace{\omega^j(u)}_{u^j}$$

$$u^{i'} = \underbrace{A^i{}_j u^j}_{\bar{u}^i}$$

MATRIX FORM:

$$\underline{u}' = \underline{A}\,\underline{u}$$

or $\underline{u} = \underline{A}^{-1}\underline{u}'$

By definition these are the $\rightarrow \bar{u}^i$ components of the linear transformation of u by A, ie $u^{i'} = \bar{u}^i$.

■ So $u \cdot v = \underline{u}^T \underline{I}\, \underline{v} = (\underline{A}^{-1}\underline{u}')^T \underline{I} (\underline{A}^{-1}\underline{v}') = (\underline{u}')^T \underbrace{(\underline{A}^{-1})^T \underline{I}\, \underline{A}^{-1}}_{} \underline{v}'$

$$\underline{G}' \equiv (\underline{A}^{-1})^T \underline{I}\, \underline{A}^{-1}$$

$$G_{i'j'} = A^{-1m}{}_i \delta_{mn} A^{-1n}{}_j = A^{-1m}{}_i A^{-1n}{}_j \delta_{mn}$$

"tensor transformation law" for $\binom{0}{2}$-tensor

[Alternatively $G_{i'j'} = e_{i'} \cdot e_{j'} = A^{-1m}{}_i A^{-1n}{}_j e_m \cdot e_n = A^{-1m}{}_i A^{-1n}{}_j \delta_{mn}$]

- Finally

$$\det(u, v, \ldots, w) = \det(\underline{u}\ \underline{v} \cdots \underline{w})$$
$$= \det(\underline{A}^{-1}\underline{u}'\ \ \underline{A}^{-1}\underline{v}' \cdots \underline{A}^{-1}\underline{w}')$$
$$= \det\ \underline{A}^{-1}(\underline{u}'\,\underline{v}' \cdots \underline{w}')$$
$$= \det A^{-1}\ \underbrace{\det(\underline{u}'\,\underline{v}' \cdots \underline{w}')}_{\equiv\ \epsilon_{i_1\cdots i_n} u^{i_1'} v^{i_2'} \cdots w^{i_n'}}$$

$\leftarrow$ determinant of product = prod of determinants

$$= \underbrace{[(\det A^{-1})\,\epsilon_{i_1\cdots i_n}]}_{\det(e_{i_1'}, \ldots, e_{i_n'}) \equiv [\det]_{i_1'\cdots i_n'}}\ u^{i_1'} \cdots w^{i_n'}$$

so
$$\det = (\det A^{-1})\,\epsilon_{i_1\cdots i_n}\,\omega^{i_1'} \otimes \cdots \otimes \omega^{i_n'}$$
$$= \underbrace{(\det A^{-1})}\ n!\ \omega^{[1'} \otimes \cdots \otimes \omega^{n']} \qquad = n!\,\omega^{[1} \otimes \cdots \otimes \omega^{n]}$$

corrects value of determinant of matrix of new components to give value of determinant tensor, which is independent of basis.

- Linear transformations. Suppose we have some other linear transformation of $\mathbb{R}^n$ into itself. In the various notations

$$\underline{u} \mapsto \underline{L}\,\underline{u}\,, \qquad u^i \mapsto L^i{}_j u^j\,, \qquad u \mapsto L(u)$$

Define the $\binom{1}{1}$-tensor $\mathbb{L}$ by
$$\mathbb{L}(f, u) = f_i L^i{}_j u^j = \underline{f}\,\underline{L}\,\underline{u}$$

so that $L = \mathbb{L}(\ , u)$ is the partial evaluation of $\mathbb{L}$ (thinking of the vector $L(u)$ as waiting for a covector argument)

Then $\mathbb{L} = L^i{}_j\, e_i \otimes \omega^j$, $\quad L^i{}_j = \mathbb{L}(\omega^i, e_j) \quad (= \delta^i{}_m L^m{}_n \delta^n{}_j = L^i{}_j)$

So
$$\underline{A}(\underline{u} \mapsto \underline{L}\,\underline{u}) \to \underbrace{\underline{A}\,\underline{u}}_{\underline{u}'} \mapsto \underline{A}\,\underline{L}\,\underbrace{\underline{u}}_{\underline{A}^{-1}\underline{u}'}$$

So
$$\underline{u}' \mapsto \underbrace{(\underline{A}\,\underline{L}\,\underline{A}^{-1})}_{\underline{L}' = \underline{A}\,\underline{L}\,\underline{A}^{-1}}\underline{u}' \qquad L^{i'}{}_{j'} = A^i{}_m L^m{}_n A^{-1\,n}{}_j = A^i{}_m A^{-1\,n}{}_j L^m{}_n$$

so we get the "tensor-transformation" law for a $\binom{1}{1}$-tensor.
Both inner products and linear transformations are represented by matrices, but their different mathematical structure is reflected in the different matrix transformation laws. Our index notation makes these differences explicit. [We need indices to hand objects with more than two indices.]

<u>Remark</u>. What is the difference between δ_{ij} and $\delta^i{}_j$?
It depends on the interpretation. The values for each index pair (i,j) are identical BUT we interpret $\delta^i{}_j$ as the components

$$\delta^i{}_j = \mathrm{EVAL}(\omega^i, e_j) = \mathrm{IDENTITY}\,(\omega^i, e_j)$$

of a tensor which does not depend on the choice of basis, i.e., has the same components no matter what basis we choose, WHILE $\delta_{ij} = G(e_i, e_j)$ are the components of a given tensor G (independent of the choice of basis) but which change under a change of basis — alternatively the component values δ_{ij} in every choice of basis do not define a single tensor but a family of tensors.

<u>Remark</u>. Are $\delta^1{}_i$ the components of a covector?
Again it depends. Since $\omega^1 = \delta^1{}_i\,\omega^i$, these are the components of the first covector in our chosen basis, so if we change the basis ω^1 will no longer (in general) have such simple components in terms of the new basis, but still it defines a unique tensor, namely ω^1. On the other hand the numerical values $\delta^1{}_i$ define different covectors in different bases. One really needs to qualify our opening question so that one of these two interpretations is clear. Then we can answer the question.

■ The dot product, duality, and index shifting can be extended in a natural way to each space $T^{(p,q)}(\mathbb{R}^n)$ which is itself a Euclidean vector space isomorphic to $(\mathbb{R}^{n^{p+q}}, \cdot)$. Such tensors have $p+q$ indices, n choices for each so $(p+q)^n$ independent components. Listing them in a certain order establishes an isomorphism with $\mathbb{R}^{n^{p+q}}$, which has its own dot product. This dot product and index shifting corresponds exactly to the ones we have established for $\binom{p}{q}$-tensors.

For example, let $W = T^{(2,0)}(\mathbb{R}^n)$, with basis $\{e_i \otimes e_j\} \equiv \{E_{ij}\}$, and $T = T^{ij} e_i \otimes e_j = T^{ij} E_{ij}$. Instead of using an index, say $A, B, C, ..$ which runs from 1 to n^2, we can use the n^2 index pairs (i,j) to label the distinct basis vectors in W and also the components of vectors in W.

I claim the dual basis can be identified with $W^{ij} \equiv \omega^i \otimes \omega^j$ and the dual space with $W^* = T^{(0,2)}(\mathbb{R}^n)$

$$W^{ij}(E_{mn}) = [\omega^i \otimes \omega^j](e_m \otimes e_n) \equiv \omega^i(e_m)\,\omega^j(e_n) = \underbrace{\delta^i{}_m \delta^j{}_n}_{\equiv I^{ij}{}_{mn}}$$

Kronecker delta on W in this notation

$$= \begin{cases} 1 & \text{if } (i,j) = (m,n) \\ 0 & \text{otherwise.} \end{cases}$$

So $S = S_{ij} W^{ij}$ is a "covector" with the evaluation given by

$$S(T) = S_{ij} T^{ij}$$

comma to separate index pairs

Let us define $$E_{ij} \cdot E_{mn} = \underbrace{\delta_{im}\delta_{jn} \equiv \delta_{ij,mn}}_{\equiv \mathcal{G}_{ij,mn}} = \begin{cases} 1 \text{ if } (i,j) = (m,n) \\ 0 \text{ otherwise.} \end{cases}$$

Then this corresponds to an inner product tensor

$$\mathcal{G} = \mathcal{G}_{ij,mn} W^{ij} \otimes W^{mn}$$

$$\mathcal{G}(T, U) = \mathcal{G}_{ij,mn} T^{ij} U^{mn} = \delta_{im}\delta_{jn} T^{ij} U^{mn} = T_{mn} U^{mn} = T^{ij} U_{ij}$$

which is how we defined the inner product previously

Note that the "$\otimes$" in $\mathcal{G} = \mathcal{G}_{ij,mn}\, W^{ij} \otimes W^{mn}$ is the tensor product for W, not $\mathbb{R}^n$, since it is holding the "covectors" (wrt W) W^{ij} and W^{mn} apart until they accept "vector" (wrt W) arguments, but this distinction doesnt matter. The $\binom{0}{4}$-tensor

$$\mathcal{G} = \mathcal{G}_{ij,mn}\, \omega^i \otimes \omega^j \otimes \omega^m \otimes \omega^n$$

can be contracted against 2 $\binom{2}{0}$-tensors to yield a real number which is exactly $\mathcal{G}(T,U) = \mathcal{G}_{ij,mn}\, T^{ij} U^{mn}$.

Our notation identifies these different interpretations. We just need to allow for "contraction" of any number of indices of a tensor with those of another

For example, what "contractions" are allowed between $T^{ij}{}_{k\ell m}$ and $S^{pq}{}_r$?

First define $[T \otimes S]^{ijpq}{}_{k\ell mn} = T^{ij}{}_{k\ell m}\, S^{pq}{}_r$

we can then contract any subset of contravanant indices with any subset of covanant indices of the same number, to yield tensors of vanous ranks less than $5+3$.

$T^{ik}{}_{j\ell m}\, S^{\ell m}{}_k \sim$ ([illegible])-tensor, for example.

Furthermore index shifting on W corresponds to index shifting of pairs of indices with the inner product G on $\mathbb{R}^n$

$$T = T^{ij} E_{ij} \to T^\flat = T_{ij} W^{ij}$$

$$T_{ij} = \mathcal{G}_{ij,mn}\, T^{mn} = \delta_{im}\delta_{jn}\, T^{mn} \quad \text{as defined above.}$$

We can repeat this for $W = \underset{\substack{\cup \\ T}}{T^{(p,q)}(\mathbb{R}^n)}$ and $W^* = \underset{\substack{\cup \\ S}}{T^{(q,p)}(\mathbb{R}^n)}$

$$T(S) = T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}\, S^{j_1 \cdots j_q}{}_{i_1 \cdots i_p}, \text{ etc.}$$

Looks like I snuck in a few new thoughts on ya.

Anyway, our extended problem with $V = \{n\times n \text{ matrices}\}$ beginning on page 43, develops matrix operations relevant to linear transformations $\sim \binom{1}{1}$-tensors instead of those relevant to inner products $\sim \binom{0}{2}$-tensors and more general linear maps from $\mathbb{R}^n$ into its dual space.

The correspondence $\qquad \underline{A} = A^i{}_j\, e^j{}_i \longrightarrow A = A^i{}_j\, \omega^j \otimes e_i$

maps the standard basis $\{e^j{}_i\}$ onto the basis $\{E^j{}_i\} = \{\omega^j \otimes e_i\}$ of $T^{(1,1)}(\mathbb{R}^n)$. $\qquad$ Now we have indices in the space V which are "1 up, 1 down" index pairs.

We'll discuss this in class.

Discussion of Problem Beginning on page 43.

(i) If $\underline{A} = A^m{}_n e^n{}_m$, then $\omega^i{}_j(\underline{A}) = \omega^i{}_j(A^m{}_n e^n{}_m) = A^n{}_n \omega^i{}_j(e^n{}_m) = A^i{}_j$, with $\underbrace{\omega^i{}_j(e^n{}_m)}_{\delta^i{}_m \delta^n{}_j}$

(ii) $AB = (A^j{}_i e^i{}_j)(B^n{}_m e^m{}_n) = A^j{}_i B^n{}_m \underbrace{e^i{}_j e^m{}_n}_{\delta^i{}_n e^m{}_j} = \underbrace{A^j{}_i B^i{}_m}_{[AB]^j{}_m = A^j{}_i B^i{}_m \checkmark} e^m{}_j$

(iii) — (iv)

$\mathcal{G}(A,B) = \mathrm{Tr}\,AB = \mathrm{Tr}[\tfrac{1}{2}(A+A^T)B] = \tfrac{1}{2}\mathrm{Tr}\,AB + \tfrac{1}{2}\underbrace{\mathrm{Tr}\,A^T B} \qquad = 0$

$G(A,B) = \mathrm{Tr}\,A^T B$ $\qquad$ or $\boxed{\mathrm{Tr}\,A^T B = \mathrm{Tr}(A^T B)^T = \mathrm{Tr}\,B^T A = -\mathrm{Tr}\,BA = -\mathrm{Tr}\,AB \to \mathrm{Tr}\,AB = 0}$

$\mathrm{Tr}(A^T B)^T$

$\mathrm{Tr}\,B^T A = \mathrm{Tr}\,AB^T$

$= -\mathrm{Tr}\,AB$

$\mathcal{G}(e^i{}_j, e^m{}_n)$

(v) $\mathrm{Tr}\, e^i{}_j e^m{}_n = \mathrm{Tr}\,\delta^i{}_n e^m{}_j = \delta^i{}_n \underbrace{\mathrm{Tr}\,e^m{}_j}_{\delta^m{}_j \text{ (think)}} = \delta^i{}_n \delta^m{}_j = \cancel{\quad} \neq \delta^{im}$ nope δ_{jn}

$\mathrm{Tr}(e^i{}_j)^T e^m{}_n = \mathrm{Tr}\, e^j{}_i e^m{}_n = \delta^i{}_n \mathrm{Tr}\, e^m{}_i = \delta^j{}_n \delta^m{}_i = \cancel{\delta_{jn}}\, \delta^{im} \delta_{jn} \checkmark$

$G(e^i{}_j, e^m{}_n) = \begin{cases} 1 & \text{if } (i,j) = (m,n) \\ 0 & \text{otherwise} \end{cases}$

(vi) Because $A - A^T = 0$ or $B + B^T = 0$ are linear conditions on the entries of the matrix and so are preserved under linear combination.

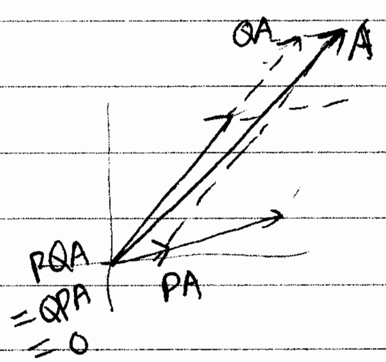
$A^1{}_1 \; A^1{}_2 \; A^1{}_3 \cdots A^1{}_n$ | n

$A^2{}_1$ | $\vdots$, 3, 2

$A^n{}_n$ | 1

above main diagonal plus main diagonal $\rightarrow \sum_{i=1}^{n} i$ for symmetric matrices

without main diagonal $\rightarrow \left(\sum_{i=1}^{n} i\right) - n$ for antisymmetric matrices.

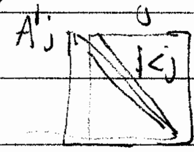


doesn't have to be orthogonal

(vii) $\sum_{i<j} 2^{-1}(A^i{}_j + A^j{}_i)(e^j{}_i + e^i{}_j) + \sum_{i<j} 2^{-1}(A^i{}_j - A^j{}_i)(e^j{}_i - e^i{}_j)$

$\sum_{i=j} A^i{}_j e^j{}_i$

~~$A^i{}_j e^j{}_i - A^j{}_i e^j{}_i - A^i{}_j e^i{}_j + A^j{}_i e^i{}_j$~~



$= \underbrace{\tfrac{1}{2}(A^i{}_j - A^j{}_i)}_{(\mathrm{ALT}\,A)^i{}_j} \underbrace{e^j{}_i}_{\text{upper }\Delta\text{ part}} + \underbrace{\tfrac{1}{2}(A^j{}_i - A^i{}_j)}_{(\mathrm{ALT}\,A)^j{}_i} \underbrace{e^i{}_j}_{\text{lower }\Delta\text{ part}}$

$\underbrace{\tfrac{1}{2}(A^i{}_j + A^j{}_i)}_{[\mathrm{SYM}\,A]^i{}_j} \underbrace{e^j{}_i}_{\text{upper }\Delta\text{ part}} + \underbrace{\tfrac{1}{2}(A^j{}_i + A^i{}_j)}_{(\mathrm{SYM}\,A)^j{}_i} \underbrace{e^i{}_j}_{\text{lower }\Delta\text{ part}}$ ✓

$j<i,\ n<m$:

$\mathcal{G}(\hat{E}^i{}_j, \hat{E}^m{}_n) = \mathrm{Tr}\, \hat{E}^i{}_j \hat{E}^m{}_n = \tfrac{1}{2} \mathrm{Tr}(e^i{}_j + e^j{}_i)(e^m{}_n + e^n{}_m)$

$= \tfrac{1}{2}[\underbrace{\mathrm{Tr}\, e^i{}_j e^m{}_n}_{\delta^i{}_n \mathrm{Tr}\, e^m{}_j} + \underbrace{\mathrm{Tr}\, e^j{}_i e^m{}_n}_{\delta^j{}_n \mathrm{Tr}\, e^m{}_i} + \underbrace{\mathrm{Tr}\, e^i{}_j e^n{}_m}_{\delta^i{}_m \mathrm{Tr}\, e^n{}_j} + \underbrace{\mathrm{Tr}\, e^j{}_i e^n{}_m}_{\delta^j{}_m \mathrm{Tr}\, e^n{}_i}]$

$= \tfrac{1}{2}(\delta^i{}_n \delta^m{}_j + \delta^j{}_n \delta^m{}_i + \delta^i{}_m \delta^n{}_j + \delta^j{}_m \delta^n{}_i)$

$= \delta^{im}\delta_{nj} + \underbrace{\delta^i{}_n \delta^m{}_j}_{0 \text{ if } j<i,\ n<m}$ ~~$i \neq j \quad m \neq n$~~

[if $i=n$ and $m=j$ then $j<i$ means $m<n$, can't happen]

$\mathcal{G}(\check{E}^i{}_j, \check{E}^m{}_n) = \mathrm{Tr}\, \check{E}^i{}_j \check{E}^m{}_n = \tfrac{1}{2} \mathrm{Tr}(e^i{}_j - e^j{}_i)(e^m{}_n - e^n{}_m)$

$= \tfrac{1}{2}[\mathrm{Tr}\, e^i{}_j e^m{}_n - \mathrm{Tr}\, e^j{}_i e^m{}_n - \mathrm{Tr}\, e^i{}_j e^n{}_m + \mathrm{Tr}\, e^j{}_i e^n{}_m]$

$= \cdots = \tfrac{1}{2}[\delta^i{}_n \delta^m{}_j - \delta^j{}_n \delta^m{}_i - \delta^i{}_m \delta^n{}_j + \delta^j{}_m \delta^n{}_i]$

$= -\delta^{im}\delta_{jn} + \underbrace{\delta^i{}_n \delta^m{}_j}_{0 \text{ as above.}}$

$\mathcal{G}(\hat{E}^i{}_j, \check{E}^m{}_n) = 0$ since $\mathrm{Tr}\, AB = 0$ if A sym, B antisym

(viii) $f(\underline{A}) = f^i{}_j \omega^j{}_i(\underline{A}) = f^i{}_j A^j{}_i \equiv \mathrm{Tr}\, \underline{F}\underline{A}$

(viii) $\mathcal{G}^i{}_j{}^m{}_n = \mathcal{G}(e^i{}_j, e^m{}_n) = \mathrm{Tr}\, e^i{}_j e^m{}_n = \mathrm{Tr}\, \delta^i{}_n e^m{}_j = \delta^i{}_n \delta^m{}_j$

$G^i{}_j{}^m{}_n = G(e^i{}_j, e^m{}_n) = \mathrm{Tr}\, e^j{}_i e^m{}_n = \mathrm{Tr}\, \delta^j{}_n e^m{}_i = \delta^j{}_n \delta^m{}_i = \delta^{im}\delta_{jn}.$

$\mathcal{G} = \delta^i{}_n \delta^m{}_j\, \omega^j{}_i \otimes \omega^n{}_m = \omega^j{}_i \otimes \omega^i{}_m = \mathrm{Tr}\, \underline{\omega} \otimes \underline{\omega}$

$G = \delta^{im}\delta_{jn}\, \omega^j{}_i \otimes \omega^n{}_m = \omega^j{}_i \otimes \omega^j{}_i = \mathrm{Tr}\, \underline{\omega}^T \otimes \underline{\omega}$

(x) Note also that the trace is a real valued linear function on V, i.e., a covector:

$$\mathrm{Tr}\, e^i{}_j = \delta^i{}_j \quad \to \quad \mathrm{Tr} = \delta^i{}_j\, \omega^j{}_i = \omega^i{}_i = \mathrm{Tr}\, \underline{\omega}\,.$$

(xi) ~~only if p=1, since it is~~

only if $p=1$ since it is not linear with more than 1 factor:

$$\det\,(A+B)(C) \neq \det AC + \det BC\,.$$

If you have trouble with indices, look at $n=2$, $n=3$ cases.

$n=2$ $\quad \{e^1{}_1, e^2{}_1, e^1{}_2, e^2{}_2\} = \left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$

$$\mathrm{Tr}\, e^1{}_1 = \mathrm{Tr}\, e^2{}_2 = 1\,, \quad \mathrm{Tr}\, e^2{}_1 = \mathrm{Tr}\, e^1{}_2 = 0\,.$$

$$\hat{E}^1{}_2 = 2^{-1/2}\begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad \check{E}^1{}_2 = 2^{-1/2}\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

$$\mathrm{Tr}(\hat{E}^1{}_2)^2 = \frac{1}{2}\,\mathrm{Tr}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \frac{1}{2}\,\mathrm{Tr}\begin{pmatrix}1&0\\0&1\end{pmatrix} = 1$$

$$\mathrm{Tr}(\check{E}^1{}_2)^2 = \frac{1}{2}\,\mathrm{Tr}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix} = \frac{1}{2}\,\mathrm{Tr}\begin{pmatrix}-1&0\\0&-1\end{pmatrix} = -1. \qquad \text{etc.}$$

Remark: another look at the generalized Kronecker deltas

$$\delta^{i_1\cdots i_p}{}_{j_1\cdots j_p} = \begin{cases} 0 \text{ if } (i_1,\ldots,i_p) \text{ is not a permutation of } (j_1,\ldots,j_p) \\ \operatorname{sgn}\sigma \text{ if } (i_1,\ldots,i_p) = (\sigma(j_1),\ldots,\sigma(j_p)) \text{ is such a permutation} \end{cases}$$

This symbol allows us to eliminate the $\sum_\sigma$ notation for the antisymmetrization operation

$$T_{[i_1\cdots i_p]} \overset{(a)}{=} \frac{1}{p!}\sum_\sigma \operatorname{sgn}\sigma\, T_{i_{\sigma(1)}\cdots i_{\sigma(p)}}$$ ← here we are permuting the index labels $(1,\ldots,p)$

$$\overset{(b)}{=} \frac{1}{p!}\sum_\pi \operatorname{sgn}\pi\, T_{\pi(i_1)\cdots\pi(i_p)}$$ ← here we are permuting the index values themselves, regardless of their values

$$\overset{(c)}{=} \frac{1}{p!}\,\delta^{j_1\cdots j_p}{}_{i_1\cdots i_p}\, T_{j_1\cdots j_p}$$

⇓

This point is tricky; an example helps understand the relation of (a) and (b):

$$T_{[ijk]} \overset{(a)}{=} \tfrac{1}{6}\left(T_{ijk}+T_{jki}+T_{kij}-T_{ikj}-T_{jik}-T_{kji}\right)$$

$$T_{[124]} = \tfrac{1}{6}\left(T_{124}+T_{241}+T_{412}-T_{142}-T_{214}-T_{421}\right)$$

$$T_{[111]} = \tfrac{1}{6}\left(T_{111}+T_{111}+T_{111}-T_{111}-T_{111}-T_{111}\right)=0$$

We can re-interpret the permutations of the ~~index~~ indices as a permutation of the index values themselves, which are any p integers, not necessarily distinct, chosen from $(1,\ldots,n)$.

In other words, we should have introduced the permutation group of n objects more abstractly, not as the concrete realization as permutations of the first n positive integers.

The equality (c) follows immediately from (b) since only the permutations of $(i_1,\ldots,i_p)$ enter the sum (with nonzero contribution) over the j indices, and the coefficient is the desired sign.

The Wedge Product: the "obvious" antisymmetrized tensor product

It only becomes obvious AFTER you understand it. For each integer value $0 \le p \le n$, we have p-vectors (antisymmetric $\binom{p}{0}$-tensors) and p-covectors (antisymmetric $\binom{0}{p}$-tensors) also called p-forms, where $\binom{0}{0}$-tensors are just scalars, i.e., real numbers.

Consider covariant antisymmetric tensors, i.e., $\mathrm{ALT}\, T = T$:

$$T = T_{i_1 \dots i_p}\, \omega^{i_1} \otimes \dots \otimes \omega^{i_p} \quad , \qquad T_{[i_1 \dots i_p]} = T_{i_1 \dots i_p}$$

$$= T_{[i_1 \dots i_p]}\, \omega^{i_1} \otimes \dots \otimes \omega^{i_p} = \frac{1}{p!} T_{j_1 \dots j_p} \underbrace{\delta^{j_1 \dots j_p}{}_{i_1 \dots i_p}\, \omega^{i_1} \otimes \dots \otimes \omega^{i_p}}_{\substack{= p!\, \omega^{[i_1} \otimes \dots \otimes \omega^{i_p]} \\ = \omega^{i_1 \dots i_p}}}$$

$$\left(= T_{i_1 \dots i_p}\, \omega^{[i_1} \otimes \dots \otimes \omega^{i_p]} \right)$$

Notice how antisymmetrizing on the lower indices is equivalent to antisymmetrizing on the upper indices. The tensors $\{\omega^{i_1 \dots i_p}\}_{i_1 < \dots < i_p}$ is a basis for the space of p-forms, but since ordered sums are inconvenient (more notation), we sum over all indices and divide by $p!$ to compensate.

Now $\omega^{i_1 \dots i_p}$ is itself an antisymmetrized tensor product of covectors, multiplied by a counting factor. Why is the counting factor (namely $p!$) included? Well, if $V = \mathbb{R}^n$ and $p = n$, then we saw above that

$$\det = \epsilon_{i_1 \dots i_n}\, \omega^{i_1} \otimes \dots \otimes \omega^{i_n} = \omega^{1 \dots n} \quad \text{is the determinant function,}$$

which is more interesting than the determinant function divided by $n!$, which is equal to the antisymmetrized tensor product with no counting factor modifying it.

It turns out to be useful to introduce an antisymmetrized tensor product, modified by some counting factor coefficient, of any number of factors which are themselves antisymmetric tensors of a given index level (all covariant or all contravariant so that we can take the antisymmetric part).

For example, $S_{ijk} = T_{ij} f_k$ are the components of $S = T \otimes f$ which are clearly antisymmetric in (i,j) if T is antisymmetric but not in all three indices. However, $\mathrm{ALT}\, S = \mathrm{ALT}\, T \otimes f$ with components $S_{[ijk]} = T_{[ij} f_{k]}$ is antisymmetric.

We would like to introduce a new product called the wedge product "$\wedge$" so that we can write $\quad T \wedge f = (\text{some factor})\ \mathrm{ALT}(T \otimes f)$, where the factor is chosen conveniently.

Suppose we make the definition

$$\text{(I)} \quad \omega^{i_1} \wedge \dots \wedge \omega^{i_p} \equiv p!\, \omega^{[i_1} \otimes \dots \otimes \omega^{i_p]} = \delta^{i_1 \dots i_p}_{j_1 \dots j_p}\, \omega^{j_1} \otimes \dots \otimes \omega^{j_p}$$
$$= \omega^{i_1 \dots i_p}$$

and extend this by linearity to the wedge product of ~~any~~ p covectors

$$f^{(1)} \wedge \dots \wedge f^{(p)} = (f^{(1)}{}_{i_1} \omega^{i_1}) \wedge \dots \wedge (f^{(p)}{}_{i_p} \omega^{i_p}) = f^{(1)}{}_{i_1} \cdots f^{(p)}{}_{i_p}\, \omega^{i_1} \wedge \dots \wedge \omega^{i_p}$$
$$= f^{(1)}{}_{i_1} \cdots f^{(p)}{}_{i_p}\, \omega^{i_1 \cdots i_p}$$
$$= f^{(1)}{}_{[i_1} \cdots f^{(p)}{}_{i_p]}\, \omega^{i_1 \cdots i_p}$$
$$= \frac{1}{p!} \underbrace{\left\{ p!\, f^{(1)}{}_{[i_1} \cdots f^{(p)}{}_{i_p]} \right\}}_{[f^{(1)} \wedge \dots \wedge f^{(p)}]_{i_1 \dots i_p}} \omega^{i_1 \dots i_p}$$

only antisymmetric part contributes to sum

[explanation: $\omega^{i_1 \dots i_p} = \omega^{[i_1 \dots i_p]} = \frac{1}{p!} \delta^{i_1 \dots i_p}_{j_1 \dots j_p}\, \omega^{j_1 \dots j_p}$ since antisymmetric,

so $f^{(1)}{}_{i_1} \cdots f^{(p)}{}_{i_p}\, \omega^{i_1 \dots i_p} = \underbrace{f^{(1)}{}_{i_1} \cdots f^{(p)}{}_{i_p} \frac{1}{p!} \delta^{i_1 \dots i_p}_{j_1 \dots j_p}}_{f^{(1)}{}_{[i_1} \cdots f^{(p)}{}_{i_p]}} \omega^{j_1 \dots j_p}$]

With this definition, then for the case $p = n$, the single independent component $\quad [f^{(1)} \wedge \dots \wedge f^{(n)}]_{1 \dots n} = n!\, f^{(1)}{}_{[1} \cdots f^{(n)}{}_{n]} = \det \begin{pmatrix} f^{(1)} \\ \vdots \\ f^{(n)} \end{pmatrix}$

is just the determinant of the matrix whose rows are the components of the covectors in this set.

Similarly for p-vectors we can define $e_{i_1} \wedge \ldots \wedge e_{i_p} = e_{i_1 \cdots i_p}$
and find that $[u_{(1)} \wedge \ldots \wedge u_{(n)}]^{1 \cdots n} = \det(\underline{u}_{(1)} \cdots \underline{u}_{(n)})$
For $V = \mathbb{R}^n$ this is just the volume of the parallellopiped they form.
So in each case the factorial factor eliminates an ugly counting factor to give something more interesting, namely the determinant.

However, we still dont know how to take the wedge product of higher rank antisymmetric tensors, our notation implicitly tells us how to do this since

$$\omega^{i_1 \cdots i_p j_1 \cdots j_q} = \omega^{i_1} \wedge \ldots \wedge \omega^{i_p} \wedge \omega^{j_1} \wedge \ldots \wedge \omega^{j_q} = \omega^{i_1 \cdots i_p} \wedge \omega^{j_1 \cdots j_q}$$

suggests how to wedge two basis tensors together in a way consistent with the notation. This can then be extended by linearity to any two antisymmetric tensors

$$T \wedge S = \left(\tfrac{1}{p!} T_{i_1 \cdots i_p} \omega^{i_1 \cdots i_p}\right) \wedge \left(\tfrac{1}{q!} S_{j_1 \cdots j_q} \omega^{j_1 \cdots j_q}\right)$$

$$= \frac{1}{p!q!} T_{i_1 \cdots i_p} S_{j_1 \cdots j_q} \underbrace{\omega^{i_1 \cdots i_p} \wedge \omega^{j_1 \cdots j_q}}_{\omega^{i_1 \cdots i_p j_1 \cdots j_q}}$$

$$= \frac{1}{p!q!} T_{[i_1 \cdots i_q} S_{j_1 \cdots j_q]} \omega^{i_1 \cdots i_p j_1 \cdots j_q}$$

only antisymmetric part contributes

$$= \frac{1}{(p+q)!} \underbrace{\left[\frac{(p+q)!}{p!q!} T_{[i_1 \cdots i_q} S_{j_1 \cdots j_q]} \right]} \omega^{i_1 \cdots i_p j_1 \cdots j_q}$$

II
$$\left\{ \begin{aligned} [T \wedge S]_{i_1 \cdots i_q j_1 \cdots j_q} &= \frac{(p+q)!}{p!q!} T_{[i_1 \cdots i_q} S_{j_1 \cdots j_q]} \\ T \wedge S &= \frac{(p+q)!}{p!q!} \, \mathrm{ALT}(T \otimes S) \end{aligned} \right.$$

In exactly the same way we could have partitioned the indices into 3 (or more) subsets and found

$$\underbrace{T}_{p\text{-form}} \wedge \underbrace{S}_{q\text{-form}} \wedge \underbrace{R}_{r\text{-form}} = \frac{(p+q+r)!}{p!q!r!} \mathrm{ALT}(T \otimes S \otimes R)$$

and so on (the pattern is clear).

Remark (aside)

Our notation assumes the wedge product is associative since no parentheses are necessary to evaluate $T\wedge S\wedge R$. Is this consistent? Do we have $T\wedge S\wedge R = (T\wedge S)\wedge R = T\wedge(S\wedge R)$? (III)

Yes, we've defined it to be true, but let's check as an exercise.

$$\underbrace{(T\wedge S)}_{p+q}\wedge \underbrace{R}_{r} = \frac{((p+q)+r)!}{(p+q)!\,r!} ALT[\underbrace{(T\wedge S)}_{\frac{(p+q)!}{p!\,r!} ALT(T\otimes S)}\otimes R] = \frac{(p+q+r)!}{p!\,q!\,r!} ALT[ALT(T\otimes S)\otimes R]$$

$$\underbrace{T}_{p}\wedge(\underbrace{S\wedge R}_{q+r}) = \frac{(p+(q+r))!}{p!\,(q+r)!} ALT[T\otimes\underbrace{(S\wedge R)}_{\frac{(q+r)!}{q!\,r!}ALT(S\otimes R)}] = \frac{(p+q+r)!}{p!\,q!\,r!} ALT[T\otimes ALT(S\otimes R)]$$

Thus the second equality of (III), given the formula (II), is equivalent to

$$ALT[ALT(T\otimes S)\otimes R] = ALT[T\otimes ALT(S\otimes R)] \quad \text{(IV)}.$$

If we had defined the wedge product by (II) as is often (usually) done, then we would need to verify (IV) in order to show that it is an associate operation, i.e., to prove (III). Let's just check that (IV) is indeed true.

$$\{ALT[ALT(T\otimes S)\otimes R]\}_{i_1\cdots i_p j_1\cdots j_q k_1\cdots k_r}$$

$$= \underbrace{\frac{1}{(p+q+r)!}\,\delta^{m_1\cdots m_p n_1\cdots n_q l_1\cdots l_r}_{i_1\cdots i_p j_1\cdots j_q k_1\cdots k_r}\,\underbrace{[ALT(T\otimes S)]_{m_1\cdots m_p n_1\cdots n_q}}_{\frac{1}{(p+q)!}\delta^{a_1\cdots a_p b_1\cdots b_q}_{m_1\cdots m_p n_1\cdots n_q}}}\, R_{l_1\cdots l_r}\ T_{a_1\cdots a_p}S_{b_1\cdots b_q}$$

$$\text{(by definition)} \equiv \delta^{[a_1\cdots a_p b_1\cdots b_q] l_1\cdots l_r}_{i_1\cdots i_p j_1\cdots j_q k_1\cdots k_r}$$

$$= \delta^{a_1\cdots a_p b_1\cdots b_q l_1\cdots l_r}_{i_1\cdots i_p j_1\cdots j_q k_1\cdots k_r}$$

since the Kronecker delta is already antisymmetric in $a_1\cdots a_p b_1\cdots b_q$, antisymmetrizing does nothing

$$= \frac{1}{(p+q+r)!}\,\delta^{a_1\cdots a_p b_1\cdots b_q l_1\cdots l_r}_{i_1\cdots i_p j_1\cdots j_q k_1\cdots k_r}\, T_{a_1\cdots a_p} S_{b_1\cdots b_q} R_{l_1\cdots l_r}.$$

Still the Remark

Exercise: Repeat for the righthand side of (IV) to obtain the same expression and therefore prove equality.

These factorials are really a nuisance right? Right. They come from summing over all indices rather than ordered indices. If we agree only to sum over ordered indices, they disappear! Recall our double vertical bar notation of page 61

$$T = \frac{1}{p!} T_{i_1 \cdots i_p} \omega^{i_1 \cdots i_p} = \sum_{i_1 < \cdots < i_p} T_{i_1 \cdots i_p} \omega^{i_1 \cdots i_p} \equiv T_{i_1 \cdots i_p} \omega^{|i_1 \cdots i_p|}$$
$$\equiv T_{|i_1 \cdots i_p|} \omega^{i_1 \cdots i_p}$$

This just tells us to only sum over those p-tuplets $(i_1, \ldots, i_p)$ whose values are ordered. Using this notation

$$[T \wedge S]_{i_1 \cdots i_p j_1 \cdots j_q} = \frac{(p+q)!}{p!\,q!} T_{[i_1 \cdots i_p} S_{j_1 \cdots j_q]} = \frac{1}{p!\,q!} \delta^{m_1 \cdots m_p n_1 \cdots n_q}_{i_1 \cdots i_p j_1 \cdots j_q} T_{m_1 \cdots m_p} S_{n_1 \cdots n_q}$$
$$= \delta^{m_1 \cdots m_p n_1 \cdots n_q}_{i_1 \cdots i_p j_1 \cdots j_q} T_{|m_1 \cdots m_p|} S_{|n_1 \cdots n_q|}$$

and similarly

$$[T \wedge S \wedge R]_{i_1 \cdots i_p j_1 \cdots j_q k_1 \cdots k_r}$$
$$= \delta^{m_1 \cdots m_p n_1 \cdots n_q l_1 \cdots l_r}_{i_1 \cdots i_p j_1 \cdots j_q k_1 \cdots k_r} T_{|m_1 \cdots m_p|} S_{|n_1 \cdots n_q|} R_{|l_1 \cdots l_q|} .$$

NO FACTORIALS IF YOU DON'T OVERCOUNT IN SUMS.

EXAMPLE: $n=3,\ p=2,\ q=1$

$$[T \wedge f]_{ijk} = \delta^{mnl}_{ijk} T_{|mn|} f_l = \delta^{231}_{ijk} T_{23} f_1 + \delta^{132}_{ijk} T_{13} f_2 + \delta^{123}_{ijk} T_{12} f_3$$

$$[T \wedge f]_{123} = \underbrace{T_{23} f_1 - T_{13} f_2 + T_{12} f_3}_{\text{"ordered sum" has alternating sign}} = \underbrace{T_{23} f_1 + T_{31} f_2 + T_{12} f_3}_{\text{"cyclic sum" has all positive signs.}}$$

Exercise. Suppose $U = (1,2,3) = u^i e_i$ and $V = (-1,1,2) = v^i e_i$ on $\mathbb{R}^3$

(i) What are the three independent components
$$(U\wedge V)^{23},\ (U\wedge V)^{31},\ (U\wedge V)^{12} \quad ?$$

(ii) If $W = (1,1,1)$, what is the single independent component of $U\wedge V\wedge W$?

(iii) Suppose $B = B^1 e_{23} + B^2 e_{31} + B^3 e_{12}$ and $E = E^1 e_1 + E^2 e_2 + E^3 e_3$
What is $[B\wedge E]^{123}$?

Exercise. On $\mathbb{R}^4$, simplify
$$(B^1 e_{234} + B^2 e_{314} + B^3 e_{124}) \wedge (E^1 e_1 + E^2 e_2 + E^3 e_3).$$

Exercise.
Simplify the following $(n=6)$, expressing as a linear combination of ordered basis tensors

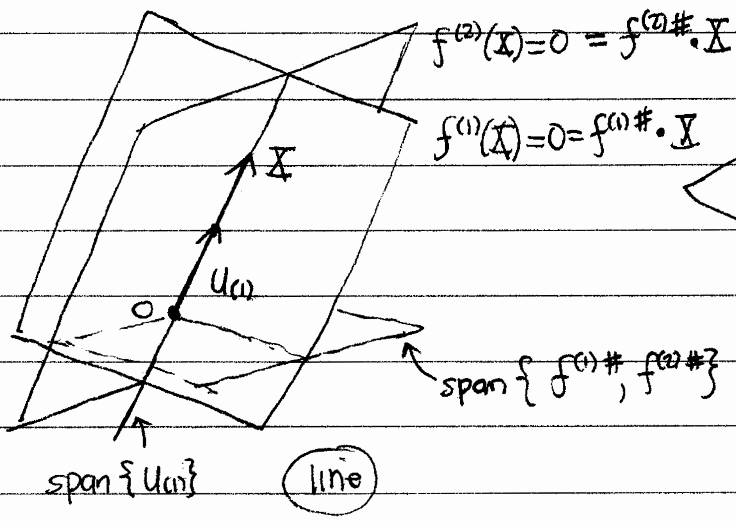
(a) $e_3 \wedge e_5 \wedge e_{24}$
(b) $e_2 \wedge e_3 \wedge e_{62}$
(c) $e_1 \wedge (e_{14} + e_{64})$
(d) $(e_1 + 3e_4 - e_6) \wedge (2e_{23} + e_{36}) \wedge e_{45}$
(e) $(e_{12} + e_{13}) \wedge (e_{34} + e_{25}) \wedge (e_{56} + e_{46})$.

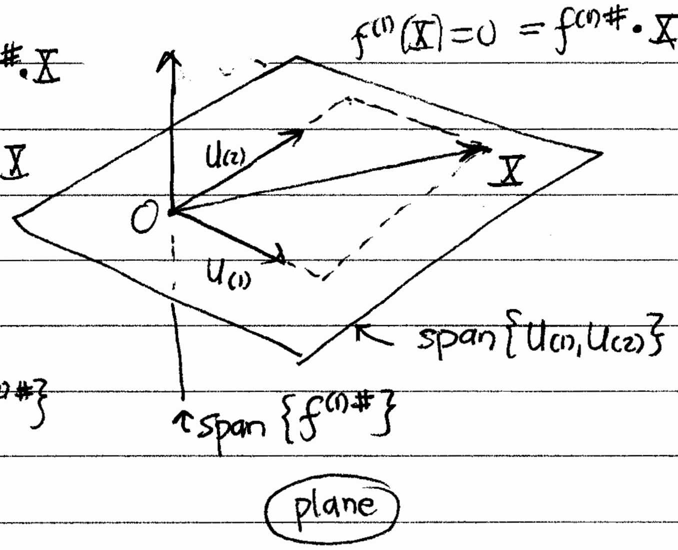
Exercise.
$$N = N^1 e_{234} - N^2 e_{134} + N^3 e_{124} - N^4 e_{123}$$
$$X = x^\alpha e_\alpha$$
what is $X\wedge N$?

SUBSPACE ORIENTATION and ANTISYMMETRIC TENSORS

If $\{u_{(1)},\ldots,u_{(p)}\}$ is a collection of p vectors in V, then span $\{u_{(1)},\ldots,u_{(p)}\}$ is the set of all possible linear combinations of these vectors — yielding a vector or linear subspace of V, whose dimension is p if the set is linearly independent. We can think of such a subspace as a "p-plane" through the origin. We would like to describe the orientation or "direction" of the p-plane.

In $\mathbb{R}^3$ there are three ways to do this: two involve only linearity, while the third uses the Euclidean inner product. The nontrivial subspaces are lines and planes through the origin.





A subspace can be specified *EXPLICITLY* by giving a basis, which may be used to parametrize it, i.e., represent a point in the subspace as an arbitrary linear combination of the basis vectors, or *IMPLICITLY* as the simultaneous solution of a system of linear equations, ie, the intersection of the zero value level surfaces of a set of linearly independent covectors.

Consider the case of a plane, that of $\{u_{(1)},u_{(2)}\}$. Any two linearly independent combinations of this basis ~~are~~ just as good as the original two — either set specifies the same plane. The 2-vector $u_{(1)} \wedge u_{(2)}$ at most changes by a scalar multiple under such a change of basis

$$u_{(i)}' = A^j{}_i u_{(j)} \rightarrow \quad u_{(1)}' \wedge u_{(2)}' = A^i{}_1 A^j{}_2 u_{(i)} \wedge u_{(j)} = A^{[i}{}_1 A^{j]}{}_2 u_{(i)} \wedge u_{(j)}$$
$$= 2!\, A^{[1}{}_1 A^{2]}{}_2 u_{(1)} \wedge u_{(2)} = \det \underline{A}\; u_{(1)} \wedge u_{(2)}$$

In fact it just changes by the determinant of the matrix of the change of basis. The condition that X belong to the plane is equivalent to

$$X = a u_{(1)} + b u_{(2)} \longrightarrow$$
$$u_{(1)} \wedge u_{(2)} \wedge X = u_{(1)} \wedge u_{(2)} \wedge (a u_{(1)} + b u_{(2)}) = 0 \qquad \text{(repeated factors in each term)}$$

or in index notation $\quad u_{(1)}^{[i} u_{(2)}^{j} X^{k]} = 0$

or $\quad \underbrace{\epsilon_{ijk} u^i_{(1)} u^j_{(2)}}_{\equiv f^{(1)}{}_k \,=\, [\circledast\, u_{(1)} \wedge u_{(2)}]_k \,=\, \frac{1}{2} \epsilon_{ijk} [u_{(1)} \wedge u_{(2)}]^{ij}} X^k = 0$

This defines a covector which specifies the same plane implicitly. One can go backwards from the covector to the 2-vector

$$\underbrace{f^{(1)}{}_k \epsilon^{kmn}}_{[\circledast f^{(1)}]^{mn}} = \epsilon_{ijk} u^i_{(1)} u^j_{(2)} \epsilon^{kmn} = (-1)^2 \underbrace{\epsilon^{mnk} \epsilon_{ijk}}_{\delta^{mn}_{ij}} u^i_{(1)} u^j_{(2)}$$
$$= [u_{(1)} \wedge u_{(2)}]^{mn}$$

The map $\circledast$ is called the "natural dual" operation.

For the case of a line, by the same reason only $f^{(1)} \wedge f^{(2)}$ is needed to specify the orientation of the line, since any two linearly independent covectors which specify the line will have a wedge product differing only by a multiple of $f^{(1)} \wedge f^{(2)}$. The natural dual of this 2-covector defines a vector $\quad u^i_{(1)} = \frac{1}{2} \epsilon^{mni} [f^{(1)} \wedge f^{(2)}]_{mn} = \epsilon^{mni} f^{(1)}{}_m f^{(2)}{}_n = [\circledast\, f^{(1)} \wedge f^{(2)}]^i$

which lies along the line since $\quad [f^{(1)} \wedge f^{(2)}]_{mn} = \epsilon_{mni} u^i_{(1)} = [\circledast\, u_{(1)}]_{mn}$

but $\quad [f^{(1)} \wedge f^{(2)}]_{mn} X^n = 0 \quad$ for X along the line, hence

(since the contraction of X with either factor vanishes)

$$0 = [f^{(1)} \wedge f^{(2)}]_{mn} X^n \epsilon^{mij}$$

$$= \epsilon_{mn\ell}\, u^{\ell}_{(i)} X^n \epsilon^{mij} = \delta^{ij}_{n\ell}\, u^{\ell}_{(1)} X^n = -[u_{(1)} \wedge X]^{ij}$$

The wedge product of 2 vectors being zero means they are linearly dependent, i.e., along the same direction.

Thus in $\mathbb{R}^3$, a p-plane is specified by the wedge product of a basis of p-vectors or by the wedge product of $(n-p)$-linearly independent covectors which implicitly give the p-plane. The natural dual map ⊛ relates the p-vector and $(n-p)$-covector to each other.

By "raising the indices" on the covectors with the Euclidean metric one makes the change $\quad 0 = f^{(i)}(X) = f^{(i)\#} \cdot X$ converting the zero evaluation of the covector on the vector to a vanishing dot product. Thus X is orthogonal to each of the vectors obtained from the covectors in this way and to the entire $(n-p)$-plane they determine — which in turn is specified by the $(n-p)$-vector which is the wedge product of these vectors. This is the third way of specifying the orientation of a p-plane — by giving its orthogonal $(n-p)$-plane.

This also leads to a "metric dual" operation which is the natural dual followed by shifting all the indices to the opposite level, i.e., the same level as before the natural dual changed the index level. Thus from the vector specifying a line, we get the 2-vector specifying the orthogonal plane, while from the 2-vector specifying a plane we get a vector orthogonal to the plane.

The same statement applies to the various covectors and p-covectors. For the case of a line, from the 2-covector one gets a vector along the line by the natural dual and a covector by lowering its index. For a plane, from the covector specifying the plane one gets the 2-vector by the natural dual and finally a 2-covector by lowering its indices.

The natural dual takes p-vectors into $(n-p)$-covectors and viceversa, while the metric dual takes p-vectors into $(n-p)$-vectors and p-covectors into $(n-p)$-covectors. Note that because ϵ_{ijk} are not the components of a tensor, the natural dual depends on the choice of basis and changes ~~All of these features may be generalized to any vector space V.~~ by a scalar factor under a change of basis. This is okay since the overall scale of any of these tensors is irrelevant to the orientation of the subspaces. However, using the metric we can convert this to a duality operation "$*$" where the overall scale is fixed so that the magnitude of the p-vector determines its p-measure. Instead of using ϵ_{ijk} for the duality operation, i.e. $\frac{1}{3!}\epsilon_{ijk}\,\omega^{ijk} = \omega^{123}$, namely the basis 3-covector which of course changes with a change of basis, one can use the "unit" 3-vector which reduces to ω^{123} for any (oriented) orthonormal basis — in particular, for the standard basis of $\mathbb{R}^3$. In other words we fix $\eta = \omega^{123}$ in the standard basis of $\mathbb{R}^3$ and then one can express the fixed 3-covector η in any other basis by the transformation law which will involve the determinant of the transformation for a 3-covector.

$$\eta_{i'j'k'} = \underbrace{A^{-1m}{}_i\, A^{-1n}{}_j\, A^{-1\ell}{}_k\, \underbrace{\eta_{mn\ell}}_{\epsilon_{mn\ell} \text{ in standard basis}}}_{\epsilon_{ijk}\,(\det \underline{A}^{-1})}$$

As long as $\det \underline{A} = 1$, one will have $\eta_{i'j'k'} = \epsilon_{ijk}$ or $\eta = \omega^{1'2'3'}$, otherwise there will be a correction factor.

Suppose one takes an orthonormal basis of $\mathbb{R}^3$ (oriented as well) adapted to a subspace, i.e., $\{e_1, e_2, e_3\}$ where $\{e_1\}$ is a basis for a 1-dimensional subspace, or $\{e_1, e_2\}$ is a basis for a 2-dimensional subspace.
For the latter case $e_1 \wedge e_2$ specifies the plane and the metric dual $*(e_1 \wedge e_2) = e_3$ (we'll see this more easily below) will give a unit normal to the plane. For the line, $* e_1 = e_2 \wedge e_3$ will give the 2-vector which specifies the orthogonal 2-plane. In each case the "magnitude" of these tensors, divided by the usual overcounting factorial factor will give the p-measure of the p-parallelopiped formed by the basis vectors. The same will extend by linearity to any adapted basis.

This is just a motivation for giving the formulas for the general case.
For p-vectors and p-covectors, $0 \le p \le n$, define:

$$[\circledast T]_{i_{p+1} \cdots i_n} = \frac{1}{p!} T^{i_1 \cdots i_p} \epsilon_{i_1 \cdots i_p i_{p+1} \cdots i_n}$$ = contraction of T with first p indices of $\omega^{1\dots n}$ divided by $p!$

$$[\circledast S]^{i_{p+1} \cdots i_n} = \frac{1}{p!} S_{i_1 \cdots i_p} \epsilon^{i_1 \cdots i_p i_{p+1} \cdots i_n}$$ = contraction of S with first p indices of $e_{1\dots n}$ divided by $p!$

<u>Exercise:</u> use the identity

$$\delta^{i_1 \cdots i_p j_{p+1} \cdots j_n}_{j_1 \cdots j_p j_{p+1} \cdots j_n} = \epsilon^{i_1 \cdots i_p j_{p+1} \cdots j_n} \epsilon_{j_1 \cdots j_p j_{p+1} \cdots j_n} = (n-p)!\, \delta^{i_1 \cdots i_p}_{j_1 \cdots j_p}$$

and the permutation result

$$\epsilon^{\overbrace{j_{p+1} \cdots j_n}^{(n-p) \text{ transpositions to get over this group of indices}} \underbrace{i_1 \cdots i_p}_{p \text{ indices to move over}}} = (-1)^{p(n-p)} \epsilon^{i_1 \cdots i_p j_{p+1} \cdots j_n}$$

$= p(n-p)$ transpositions

to show that

$$\circledast \circledast T = (-1)^{p(n-p)} T \quad \text{for a p-vector } T.$$

What is $(-1)^{p(3-p)}$ for all values of p: $0 \le p \le 3$?
What is $(-1)^{p(4-p)}$ for all values of p: $0 \le p \le 4$?

Note that if $T = \frac{1}{p!} T^{i_1\cdots i_p} e_{i_1\cdots i_p}$ and

$$\circledast T = \frac{1}{(n-p)!} {}^{\circledast}T_{i_{p+1}\cdots i_n} \omega^{i_{p+1}\cdots i_n} = \frac{1}{p!}\frac{1}{(n-p)!} T^{i_1\cdots i_p} \epsilon_{i_1\cdots i_p i_{p+1}\cdots i_n} \omega^{i_{p+1}\cdots i_n}$$

then by linearity of the natural dual

$$\circledast \quad = \frac{1}{p!} T^{i_1\cdots i_p} \circledast e_{i_1\cdots i_p}$$

so equating the two expressions for ${}^{\circledast}T$ one gets

$$\circledast e_{i_1\cdots i_p} = \frac{1}{(n-p)!} \epsilon_{i_1\cdots i_p i_{p+1}\cdots i_n} \omega^{i_{p+1}\cdots i_n} = \epsilon_{i_1\cdots i_p |i_{p+1}\cdots i_n} \omega^{|i_{p+1}\cdots i_n|}$$

Similarly one finds

$$\circledast \omega^{i_1\cdots i_p} = \qquad \epsilon^{i_1\cdots i_p i_{p+1}\cdots i_n} e_{|i_{p+1}\cdots i_n|}.$$

EXERCISE. Verify the formula for $\circledast e_{i_1\cdots i_p}$ using the component relations $[e_{i_1\cdots i_p}]^{j_1\cdots j_p} = \delta^{j_1\cdots j_p}{}_{i_1\cdots i_p}$

$$[\omega^{1\cdots n}]_{j_1\cdots j_n} = \delta^{1\cdots n}_{j_1\cdots j_n} = \epsilon_{j_1\cdots j_n}$$

and the component formula for ${}^{\circledast}T$.

Like the wedge product in practice, the natural dual in practice is simple. For $n=3$ one has:

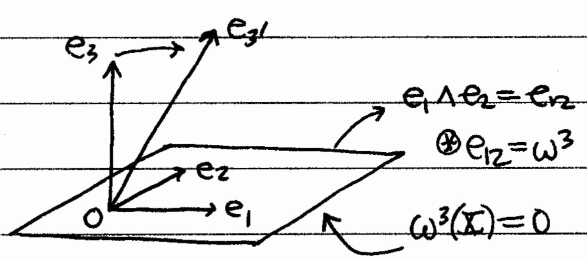
	p-vectors to p-covectors $\circledast : \Lambda^{(p)}(V) \to \Lambda^{(n-p)}(V)^*$	p-covectors to p-vectors $\circledast : \Lambda^{(p)}(V)^* \to \Lambda^{(n-p)}(V)$
$p=0$	$\circledast 1 = \epsilon_{123}\omega^{123} = \omega^{123}$	$\circledast 1 = \epsilon^{123} e_{123} = e_{123}$
	$\circledast e_1 = \epsilon_{123}\omega^{23} = \omega^{23}$	$\circledast \omega^1 = \epsilon^{123} e_{23} = e_{23}$
$p=1$	$\circledast e_2 = \epsilon_{231}\omega^{31} = \omega^{31}$	$\circledast \omega^2 = \epsilon^{231} e_{31} = e_{31}$
	$\circledast e_3 = \epsilon_{312}\omega^{12} = \omega^{12}$	$\circledast \omega^3 = \epsilon^{312} e_{12} = e_{12}$
	$\circledast e_{23} = \epsilon_{231}\omega^1 = \omega^1$	$\circledast \omega^{23} = \epsilon^{231} e_1 = e_1$
$p=2$	$\circledast e_{31} = \epsilon_{312}\omega^2 = \omega^2$	$\circledast \omega^{31} = \epsilon^{312} e_2 = e_2$
	$\circledast e_{12} = \epsilon_{123}\omega^3 = \omega^3$	$\circledast \omega^{12} = \epsilon^{123} e_3 = e_3$
$p=3$	$\circledast e_{123} = \epsilon_{123} = 1$	$\circledast \omega^{123} = \epsilon^{123} = 1$

EXERCISE: (i) If $n=3$ and $B = B_{23}\omega^{23} + B_{31}\omega^{31} + B_{12}\omega^{12}$, what is $^{\circledast}B$.
If $E = E^i e_i$, what is $E \wedge {}^{\circledast}B$?

(ii) If $n=4$, what is $^{\circledast}[\omega^{12} + \omega^{34}]$?
What is $(2e_{12} + 3e_{13} - e_{23}) \wedge {}^{\circledast}[\omega^{12} + \omega^{34}]$?
What is $^{\circledast}[e_{123} - e_{412} + 2e_{431}]$?

(iii) Repeat (i) for $n=4$.

A <u>decomposable</u> p-vector or p-covector is one which can be represented as the wedge product of p vectors or covectors.
An <u>adapted basis</u> of a vector space V, adapted to a p-dim subspace W, is a basis of V such that the first p basis vectors are a basis of W. Each adapted basis determines a direct sum of V into W and a complementary subspace which is the span of the last $(n-p)$ basis vectors. Although this changes with a change of adapted basis, the last $(n-p)$ dual basis covectors still determine the given subspace W.



In the $\mathbb{R}^3$ example in this diagram the plane of e_1 and e_2 is determined by $\omega^3(X) = 0$. If e_3 is changed to $e_{3'}$, ω^3 can at most change to $\omega^{3'} = c\,\omega^3$ to preserve $\omega^{3'}(e_{3'}) = 1$, but its orientation must stay the same to maintain $\omega^{3'}(e_1) = \omega^{3'}(e_2) = 0$.
In general if $\{e_1, \dots, e_p\}$ determine a p-dimensional subspace, then $*[e_1 \wedge \dots \wedge e_p] = \omega^{p+1 \dots n}$ can at most change by a determinant factor since $\{\omega^{p+1'}, \dots, \omega^{n'}\}$ must be linearly independent linear combinations of $\{\omega^{p+1}, \dots, \omega^n\}$ alone so that the duality relations giving zero along $\{e_{1'}, \dots, e_{p'}\}$ are preserved [these must be linear combinations of $\{e_1, \dots, e_p\}$ only to be an adapted basis].

Exercise. *Using the idea of an adapted basis, explain why the natural dual of a decomposable p-vector must itself be decomposable.*

Inner product for antisymmetric tensors

If we have an inner product on V, we have shown how to get an inner product on any of the tensor spaces over V. If T and S are both $\binom{p}{q}$-tensors, then their inner product is the scalar

$$G_{im}\cdots G^{jn}\cdots T^{i\cdots}{}_{j\cdots} S^{m\cdots}{}_{n\cdots} = T_{m\cdots}{}^{}_{j\cdots} S^{m\cdots j\cdots} = T^{i\cdots n\cdots} S_{i\cdots n\cdots}.$$

For antisymmetric tensors this overcounts the number of independent component terms in these sums, so it is natural to divide by the number of repetitions in the sum. For p-vectors and p-covectors, define

p-vectors: $$\langle T,S\rangle = \frac{1}{p!} G_{i_1j_1}\cdots G_{i_pj_p} T^{i_1\cdots i_p} S^{j_1\cdots j_p}$$
$$= T_{j_1\cdots j_p} S^{|j_1\cdots j_p|}$$

p-covectors: $$\langle T,S\rangle = \frac{1}{p!} G^{i_1j_1}\cdots G^{i_pj_p} T_{i_1\cdots i_p} S_{j_1\cdots j_p}$$
$$= T^{|j_1\cdots j_p|} S_{j_1\cdots j_p}.$$

For the Euclidean metric, the self inner product of a p-vector is just the sum of the squares of its ordered-indexed components.

Exercise. What is the self-inner product on $\mathbb{R}^3$ (Euclidean metric) of (i) $E^1e_1+E^2e_2+E^3e_3$ (ii) $B^1e_{23}+B^2e_{31}+B^3e_{12}$ (iii) $3\,e_{123}$? (iv) $F=F^{ij}e_{|ij|}$.

The unit n-form on an oriented vector space with inner product

On page 54 we used the alternating symbols to give a compact expression for the determinant of any matrix $\underline{A}$:

$$\det \underline{A} = \epsilon_{i_1 \cdots i_n} A^{i_1}{}_1 \cdots A^{i_n}{}_n$$

$$\det \underline{A}\ \epsilon_{j_1 \cdots j_n} = \epsilon_{i_1 \cdots i_n} A^{i_1}{}_{j_1} \cdots A^{i_n}{}_{j_n}$$

The index level of the alternating symbol is just a convenience here to use our summation convention. If we apply these to the matrix $\underline{G} = (G_{ij})$ of components of an inner product, we can write

$$\det \underline{G}\ \epsilon_{j_1 \cdots j_n} = \epsilon^{i_1 \cdots i_n} G_{i_1 j_1} \cdots G_{i_n j_n}$$

$$\det \underline{G}^{-1} \epsilon^{j_1 \cdots j_n} = \epsilon_{i_1 \cdots i_n} G^{i_1 j_1} \cdots G^{i_n j_n}$$

Obviously we get into trouble if we try to extend our index-shifting convention to the alternating symbol, since it is off by the determinant factor.

However, suppose we have an oriented vector space and define

$$\eta_{i_1 \cdots i_n} = \pm |\det \underline{G}|^{1/2} \epsilon_{i_1 \cdots i_n} \quad \begin{cases} + \text{ sign for an oriented basis} \\ - \text{ sign for oppositely oriented basis} \end{cases}$$

and define all other index positions for this object to be obtained by the usual rules for index raising. In particular

$$\begin{aligned} \eta^{i_1 \cdots i_n} &= G^{i_1 j_1} \cdots G^{i_n j_n} \eta_{j_1 \cdots j_n} \\ &= G^{i_1 j_1} \cdots G^{i_n j_n} \epsilon_{j_1 \cdots j_n} (\pm |\det \underline{G}|^{1/2}) \\ &= (\det \underline{G}^{-1})\, \epsilon^{i_1 \cdots i_n} \, (\pm |\det \underline{G}|^{1/2}) \\ &= \pm (\operatorname{sgn} \det \underline{G})\, |\det G|^{-1/2} \epsilon^{i_1 \cdots i_n} . \end{aligned}$$

I claim that $\eta = \eta_{|i_1 \cdots i_n|} \omega^{|i_1 \cdots i_n|} = \eta_{1 \cdots n}\, \omega^{1 \cdots n}$ is a uniquely defined tensor, independent of which particular basis we use to define it.

First recall $\quad G_{i'j'} = A^{-1\,m}{}_{i} A^{-1\,n}{}_{j}\, G_{mn} \leftrightarrow \underline{G}' = (\underline{A}^{-1})^T \underline{G}\, \underline{A}^{-1}$

so $\quad \det \underline{G}' = (\det A^{-1})^2 \det \underline{G} \quad$ (why?)

and $\quad |\det \underline{G}'|^{1/2} = (\operatorname{sgn} \det \underline{A}) \det \underline{A}^{-1} \; |\det \underline{G}|^{1/2}$.

According to the bottom of page 54, $\det \underline{G}$ acts like a scalar density of weight $W=2$, while $|\det \underline{G}|^{1/2}$ acts like a scalar density of weight $W=1$ except that it changes sign under a change of orientation ($\det A<0$), so it is called an "oriented" scalar density.

Next evaluate

$$A^{-1\,j_1}{}_{i_1} \cdots A^{-1\,j_n}{}_{i_n}\, \eta_{j_1 \cdots j_n}$$

$$= \underbrace{(\pm 1)\, |\det \underline{G}|^{1/2}\, \underbrace{\epsilon_{j_1 \cdots j_n} A^{-1\,j_1}{}_{i_1} \cdots A^{-1\,j_n}{}_{i_n}}_{(\det A^{-1})\, \epsilon_{i_1 \cdots i_n}}}_{(\operatorname{sgn} \det \underline{A})\, |\det \underline{G}'|^{1/2}} = \overbrace{(\operatorname{sgn} \det \underline{A})(\pm 1)}^{\text{orientation sign for new basis}} |\det \underline{G}'|^{1/2} \epsilon_{i_1 \cdots i_n} \equiv \eta_{i_1' \cdots i_n'}$$

If $\det A < 0$ this switches the orientation sign as it should, so in fact the transformation law for a $\binom{0}{n}$-tensor holds, i.e., the above component definition defines the same tensor for every choice of basis:

$$\eta = \frac{1}{n!}\, \eta_{i_1 \cdots i_n}\, \omega^{i_1 \cdots i_n} = \eta_{1 \cdots n}\, \omega^{1 \cdots n}$$

This is called the unit n-form for the oriented inner product vector space. It does two things: 1) it carries the orientation information, with $c\eta$ positively oriented if $c>0$ and negatively oriented if $c<0$
2) it measures n-volume by setting the scale as explained above.

An orthonormal basis with respect to a given inner product is one for which each basis vector is a unit vector (with sign ± 1 : $G_{ii} = G(e_i, e_i) = \pm 1$) orthogonal to the rest ($G_{ij} = 0$, $i \neq j$). The difference in the number of positive and negative signs is called the signature and is fixed for a given inner product (accept as a fact for now). A "positive-definite" inner product has all positive signs, i.e. signature n, while a "negative-definite" inner product has all negative signs, i.e., signature $-n$. An "indefinite" inner product has signature s between these two extreme values. A "Lorentz" inner product has only one negative sign or only one positive sign (the choice depends on prejudice) and so has signature $|s| = (n-1) - 1 = n-2$.

Useful observation: For an orthonormal basis, $|\det G|^{1/2} = 1$, so $\eta = \omega^{1\cdots n}$ if the basis is positively-oriented (same orientation as the chosen one) and $\eta = -\omega^{1\cdots n}$ otherwise.

[On $\mathbb{R}^n$ with the standard inner product and orientation, $\eta = \omega^{1\cdots n}$.]

Thus η is the n-covector which assigns unit volume to a unit hypercube—
~~the paralleloped formed by an orthonormal basis~~ oops!

So What? Well, now we can define a metric duality operation that has tensor character by using $\eta_{i_1 \cdots i_n}$ instead of $\epsilon_{i_1 \cdots i_n}$. We will obtain a unique tensor by taking the metric dual, independent of the choice of basis. This will automatically tell us both about p-measures for p-parallelopipeds in n-dimensional V, as well as orientation information that generalizes our "counterclockwise" orientation in a plane and its connection to the right handed normal in $\mathbb{R}^3$ (inner and outer orientations).

Metric Duality operation: $\begin{cases} * : \Lambda^{(p)}(V) \to \Lambda^{(n-p)}(V) \\ * : \Lambda^{(p)}(V)^* \to \Lambda^{(n-p)}(V)^* \end{cases}$

We modify the natural dual using $\eta_{i_1 \cdots i_n}$ in place of $\epsilon_{i_1 \cdots i_n}$ and then shift all the indices back to their original level using our inner product (metric).

p-vector: $[{}^*T]_{i_{p+1} \cdots i_n} = \frac{1}{p!} T_{i_1 \cdots i_p} \eta^{i_1 \cdots i_p}{}_{i_{p+1} \cdots i_n}$

p-covector: $[{}^*T]^{i_{p+1} \cdots i_n} = \frac{1}{p!} T^{i_1 \cdots i_p} \eta_{i_1 \cdots i_p}{}^{i_{p+1} \cdots i_n}$

which is equivalent to

$$\boxed{\begin{aligned} * e_{i_1 \cdots i_p} &= \eta_{i_1 \cdots i_p}{}^{i_{p+1} \cdots i_n} e_{|i_{p+1} \cdots i_n|} \\ * \omega^{i_1 \cdots i_p} &= \eta^{i_1 \cdots i_p}{}_{i_{p+1} \cdots i_n} \omega^{|i_{p+1} \cdots i_n|} \end{aligned}}$$

In an oriented orthonormal basis, then $\eta_{1 \cdots n} = 1$, and raising each index multiplies it by the sign of the corresponding basis vector.

example $\mathbb{R}^4$, standard basis, standard orientation, but inner product is instead $\underline{G} = \text{diag}(1,1,1,-1)$, i.e., e_4 has a negative sign.

Then $\eta_{1234} = 1 = \eta^1{}_{234} = \eta^{12}{}_{34} = -\eta_{123}{}^4 = -\eta_{12}{}^{34}$ etc

since $\underline{G}$ is diagonal, so $\eta_{123}{}^4 = \eta_{1234} G^{44} = -\eta_{1234}$, etc.

example $\mathbb{R}^3$, standard basis, inner product, orientation:

$*: \Lambda^{(p)}(V) \to \Lambda^{(n-p)}(V)$	$*: \Lambda^{(p)}(V)^* \to \Lambda^{(n-p)}(V)^*$
$*1 = e_{123}$	$*1 = \omega^{123}$
$*e_1 = e_{23}$	$*\omega^1 = \omega^{23}$
$*e_2 = e_{31}$	$*\omega^2 = \omega^{31}$
$*e_3 = e_{12}$	$*\omega^3 = \omega^{12}$
$*e_{23} = e_1$	$*\omega^{23} = \omega^1$
$*e_{31} = e_2$	$*\omega^{31} = \omega^2$
$*e_{12} = e_3$	$*\omega^{12} = \omega^3$
$*e_{123} = 1$	$*\omega^{123} = 1$

exercise: note that for each of the basis tensors T:

$$T \wedge {}^*T = e_{123}$$

This is no accident.

Has anyone noticed I've changed the spelling of parallelopiped?

Linear maps

Suppose V is an n-dimensional vector space with basis $\{e_i\}_{i=1,\dots,n}$ and W is an m-dimensional vector space with basis $\{E_\alpha\}_{\alpha=1,\dots,m}$. Let $\{\omega^i\}$ and $\{W^\alpha\}$ be the respective dual bases. Let $A: V \longrightarrow W$ be a linear map. Then by linearity

$$A(v) = A(v^i e_i) = v^i A(e_i),$$

i.e. the map is completely determined by its values on the basis vectors. For each i, $A(e_i) \in W$ can be expressed in terms of its components with respect to $\{E_\alpha\}$

$$A(e_i) = A^\alpha{}_i E_\alpha, \quad A^\alpha{}_i = W^\alpha(A(e_i)).$$

Thus

$$w = A(v) = v^i A^\alpha{}_i E_\alpha = \underbrace{[A^\alpha{}_i v^i]}_{w^\alpha} E_\alpha$$

becomes $\boxed{\begin{array}{l} w^\alpha = A^\alpha{}_i v^i \\ A(e_i) = A^\alpha{}_i E_\alpha \end{array}}$ in components, equivalent to on the basis vectors.

The matrix $\underline{A} = (A^\alpha{}_i)$ is called the matrix representation of A with respect to the bases $\{e_i\}$ and $\{E_\alpha\}$ of V and W respectively. If either basis changes, the matrix will change in an "obvious" way (obvious when you see it).

$$e_{i'} = B^{-1j}{}_i e_{j'}, \quad \omega^{i'} = B^i{}_j \omega^j$$

$$E_{\alpha'} = C^{-1\beta}{}_\alpha E_\beta, \quad W^{\alpha'} = C^\alpha{}_\beta W^\beta$$

$$A^{\alpha'}{}_{i'} = W^{\alpha'}(A(e_{i'})) = \cancel{B^{-1j}{}_i A(e_j)} \; C^\alpha{}_\beta W^\beta(A(e_j B^{-1j}{}_i))$$

$$= C^\alpha{}_\beta W^\beta(A(e_j)) B^{-1j}{}_i = C^\alpha{}_\beta A^\beta{}_j B^{-1j}{}_i$$

or $\underline{A}' = \underline{C}\,\underline{A}\,\underline{B}^{-1}$

When $V=W$ and $e_i = E_i$, this reduces to the more familiar result $\underline{A}' = \underline{C}\,\underline{A}\,\underline{C}^{-1}$.

For a given vector space V each space of tensors of a given "index type" $T^{(p,q)}(V)$ or subspaces with certain symmetries like $\Lambda^{(p)}(V)$ and $\Lambda^{(p)}(V)^*$ is a vector space in its own right. However, instead of labeling the basis vectors in these spaces by a subscript label taking values between 1 and the dimension of the space, we use collections of indices associated with the underlying space V. The linear operations we have introduced all correspond to various linear maps between these spaces which can be expressed either in "component" form or as a relation between the new and old basis vectors, which defines the "matrix" of the linear transformation — but matrix in this generalized sense of one index corresponding to a collection of indices.

The "index-shifting" maps associated with an inner product or "metric" G are a perfect example. Considering the "lowering" map on V:

$$\flat : V \to V^* \qquad [X^\flat]_i \equiv X_i = G_{ij} X^j \qquad \text{[component relation]}$$

or
$$X^\flat = X_i \omega^i = G_{ij} X^j \omega^i$$
$$= (X^j e_j)^\flat = X^j e_j{}^\flat \qquad \text{by linearity}$$

which means that
$$e_j{}^\flat = G_{ji}\omega^i \qquad \text{[basis relation]}.$$

Similarly
$$\# : V^* \to V \qquad [f^\#]^i \equiv f^i = G^{ij} f_j \qquad \text{[component relation]}$$
$$\omega^{j\#} = G^{ji} e_i \qquad \text{[basis relation]}.$$

(Exercise: verify this as above)

The index shifting maps can be extended to any collection of indices for any space of tensors of a given type. The $\flat$ and $\#$ notation will always indicate shifting all the indices down and up respectively

In particular for p-vectors and p-covectors one can translate the component relations

$$T_{i_1\cdots i_p} = G_{i_1j_1}\cdots G_{i_pj_p} T^{j_1\cdots j_p}$$
$$T^{i_1\cdots i_p} = G^{i_1j_1}\cdots G^{i_pj_p} T_{j_1\cdots j_p}$$

to the basis relations

$$e_{i_1\cdots i_p}{}^{\flat} = G_{i_1j_1}\cdots G_{i_pj_p}\, \omega^{j_1\cdots j_p}$$
$$\omega^{i_1\cdots i_p}{}^{\#} = G^{i_1j_1}\cdots G^{i_pj_p}\, e_{j_1\cdots j_p} \qquad \text{(exercise: verify these)}$$

for the maps $\flat : \Lambda^{(p)}(V) \to \Lambda^{(p)}(V)^*$, $\# : \Lambda^{(p)}(V)^* \to \Lambda^{(p)}(V)$.

In fact it is natural to interpret $\Lambda^{(p)}(V)^*$ as the dual space to $\Lambda^{(p)}(V)$ since the natural contraction

$$T_{i_1\cdots i_p} S^{|i_1\cdots i_p|} = \frac{1}{p!} T_{i_1\cdots i_p} S^{i_1\cdots i_p}$$

is linear both in the p-vector S and the p-covector T, so fixing either factor produces a real-valued linear function of the other.

$\{\omega^{|i_1\cdots i_p|}\}$ is the basis dual to $\{e_{|i_1\cdots i_p|}\}$, and

$\omega^{|i_1\cdots i_p|}$ contracted on $e_{|j_1\cdots j_p|} = \delta^{|i_1\cdots i_p|}_{|j_1\cdots j_p|}$ (Kronecker delta for these two spaces)

is the duality relation.

The inner product on $\Lambda^{(p)}(V)$

$$\langle T, S\rangle = T^{i_1\cdots i_p} G_{i_1j_1}\cdots G_{i_pj_p} S^{j_1\cdots j_p} \equiv T_{j_1\cdots j_p} S^{j_1\cdots j_p}$$

induces the inner product on $\Lambda^{(p)}(V)$

$$\langle T, S\rangle = T_{i_1\cdots i_p} G^{i_1j_1}\cdots G^{i_pj_p} T_{j_1\cdots j_p} \equiv T^{j_1\cdots j_p} S_{j_1\cdots j_p}$$

for which the above relations are the component and basis relations for the two index shifting maps between these two spaces (for each p).

The "matrix" of this inner product is

$$\langle e_{i_1\cdots i_p}, e_{j_1\cdots j_p}\rangle = \delta^{k_1\cdots k_p}_{i_1\cdots i_p} G_{k_1j_1}\cdots G_{k_pj_p} .$$

Both the natural dual and the metric dual are linear maps among these spaces.

$$^{\circledast} e_{i_1\cdots i_p} = \epsilon_{i_1\cdots i_p i_{p+1}\cdots i_n}\, \omega^{|i_{p+1}\cdots i_n|}$$

$$* e_{i_1\cdots i_p} = \underbrace{\eta_{i_1\cdots i_p i_{p+1}\cdots i_n}}_{\eta_{1\cdots n}\,\epsilon_{i_1\cdots i_n}}\; \underbrace{\omega^{|i_{p+1}\cdots i_n|\#}}_{= G^{i_{p+1}i_{p+1}}\cdots G^{i_n i_n}\, e_{i_{p+1}\cdots i_n}}$$

For an orthonormal frame $G_{ii} = G^{ii} = \mathrm{sgn}(e_i)$ so

$$= \underbrace{G^{i_{p+1}i_{p+1}}\cdots G^{i_n i_n}}_{\text{product of signs.}} e_{i_{p+1}\cdots i_n}$$

$$\underbrace{\eta_{1\cdots n}}_{1 \text{ for oriented orthonormal frame.}} \epsilon_{i_1\cdots i_n}$$

Similar basis relations hold for $\omega^{i_1\cdots i_p}$. In an oriented orthonormal frame, the natural and metric duals are very closely related.

The metric dual turns out to be very closely related to the inner products for p-vectors and p-covectors. The following calculation establishes that simple relationship.

$$\underbrace{\overset{p\text{-covector}}{T} \wedge \underbrace{{}^*\overset{p\text{-covector}}{S}}_{n-p\text{-covector}}}_{n\text{-covector}} = \left(\frac{1}{p!} T_{i_1\cdots i_p}\,\omega^{i_1\cdots i_p}\right)\wedge\left(\frac{1}{(n-p)!\,p!} S^{j_1\cdots j_p}\,\eta_{j_1\cdots j_p i_{p+1}\cdots i_n}\,\omega^{i_{p+1}\cdots i_n}\right)$$

$$= \left(\frac{1}{p!}\right)^2 T_{i_1\cdots i_p} S^{j_1\cdots j_p}\,\frac{1}{(n-p)!}\,\underbrace{\eta_{j_1\cdots j_p i_{p+1}\cdots i_n}}_{\eta_{1\cdots n}\,\epsilon_{j_1\cdots j_p i_{p+1}\cdots i_n}}\;\underbrace{\omega^{i_1\cdots i_p i_{p+1}\cdots i_n}}_{\epsilon^{i_1\cdots i_n}\,\omega^{1\cdots n}}$$

$$= \underbrace{\left(\frac{1}{p!}\right)^2 T_{i_1\cdots i_p} S^{j_1\cdots j_p}\,\underbrace{\frac{1}{(n-p)!}\,\delta^{i_1\cdots i_n}_{j_1\cdots j_p i_{p+1}\cdots i_n}}_{\delta^{i_1\cdots i_p}_{j_1\cdots j_p}}}_{\frac{1}{p!}T_{i_1\cdots i_p}S^{i_1\cdots i_p} = \langle T, S\rangle}\;\underbrace{\eta_{1\cdots n}\,\omega^{1\cdots n}}_{\eta} = \langle T, S\rangle\,\eta$$

$$\boxed{\overset{p}{T}\wedge{}^*\overset{p}{S} = \langle T, S\rangle\,\eta}\quad\Big\}\ \text{same for } p\text{-vectors with } \eta^\# \text{ in place of } \eta$$

$$T\wedge{}^*T = \langle T, T\rangle\,\eta$$

in an oriented ON frame: $e_{i_1\cdots i_p}\wedge{}^*e_{i_1\cdots i_p} = \underbrace{\langle e_{i_1\cdots i_p}, e_{i_1\cdots i_p}\rangle}_{\text{product of signs of } e_{i_1},\dots,e_{i_p}}\,\overset{\text{product of all signs}\,\times\,\omega^{1\cdots n}}{\widetilde{\eta^\#}}$

${}^*e_{i_1\cdots i_p}$ is what you wedge into $e_{i_1\cdots i_p}$ on the right to get product of signs associated with "complementary indices" times $\omega^{1\cdots n}$

Exercise. Why "complementary indices"?

Exercise. Use the component definitions to show that

$$\langle {}^*T, {}^*S\rangle = \langle T, S\rangle \quad \text{for 2 } p\text{-vectors } T \text{ and } S.$$

Exercise. On $\mathbb{R}^3$ define ${}^*(X\wedge Y) = X\times Y$, for two vectors X and Y. What are the components $[X\times Y]^i$ (and verify)?

exercise Evaluate $**T$ for a p-covector.

answer:

$$[*T]_{i_{p+1}\cdots i_n} = \frac{1}{p!} T_{i_1\cdots i_p}\,\eta^{i_1\cdots i_p}{}_{i_{p+1}\cdots i_n}$$

$$[*(*T)]_{j_{n-p+1}\cdots j_n} = \frac{1}{(n-p)!}(*T)_{j_1\cdots j_{n-p}}\,\eta^{j_1\cdots j_{n-p}}{}_{j_{n-p+1}\cdots j_n}$$

$$= \frac{1}{(n-p)!}\,\frac{1}{p!}\, T_{i_1\cdots i_p}\underbrace{\eta^{i_1\cdots i_p}{}_{j_1\cdots j_{n-p}}\,\eta^{j_1\cdots j_{n-p}}{}_{j_{n-p+1}\cdots j_n}}_{\eta^{i_1\cdots i_p j_1\cdots j_{n-p}}\,\eta_{j_1\cdots j_{n-p}\,j_{n-p+1}\cdots j_n}}$$

move $(n-p)$ indices across p indices requires p transpositions for each of the $(n-p)$ indices, which leads to a sign change of $(-1)^{p(n-p)}$

$$(-1)^{p(n-p)}\,\eta^{i_1\cdots i_p j_1\cdots j_{n-p}}\,\eta_{j_{n-p+1}\cdots j_n\, j_1\cdots j_{n-p}}$$

summed.

$$\to \operatorname{sgn}\det(G_{mn})\,\underbrace{\delta^{i_1\cdots i_p j_1\cdots j_{n-p}}_{j_{n-p+1}\cdots j_n j_1\cdots j_{n-p}}}_{(n-p)!\,\delta^{i_1\cdots i_p}_{j_{n-p+1}\cdots j_{n-p}}}$$

PART I:
p.89 in an oriented basis:

$\eta^{i_1\cdots i_n} = \operatorname{sgn}\det(G_{mn})\,|\det(G_{mn})|^{1/2}\,\epsilon^{i_1\cdots i_n}$

$\eta_{i_1\cdots i_n} = |\det(G_{mn})|^{1/2}\,\epsilon_{i_1\cdots i_n}$

$\eta^{i_1\cdots i_n}\,\eta_{j_1\cdots j_n} = \operatorname{sgn}\det(G_{mn})\,\underbrace{\epsilon^{i_1\cdots i_n}\epsilon_{j_1\cdots j_n}}_{\delta^{i_1\cdots i_n}_{j_1\cdots j_n}}$

$$= \underbrace{\frac{1}{p!}\,T_{i_1\cdots i_p}\,\delta^{i_1\cdots i_p}_{j_{n-p+1}\cdots j_n}}_{T_{j_{n-p+1}\cdots j_n}}\,(-1)^{p(n-p)}\,\operatorname{sgn}\det(G_{mn})$$

i.e. $$\boxed{**T = \underbrace{(-1)^{p(n-p)}}_{}\,\operatorname{sgn}\det(G_{mn})}$$

$$(-1)^{pn}\underbrace{(-1)^{p^2}}_{(-1)^{p}=(-1)^{-p}} = (-1)^{pn-p} = (-1)^{(n-1)p}$$

so if n is odd, this sign is positive. ($n=3$ is odd) independent of p.

If n is even, it is $(-1)^p$. ($n=4$ is even.)

For the usual Euclidean inner product on $\mathbb{R}^3$: $**T = T$.