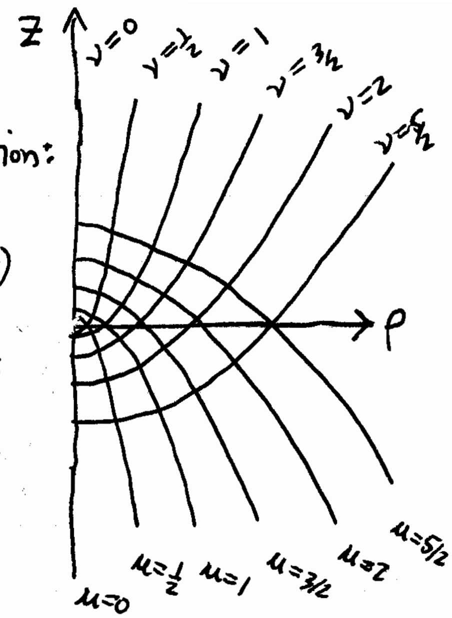
MATH 5600 Spring 91 Differential Geometry Take Home Final

PARABOLOIDAL COORDINATES ON $\mathbb{R}^3$

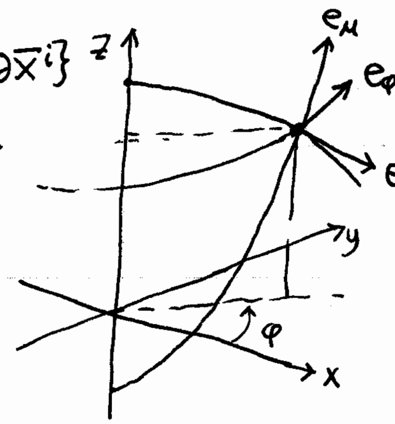
$$x = \rho\cos\varphi = \mu\nu\cos\varphi$$
$$y = \rho\sin\varphi = \mu\nu\sin\varphi$$
$$z = \underbrace{z}_{\text{cylindrical}} = \underbrace{\tfrac{1}{2}(\mu^2-\nu^2)}_{\text{paraboloidal}}$$

ρz plane transformation:

$$\rho = \mu\nu$$
$$z = \tfrac{1}{2}(\mu^2-\nu^2)$$



Revolve around z-axis to get 3-dim picture. μ,ν coord surfaces are parabolas of revolution. The φ coordinate surfaces are still the ρz half planes. The μ and ν coordinate lines are parabolas, while the φ coordinate lines are still circles about the z-axis. From the figure one can see that $\{e_\mu, e_\nu, e_\varphi\} \equiv \{\partial/\partial\mu, \partial/\partial\nu, \partial/\partial\varphi\} \equiv \{\partial/\partial\bar{x}^i\}$ is a righthanded frame ($e_\mu \times e_\nu$ is along e_φ).



The coordinate ranges are

$$\mu \ge 0, \quad \nu \ge 0, \quad 0 \le \varphi \le 2\pi .$$

1) Show that the transformation between ρ and z and μ and ν may be inverted to obtain

$$\mu = \sqrt{z + \sqrt{z^2+\rho^2}}\ , \quad \nu = \sqrt{-z + \sqrt{z^2+\rho^2}}$$

so the coordinate map is

$$\mu = \sqrt{z + \sqrt{x^2+y^2+z^2}}\ , \quad \nu = \sqrt{-z + \sqrt{x^2+y^2+z^2}}\ , \quad \varphi = \tan^{-1}\frac{y}{x} + \begin{cases} 0 & \text{I, IV} \\ \pi & \text{II} \\ -\pi & \text{IV} \end{cases}$$

(quads:)

2) Compute the transformation matrix $A^{-1}(\bar{x})^i{}_j = \dfrac{\partial x^i}{\partial \bar{x}^j}$ by evaluating the differentials $dx^i = A^{-1}(\bar{x})^i{}_j\, d\bar{x}^j$.

3) Since $\dfrac{\partial}{\partial \bar{x}^i} = A^{-1}(\bar{x})^j{}_i \dfrac{\partial}{\partial x^j}$, the columns of $\underline{A}^{-1}(\bar{x})$ represent the Cartesian coordinate components of the new coordinate frame vectors. Their dot products, considered as vectors in $\mathbb{R}^3$ give the dot products $\bar{g}_{ij} = \bar{e}_i \cdot \bar{e}_j$ of the new coordinate frame vectors. Show that they are orthogonal and evaluate their lengths, namely

$$(\bar{g}_{ij}) = [\underline{A}^{-1}(\bar{x})]^T \underline{A}^{-1}(\bar{x}).$$

Using these results, express the metric $g = \bar{g}_{ij}\, d\bar{x}^i \otimes d\bar{x}^j$
in this orthogonal coordinate system.

4) Evaluate the oriented unit volume 3-form

$$\eta = dx \wedge dy \wedge dz = [\det \underline{A}^{-1}(\bar{x})]\; \underbrace{d\bar{x}^1 \wedge d\bar{x}^2 \wedge d\bar{x}^3}_{d\mu \wedge d\nu \wedge d\varphi}$$

Since $[\det \underline{A}^{-1}(\bar{x})]$ is positive, these are oriented coordinates and $[\det \underline{\bar{g}}]^{1/2} = [\det \underline{A}^{-1}(\bar{x})]$.

5) Introduce the associated orthonormal frame

$$\{\bar{e}_{\hat{i}}\} = \{e_{\hat{\mu}}, e_{\hat{\nu}}, e_{\hat{\varphi}}\} \qquad \bar{e}_{\hat{i}} = (\bar{g}_{ii})^{-1/2}\, \bar{e}_i$$

$$\{\bar{\omega}^{\hat{i}}\} = \{\omega^{\hat{\mu}}, \omega^{\hat{\nu}}, \omega^{\hat{\varphi}}\} \qquad \bar{\omega}^{\hat{i}} = (\bar{g}_{ii})^{1/2}\, \bar{\omega}^i \quad , \quad \bar{\omega}^i \equiv d\bar{x}^i$$

Let $\underline{a}(\bar{x})$ be the transformation matrix between the old and new orthonormal frames:

$$\bar{e}_{\hat{i}} = a(\bar{x})^{-1\,j}{}_i\, \cancel{\delta x}\,{}_j$$

$$\bar{\omega}^{\hat{i}} = a(x)^i{}_j\, dx^j$$

Then this orthogonal matrix is

$$a(\bar{x})^{-1\,i}{}_j = (\bar{g}_{jj})^{-1/2} A(\bar{x})^{-1\,i}{}_j \qquad (\text{normalize columns of } A(\bar{x})^{-1})$$

What is it?

Take its transpose to obtain $\underline{a}(x)$.

Get $\underline{A}(x)$ by dividing the rows of $\underline{a}(x)$ by the same normalizing factors used to divide the columns of $\underline{A}(\bar{x})^{-1}$:

$$A(\bar{x})^i{}_j = a(\bar{x})^i{}_j\, (\bar{g}_{ii})^{-1/2} \; = \; \frac{\partial \bar{x}^i}{\partial x^j}(x)$$

6) By differentiating the coordinate map of part 1) and re-expressing its matrix of entries in terms of the new coordinates, verify that $A(\bar{x})$ has the value obtained in 5). [Check also that $\underline{A}(\bar{x})\, \underline{A}^{-1}(\bar{x}) = \underline{I}$]

7) Compute the independent structure functions of the orthonormal frame $\{\bar{C}^{\hat{i}}{}_{\hat{j}\hat{k}}\}_{j<k}$ defined by: $[\bar{e}_{\hat{j}}, \bar{e}_{\hat{k}}] = \bar{C}^{\hat{i}}{}_{\hat{j}\hat{k}}\, \bar{e}_{\hat{i}}$.

8) Compute the components of the covariant derivative in the coordinate and associated orthonormal frame using the formulas

$$\underline{A}\, d\underline{A}^{-1} = \underline{\bar{\omega}} = (\bar{\Gamma}^i{}_{kj}\, d\bar{x}^j)$$

$$\underline{a}\, d\underline{a}^{-1} = \underline{\hat{\omega}} = (\bar{\Gamma}^{\hat{i}}{}_{\hat{k}\hat{j}}\, \bar{\omega}^{\hat{k}})$$

9) Verify these results using the formulas involving the derivatives of the metric and the structure functions.

10) Now for something new, well not new, but a putting together of things we already know. Consider the coordinate frame formula:

$$R^i{}_{jmn} = \partial_m \Gamma^i{}_{nj} - \partial_n \Gamma^i{}_{mj} + \Gamma^i{}_{m\ell}\Gamma^\ell{}_{nj} - \Gamma^i{}_{n\ell}\Gamma^\ell{}_{mj}$$

$$= 2\,\partial_{[m}\Gamma^i{}_{n]j} + 2\Gamma^i{}_{[m\underset{\substack{\uparrow\\ \text{no antisym}\\ \text{on } \ell}}{\ell}}\Gamma^\ell{}_{n]j} = R^i{}_{j[mn]}$$

$$\Omega^i{}_j \equiv \tfrac{1}{2} R^i{}_{jmn}\, dx^{mn} = \tfrac{1}{2}\Big[\underbrace{2\,\partial_{[m}\Gamma^i{}_{n]j}\, dx^{mn}}_{[d\omega^i{}_j]_{mn}} + \underbrace{2\Gamma^i{}_{[m\underset{\substack{\uparrow\\ \text{no antisym on } \ell}}{\ell}}\Gamma^\ell{}_{n]j}\, dx^{mn}}_{[\omega^i{}_\ell \wedge \omega^\ell{}_j]_{mn}}\Big]$$

$$= d\omega^i{}_j + \omega^i{}_\ell \wedge \omega^\ell{}_j$$

By introducing a curvature 2-form , matrix $\underline{\Omega} = (\Omega^i{}_j)$ one can more efficiently compute the curvature tensor components

$$\underset{\substack{\uparrow\\ \text{readoff}}}{\underline{\Omega}} = d\underline{\omega} + \underbrace{\underline{\omega}\wedge\underline{\omega}}_{\substack{\text{matrix product of matrix indices}\\ \text{wedge product of 1 form matrix entries.}}}$$

readoff curvature tensor components — matrix indices give left pair of tensor indices, coefficients of dx^{mn} give second pair.

If we use this in the new coordinate frame:

$$\underline{\bar{\Omega}} = d\underline{\bar{\omega}} + \underline{\bar{\omega}}\wedge\underline{\bar{\omega}} = \underbrace{d(\underline{A}\, d\underline{A}^{-1})}_{d\underline{A}\wedge d\underline{A}^{-1} + \underline{A}\, \underbrace{d^2\underline{A}^{-1}}_{=0}} + \underline{A}\, d\underline{A}^{-1} \wedge \underline{A}\, d\underline{A}^{-1}$$

But $\underline{A}\,\underline{A}^{-1} = I \rightarrow [d\underline{A}\,\underline{A}^{-1} + \underline{A}\,d\underline{A}^{-1} = 0]\,A$

$$d\underline{A} + \underline{A}\,d\underline{A}^{-1}\underline{A} = 0 \rightarrow d\underline{A} = -\underline{A}\,d\underline{A}^{-1}\underline{A}$$

so $\quad d\underline{A} \wedge d\underline{A}^{-1} = -\underline{A}\,d\underline{A}^{-1}\underline{A} \wedge d\underline{A}^{-1} = -\underline{A}\,d\underline{A}^{-1} \wedge \underline{A}\,d\underline{A}^{-1}$

wedge can be anywhere between differentials since matrices are just functions and can be put anywhere as factors with respect to the wedge product.

hence $\underline{\Omega} = 0$.

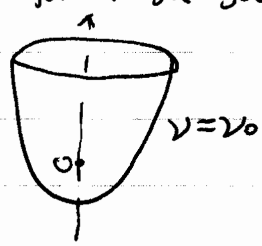
Of course we knew the curvature tensor to be zero, but this matrix method most efficiently achieves this result.

[Note: if $\underline{\omega} \wedge \underline{\omega}$ bothers you, here is an example of 2x2 matrices of 1-forms

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \wedge \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \alpha\wedge A + \beta\wedge C & \alpha\wedge B + \beta\wedge D \\ \gamma\wedge A + \delta\wedge C & \gamma\wedge B + \delta\wedge D \end{pmatrix}$$

where all the entries are assumed to be 1-forms. (or, in fact, p-forms).]

11) You only had to follow 10), not do anything. Now, from your results for $\bar{\Gamma}^i{}_{jk}$ you can read off the components of the covariant derivative



for the 2-dim spaces (parabolas of revolution) with metric $\quad {}^{(2)}g = g\big|_{\substack{\nu=\nu_0 \\ d\nu=0}}$ on which μ, φ are local coordinates. [These components can only be defined by the 2-dimensional formula in terms of the metric derivatives.]

What is the 2-dimensional matrix $\quad {}^{(2)}\underline{\omega} = ({}^{(2)}\omega^\alpha{}_\beta) = ({}^{(2)}\Gamma^\alpha{}_{\gamma\beta}\,d\bar{x}^\gamma)$

where $\alpha, \beta, \ldots = \mu, \phi$ indices. (1, 3 in numbers)?

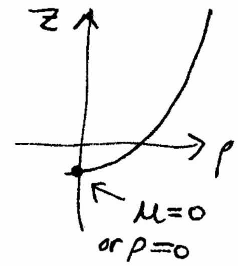
Now compute the 2-curvature 2-form matrix

$${}^{(2)}\underline{\Omega} = d{}^{(2)}\underline{\omega} + {}^{(2)}\underline{\omega} \wedge {}^{(2)}\underline{\omega} \quad = \left(\tfrac{1}{2}\,{}^{(2)}R^\alpha{}_{\beta\gamma\delta}\,d\bar{x}^\gamma \wedge d\bar{x}^\delta\right)$$

Read off the two components $\quad {}^{(2)}R^\mu{}_{\varphi\mu\varphi}\,,\ {}^{(2)}R^\varphi{}_{\mu\mu\varphi}\,.$

Does $\;{}^{(2)}R_{\mu\varphi\mu\varphi} = -{}^{(2)}R_{\varphi\mu\mu\varphi}$?

12) Evaluate ${}^{(2)}R^{\hat\mu}{}_{\hat\varphi\hat\mu\hat\varphi} = (g_{\varphi\varphi})^{-1}\,{}^{(2)}R^{\mu}{}_{\varphi\mu\varphi}$.
What is the value at $\mu=0$, the vertex of the parabola of revolution?
The parabola which is revolved is



$$\rho = \mu\nu_0 \quad\rightarrow\quad \mu = \rho/\nu_0$$
$$z = \tfrac{1}{2}(\mu^2-\nu_0^2) \;=\; \tfrac{1}{2}(\rho^2/\nu_0^2-\nu_0^2)$$

The curvature of this parabola at any point comes from the multivariable calculus formula

$$K = \frac{|d^2z/d\rho^2|}{[1+(dz/d\rho)^2]^{3/2}}\,.$$

Evaluate $K(\rho=0)$ and compare it to the value of ${}^{(2)}R^{\hat\mu}{}_{\hat\varphi\hat\mu\hat\varphi}$.
Do you notice any relationship?

13) Show that the μ coordinate lines are geodesics on these parabolas, but that the φ coordinate lines are not.

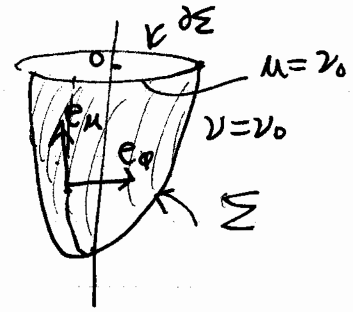
14) · What is the single independent structure function $C^{\hat\varphi}{}_{\hat\mu\hat\varphi}$ for the 2-dim orthonormal frame?
Use it to compute the components of the covariant derivative in the orthonormal frame: ${}^{(2)}\Gamma^{\hat\alpha}{}_{\hat\beta\hat\gamma}$.

Use them to show that $e_{\hat\mu}$ and $e_{\hat\varphi}$ are parallel transported along the μ coordinate lines.

15) All of these computations (with the exception of the curvature 2-form notation) have been done with either cylindrical or spherical coordinates in the notes, so you should have no problem if you understood them.

16) Let $\begin{cases} \underline{X} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + (x^2+y^2+z^2)\frac{\partial}{\partial z} \\ \underline{X}^\flat = -y\,dx + x\,dy + (x^2+y^2+z^2)\,dz \end{cases}$

Evaluate $\underline{X}^\flat$ in paraboloidal coordinates. Find $\underline{X}$ in these coordinates.
Evaluate $\nabla_{e_\mu} \underline{X}$.

17) 

Let Σ be the 2-surface $\begin{cases} \nu=\nu_0 \\ 0 \le \mu \le \nu_0 \end{cases}$

parametrized by $\{\mu, \varphi\}$.
What choice of normal does this inner orientation imply by the right hand rule? (inward/upper or outerward/downer (?))

Looking down from above, what is the induced orientation of $\partial\Sigma$: clockwise or counterclockwise?
As on pages 152-153, verify Stokes thm $\int_{\partial\Sigma} \underline{X}^\flat = \int_\Sigma d\underline{X}^\flat$

for $\underline{X}^\flat$ of part 16).

18) That's all folks. Have fun. Stop by if you have any difficulty.
I need your work by 5pm Friday May 3 to make up grades for Monday.
W: 645-7335 h: 527-4641. [Slippage to Monday noon if you really have a time problem].

TAKEHOME FINAL WORKED PROBLEMS

1) $\rho = \mu\nu \longrightarrow \nu = \rho/\mu$ or $\mu = \rho/\nu$

$z = \frac{1}{2}(\mu^2-\nu^2) = \frac{1}{2}(\mu^2 - \rho^2/\mu^2)$

$\mu^4 - 2z\mu^2 - \rho^2 = 0$

$\mu^2 = \dfrac{2z \pm \sqrt{4z^2+4\rho^2}}{2} = z + \sqrt{z^2+\rho^2}$

≥ 0

$\mu = \sqrt{z + \sqrt{z^2+\rho^2}} \qquad (\mu \geq 0)$

$z = \frac{1}{2}(\rho^2/\nu^2 - \nu^2)$

$2z + \nu^2 = \rho^2/\nu^2$

$\nu^4 + 2z\nu^2 - \rho^2 = 0$

$\nu^2 = \dfrac{-2z \pm \sqrt{4z^2+4\rho^2}}{2}$

$= -z + \sqrt{z^2+\rho^2}$ (ditto)

$\nu = \sqrt{-z+\sqrt{z^2+\rho^2}} \qquad (\nu \geq 0)$

Note $\underbrace{z^2+\rho^2}_{z^2+x^2+y^2 = r^2} = \dfrac{4\mu^2\nu^2 + \mu^4 - 2\mu^2\nu^2 + \nu^4}{4} = \left(\dfrac{\mu^2+\nu^2}{2}\right)^2$

so $r = \sqrt{z^2+\rho^2} = \dfrac{\mu^2+\nu^2}{2}$.

2) $x = \mu\nu\cos\varphi \qquad dx = \nu\cos\varphi\, d\mu + \mu\cos\varphi\, d\nu - \mu\nu\sin\varphi\, d\varphi$

$y = \mu\nu\sin\varphi \qquad dy = \nu\sin\varphi\, d\mu + \mu\sin\varphi\, d\nu + \mu\nu\cos\varphi\, d\varphi$

$z = \frac{1}{2}(\mu^2-\nu^2) \qquad dz = \mu\, d\mu - \nu\, d\mu$

$$\underline{A}^{-1}(\bar{x}) = \begin{bmatrix} \nu\cos\varphi & \mu\cos\varphi & -\mu\nu\sin\varphi \\ \nu\sin\varphi & \mu\sin\varphi & \mu\nu\cos\varphi \\ \mu & -\nu & 0 \end{bmatrix}$$

3) $g_{\mu\mu} = \mu^2+\nu^2 \qquad g_{\nu\nu} = \mu^2+\nu^2 \qquad g_{\varphi\varphi} = \mu^2\nu^2$

$g = \delta_{ij}\, dx^i \otimes dx^j = (\mu^2+\nu^2)[d\mu\otimes d\mu + d\nu\otimes d\nu] + \mu^2\nu^2 d\varphi\otimes d\varphi$

4) Row expansion on last row

$\det \underline{A}^{-1}(\bar{x}) = \mu \begin{vmatrix} \mu\cos\varphi & -\mu\nu\sin\varphi \\ \mu\sin\varphi & \mu\nu\cos\varphi \end{vmatrix} - (-\nu) \begin{vmatrix} \nu\cos\varphi & -\mu\nu\sin\varphi \\ \nu\sin\varphi & \mu\nu\cos\varphi \end{vmatrix}$

$= \mu(\mu^2\nu) + \nu(\mu\nu^2) = \mu\nu(\mu^2+\nu^2) \geq 0$ so positively oriented.

$\eta = \underbrace{\mu\nu(\mu^2+\nu^2)}_{(g_{\mu\mu}g_{\nu\nu}g_{\varphi\varphi})^{1/2}} d\mu\wedge d\nu\wedge d\varphi$

5) $e_{\hat\mu} = \dfrac{1}{(\mu^2+\nu^2)^{1/2}}\dfrac{\partial}{\partial\mu}, \quad e_{\hat\nu} = \dfrac{1}{(\mu^2+\nu^2)^{1/2}}\dfrac{\partial}{\partial\nu}, \quad e_{\hat\varphi} = \dfrac{1}{(\mu\nu)}\dfrac{\partial}{\partial\varphi}$

$\omega^{\hat\mu} = (\mu^2+\nu^2)^{1/2} d\mu, \quad \omega^{\hat\nu} = (\mu^2+\nu^2)^{1/2} d\nu, \quad \omega^{\hat\varphi} = \mu\nu\, d\varphi$

(note $\eta = \omega^{\hat\mu}\wedge\omega^{\hat\nu}\wedge\omega^{\hat\varphi} = \omega^{\hat\mu\hat\nu\hat\varphi}$)

$$\underline{a}^{-1}(\bar{x}) = \begin{bmatrix} \frac{\nu}{(\mu^2+\nu^2)^{1/2}}\cos\varphi & \frac{\mu}{(\mu^2+\nu^2)^{1/2}}\cos\varphi & -\sin\varphi \\ \frac{\nu}{(\mu^2+\nu^2)^{1/2}}\sin\varphi & \frac{\mu}{(\mu^2+\nu^2)^{1/2}}\sin\varphi & \cos\varphi \\ \frac{\mu}{(\mu^2+\nu^2)^{1/2}} & \frac{-\nu}{(\mu^2+\nu^2)^{1/2}} & 0 \end{bmatrix}$$

$$\underline{a}(\bar{x}) = \begin{bmatrix} \frac{\nu}{(\mu^2+\nu^2)^{1/2}}\cos\varphi & \frac{\nu}{(\mu^2+\nu^2)^{1/2}}\sin\varphi & \frac{\mu}{(\mu^2+\nu^2)^{1/2}} \\ \frac{\mu}{(\mu^2+\nu^2)^{1/2}}\cos\varphi & \frac{\mu}{(\mu^2+\nu^2)^{1/2}}\sin\varphi & \frac{-\nu}{(\mu^2+\nu^2)^{1/2}} \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix}$$

$$\underline{A}(\bar{x}) = \begin{bmatrix} \frac{\nu}{\mu^2+\nu^2}\cos\varphi & \frac{\nu}{\mu^2+\nu^2}\sin\varphi & \frac{\mu}{\mu^2+\nu^2} \\ \frac{\mu}{\mu^2+\nu^2}\cos\varphi & \frac{\mu}{\mu^2+\nu^2}\sin\varphi & \frac{-\nu}{\mu^2+\nu^2} \\ -\frac{1}{\mu\nu}\sin\varphi & \frac{1}{\mu\nu}\cos\varphi & 0 \end{bmatrix} = \left(\frac{\partial \bar{x}^i}{\partial x^j}(x(\bar{x}))\right)$$

6) Now recall that $2\sqrt{z^2+\rho^2} = \mu^2+\nu^2$, so for example,

$$\mu = (z + (z^2+\rho^2)^{1/2})^{1/2}$$

$$d\mu = \frac{1}{2(\quad)^{1/2}}\left[dz + \frac{1}{2}\frac{[2z\,dz+2x\,dx+2y\,dy]}{(z^2+\rho^2)^{1/2}}\right] = \frac{1}{2\mu}\left[\frac{(\mu^2+\nu^2)\,dz + [\overbrace{2z}^{\mu^2-\nu^2}dz+2x\,dx+2y\,dy]}{\mu^2+\nu^2}\right]$$

$$= \frac{1}{2\mu}\left[\frac{2x\,dx+2y\,dy+2\mu^2\,dz}{\mu^2+\nu^2}\right] = \frac{x\,dx+y\,dy+\mu^2\,dz}{\mu(\mu^2+\nu^2)}$$

$$= \frac{\mu\nu\cos\varphi\,dx + \mu\nu\sin\varphi\,dy + \mu^2\,dz}{\mu(\mu^2+\nu^2)}$$

and these components of $d\mu$ are exactly the first row of $\underline{A}(\bar{x})$. Second row similar.

The last row comes from $d\varphi = -\frac{\sin\varphi}{\rho}dx + \frac{\cos\varphi}{\rho}dy$ which is the cylindrical coordinate result with ρ then replaced by $\mu\nu$.

7) $$[e_{\hat\nu}, e_{\hat\varphi}] = \left[(\mu^2+\nu^2)^{-1/2}\frac{\partial}{\partial\nu}, (\mu\nu)^{-1}\frac{\partial}{\partial\varphi}\right] = (\mu^2+\nu^2)^{-1/2}\frac{1}{\mu}\left(-\frac{1}{\nu^2}\right)\frac{\partial}{\partial\varphi}$$

$$= -\frac{1}{(\mu^2+\nu^2)^{1/2}\nu}e_{\hat\varphi} \qquad C^{\hat\varphi}{}_{\hat\nu\hat\varphi} = -\frac{1}{(\mu^2+\nu^2)^{1/2}\nu}$$

$$[e_{\hat\mu}, e_{\hat\varphi}] = \left[(\mu^2+\nu^2)^{-1/2}\frac{\partial}{\partial\mu}, (\mu\nu)^{-1}\frac{\partial}{\partial\varphi}\right] = \ldots = -\frac{1}{(\mu^2+\nu^2)^{1/2}\mu}e_{\hat\varphi} \qquad C^{\hat\varphi}{}_{\hat\mu\hat\varphi} = -\frac{1}{(\mu^2+\nu^2)^{1/2}\mu}$$

$$[e_{\hat\mu}, e_{\hat\nu}] = \left[(\mu^2+\nu^2)^{-1/2}\frac{\partial}{\partial\mu}, (\mu^2+\nu^2)^{-1/2}\frac{\partial}{\partial\nu}\right] = (\mu^2+\nu^2)^{-1}\frac{\partial}{\partial\mu}\left(\ln(\mu^2+\nu^2)^{-1/2}\right)\frac{\partial}{\partial\nu} - \ldots$$

$$= -\frac{1}{2}(\mu^2+\nu^2)^{-1}\frac{2\mu}{(\mu^2+\nu^2)}\frac{\partial}{\partial\nu} + \frac{1}{2}(\mu^2+\nu^2)^{-1}\frac{2\nu}{\mu^2+\nu^2}\frac{\partial}{\partial\mu} = \frac{1}{(\mu^2+\nu^2)^{3/2}}\left[-\mu e_{\hat\nu} + \nu e_{\hat\mu}\right]$$

$$C^{\hat\mu}{}_{\hat\mu\hat\nu} = \frac{\nu}{(\mu^2+\nu^2)^{3/2}}, \quad C^{\hat\nu}{}_{\hat\mu\hat\nu} = \frac{-\mu}{(\mu^2+\nu^2)^{3/2}}$$

8) a)

$$\underline{\overline{\omega}} = \underline{A}\, d\underline{A}^{-1} = \begin{bmatrix} \frac{\nu}{(2r)}c & \frac{\nu}{(2r)}s & \frac{\mu}{(2r)} \\ \frac{\mu}{(2r)}c & \frac{\mu}{(2r)}s & \frac{-\nu}{(2r)} \\ -\frac{1}{\rho}s & \frac{1}{\rho}c & 0 \end{bmatrix} \underbrace{d\begin{bmatrix} \nu c & \mu c & -\mu\nu s \\ \nu s & \mu s & \mu\nu c \\ \mu & -\nu & 0 \end{bmatrix}}$$

letting $(2r) = \mu^2+\nu^2$, $\rho = \mu\nu$

$$\begin{pmatrix} 0 & c & -\nu s \\ 0 & s & \nu c \\ 1 & 0 & 0 \end{pmatrix} d\mu + \begin{pmatrix} c & 0 & -\mu s \\ s & 0 & \mu c \\ 0 & -1 & 0 \end{pmatrix} d\nu + \begin{pmatrix} -\nu s & -\mu s & -\mu\nu c \\ \nu c & \mu c & -\mu\nu s \\ 0 & 0 & 0 \end{pmatrix} d\varphi$$

$$= \begin{bmatrix} \frac{\mu}{(2r)} & \frac{\nu}{(2r)} & 0 \\ \frac{-\nu}{(2r)} & \frac{\mu}{(2r)} & 0 \\ 0 & 0 & \cancel{\frac{1}{\mu}} \end{bmatrix} d\mu + \begin{bmatrix} \frac{\nu}{(2r)} & \frac{-\mu}{(2r)} & 0 \\ \frac{\mu}{(2r)} & \frac{\nu}{(2r)} & 0 \\ 0 & 0 & \cancel{\frac{1}{\nu}} \end{bmatrix} d\nu + \begin{bmatrix} 0 & 0 & -\frac{\mu\nu^2}{(2r)} \\ 0 & 0 & -\frac{\mu^2\nu}{(2r)} \\ \frac{\nu}{\rho} & \frac{\mu}{\rho} & 0 \end{bmatrix} d\varphi$$

oops ↗   oops ↗

or 9a) $\Gamma_{ljk} = \frac{1}{2}(g_{lj,k} - g_{jk,l} + g_{kl,j})$ $\Gamma^l{}_{jk} = g^{ll}\Gamma_{ljk} = (g_{ll})^{-1}\Gamma_{ljk}$ (orthog coords)

$g_{\mu\mu} = g_{\nu\nu} = \mu^2+\nu^2$, $g_{\varphi\varphi} = \mu\nu$

$\Gamma_{\mu\mu\mu} = \frac{1}{2}(g_{\mu\mu,\mu} - \cancel{g_{\mu\mu,\mu}} + \cancel{g_{\mu\mu,\mu}}) = \mu$ $\Gamma^\mu{}_{\mu\mu} = \frac{\mu}{\mu^2+\nu^2} = [\omega^\mu{}_\mu]_\mu$ ✓

$\Gamma_{\mu\mu\nu} = \frac{1}{2}(g_{\mu\mu,\nu} - \cancel{g_{\mu\nu,\mu}} + \cancel{g_{\nu\mu,\mu}}) = \nu$ $\Gamma^\mu{}_{\mu\nu} = \frac{\nu}{\mu^2+\nu^2} = [\omega^\mu{}_\nu]_\mu$ ✓

$\Gamma_{\nu\mu\mu} = \frac{1}{2}(\cancel{g_{\nu\mu,\mu}} - g_{\mu\mu,\nu} + \cancel{g_{\mu\nu,\mu}}) = -\nu$ $\Gamma^\nu{}_{\mu\mu} = \frac{-\nu}{\mu^2+\nu^2} = [\omega^\nu{}_\mu]_\mu$ ✓

$\Gamma_{\nu\mu\nu} = \frac{1}{2}(\cancel{g_{\nu\mu,\nu}} - \cancel{g_{\mu\nu,\nu}} + g_{\nu\nu,\mu}) = \mu$ $\Gamma^\nu{}_{\mu\nu} = \frac{\mu}{\mu^2+\nu^2} = [\omega^\mu{}_\nu]_\mu$ ✓

same for next four (let's pretend we did it)

$\Gamma^\mu{}_{\nu\mu} = \frac{\nu}{\mu^2+\nu^2}$

$\Gamma^\mu{}_{\nu\nu} =$

$\Gamma^\nu{}_{\nu\mu} =$

$\Gamma^\nu{}_{\nu\nu} =$

$\Gamma_{\varphi\varphi\mu} = \frac{1}{2}(g_{\varphi\varphi,\mu} - \cancel{g_{\varphi\mu,\varphi}} + \cancel{g_{\mu\varphi,\varphi}}) = \mu\nu^2$, $\Gamma^\varphi{}_{\varphi\mu} = \frac{1}{\mu} = \frac{\nu}{\rho}$ ✓

$\Gamma_{\varphi\varphi\nu} = \frac{1}{2}(g_{\varphi\varphi,\nu} - \quad) = \nu\mu^2$, $\Gamma^\varphi{}_{\varphi\nu} = \frac{1}{\nu} = \frac{\mu}{\rho}$ ✓

$\Gamma_{\mu\varphi\varphi} = \frac{1}{2}(\cancel{g_{\mu\varphi,\varphi}} - g_{\varphi\varphi,\mu} + \cancel{g_{\varphi\mu,\varphi}}) = -\mu\nu^2$, $\Gamma^\mu{}_{\varphi\varphi} = \frac{-\mu\nu^2}{\mu^2+\nu^2}$ ✓

$\Gamma_{\nu\varphi\varphi} = \frac{1}{2}(\ \backslash\ - g_{\varphi\varphi,\nu}\ \backslash\ ) = -\nu\mu^2$, $\Gamma^\nu{}_{\varphi\varphi} = \frac{-\nu\mu^2}{\mu^2+\nu^2}$ ✓

sloppiness in mental multiplication forgot these:

$\Gamma_{\varphi\mu\varphi} = \frac{1}{2}(\cancel{g_{\varphi\mu,\varphi}} - \cancel{g_{\mu\varphi,\varphi}} + g_{\varphi\varphi,\mu}) = \mu\nu^2$, $\Gamma^\varphi{}_{\mu\varphi} = \frac{1}{\mu}$

$\Gamma_{\varphi\nu\varphi} = \qquad\qquad = \mu^2\nu$, $\Gamma^\varphi{}_{\nu\varphi} = \frac{1}{\nu}$

8b) $\mathcal{M} \equiv \frac{\mu}{(\mu^2+\nu^2)^{1/2}}$, $\mathcal{V} \equiv \frac{\nu}{(\mu^2+\nu^2)^{1/2}}$, $\mathcal{M}^2+\mathcal{V}^2 = 1$.

$$\frac{\partial \mathcal{M}}{\partial \mu} = \frac{(\mu^2+\nu^2)^{1/2}\,1 - \mu\cdot\frac{1}{2}\frac{2\mu}{(\ )^{1/2}}}{(\mu^2+\nu^2)} = \frac{(\mu^2+\nu^2)-\mu^2}{(\mu^2+\nu^2)^{3/2}} = \frac{\nu^2}{(\mu^2+\nu^2)^{3/2}}$$

$$\frac{\partial \mathcal{M}}{\partial \nu} = -\frac{1}{2}\frac{\mu\cdot 2\nu}{(\ )^{3/2}} = \frac{-\mu\nu}{(\mu^2+\nu^2)^{3/2}}$$

$$\frac{\partial \mathcal{V}}{\partial \mu} = \cdots \quad \frac{-\mu\nu}{(\mu^2+\nu^2)^{3/2}}$$

$$\frac{\partial \mathcal{V}}{\partial \nu} = \quad ---- \quad = \frac{\mu^2}{(\mu^2+\nu^2)^{3/2}}$$

$$\underline{\hat{\omega}} = \underline{a}\, d\underline{a}^{-1} = \begin{bmatrix} \mathcal{V}c & \mathcal{V}s & \mathcal{M} \\ \mathcal{M}c & \mathcal{M}s & -\mathcal{V} \\ -s & c & 0 \end{bmatrix} d \underbrace{\begin{bmatrix} \mathcal{V}c & \mathcal{M}c & -s \\ \mathcal{V}s & \mathcal{M}s & c \\ \mathcal{M} & -\mathcal{V} & 0 \end{bmatrix}}$$

$$\begin{bmatrix} -\mu\nu c & \nu^2 c & 0 \\ -\mu\nu s & \nu^2 s & 0 \\ \nu^2 & \mu\nu & 0 \end{bmatrix} \frac{d\mu}{(\mu^2+\nu^2)^{3/2}} + \begin{bmatrix} \mu^2 c & -\mu\nu c & 0 \\ \mu^2 s & -\mu\nu s & 0 \\ -\mu\nu & -\mu^2 & 0 \end{bmatrix} \frac{d\nu}{(\mu^2+\nu^2)^{3/2}} + \begin{bmatrix} -\mathcal{V}s & -\mathcal{M}s & -c \\ \mathcal{V}c & \mathcal{M}c & -s \\ 0 & 0 & 0 \end{bmatrix} d\varphi$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\mathcal{V}}{(\mu^2+\nu^2)} \underbrace{\left[(\mu^2+\nu^2)^{1/2} d\mu\right]}_{\omega^{\hat{\mu}}} + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\mathcal{M}}{(\mu^2+\nu^2)} \underbrace{\left[(\mu^2+\nu^2)^{1/2} d\nu\right]}_{\omega^{\hat{\nu}}} + \begin{bmatrix} 0 & 0 & -\mathcal{V} \\ 0 & 0 & -\mathcal{M} \\ \mathcal{V} & \mathcal{M} & 0 \end{bmatrix} \frac{1}{\mu\nu} \underbrace{\left[\mu\nu\, d\varphi\right]}_{\omega^{\hat{\varphi}}}$$

$$\Gamma^{\hat{\mu}}{}_{\hat{\mu}\hat{\nu}} = \frac{\nu}{(\mu^2+\nu^2)^{3/2}} = -\Gamma^{\hat{\nu}}{}_{\hat{\mu}\hat{\mu}}, \quad \Gamma^{\hat{\mu}}{}_{\hat{\nu}\hat{\nu}} = -\frac{\mu}{(\mu^2+\nu^2)^{3/2}} = -\Gamma^{\hat{\nu}}{}_{\hat{\nu}\hat{\mu}},$$

$$\Gamma^{\hat{\mu}}{}_{\hat{\varphi}\hat{\varphi}} = -\frac{1}{\mu(\mu^2+\nu^2)^{1/2}} = -\Gamma^{\hat{\varphi}}{}_{\hat{\mu}\hat{\varphi}}, \quad \Gamma^{\hat{\nu}}{}_{\hat{\varphi}\hat{\varphi}} = -\frac{1}{\nu(\mu^2+\nu^2)^{1/2}} = -\Gamma^{\hat{\varphi}}{}_{\hat{\varphi}\hat{\nu}}$$

9b) or $\Gamma^{\hat{i}}{}_{\hat{j}\hat{k}} = \Gamma_{\hat{i}\hat{j}\hat{k}} = \frac{1}{2}\left(C_{\hat{i}\hat{j}\hat{k}} - C_{\hat{j}\hat{k}\hat{i}} + C_{\hat{k}\hat{i}\hat{j}}\right)$

$$C_{\hat{\varphi}\hat{\mu}\hat{\varphi}} = -\frac{1}{\mu(\mu^2+\nu^2)^{1/2}}, \quad C_{\hat{\varphi}\hat{\nu}\hat{\varphi}} = -\frac{1}{\nu(\mu^2+\nu^2)^{1/2}}, \quad C_{\hat{\mu}\hat{\nu}\hat{\mu}} = -\frac{\nu}{(\mu^2+\nu^2)^{3/2}}, \quad C_{\hat{\nu}\hat{\mu}\hat{\nu}} = -\frac{\mu}{(\mu^2+\nu^2)^{3/2}}$$

$$\Gamma^{\hat{\mu}}{}_{\hat{\mu}\hat{\nu}} = \frac{1}{2}\left(C_{\hat{\mu}\hat{\mu}\hat{\nu}} - C_{\hat{\mu}\hat{\nu}\hat{\mu}} + \cancel{C_{\hat{\nu}\hat{\mu}\hat{\mu}}}\right) = C_{\hat{\mu}\hat{\mu}\hat{\nu}} = \frac{\nu}{(\ )^{3/2}} \checkmark$$

$$\Gamma^{\hat{\mu}}{}_{\hat{\nu}\hat{\nu}} = \frac{1}{2}\left(\cancel{C_{\hat{\mu}\hat{\nu}\hat{\nu}}} - C_{\hat{\nu}\hat{\nu}\hat{\mu}} + C_{\hat{\nu}\hat{\mu}\hat{\nu}}\right) = C_{\hat{\nu}\hat{\mu}\hat{\nu}} = \frac{-\mu}{(\ )^{3/2}} \checkmark$$

$$\Gamma^{\hat{\mu}}{}_{\hat{\varphi}\hat{\varphi}} = \frac{1}{2}\left(\cancel{C_{\hat{\mu}\hat{\varphi}\hat{\varphi}}} - C_{\hat{\varphi}\hat{\varphi}\hat{\mu}} + C_{\hat{\varphi}\hat{\mu}\hat{\varphi}}\right) = C_{\hat{\varphi}\hat{\mu}\hat{\varphi}} = -\frac{1}{\mu(\ )^{1/2}} \checkmark$$

$$\Gamma^{\hat{\nu}}{}_{\hat{\varphi}\hat{\varphi}} = \frac{1}{2}\left(\cancel{C_{\hat{\nu}\hat{\varphi}\hat{\varphi}}} - C_{\hat{\varphi}\hat{\varphi}\hat{\nu}} + C_{\hat{\varphi}\hat{\nu}\hat{\varphi}}\right) = C_{\hat{\varphi}\hat{\nu}\hat{\varphi}} = -\frac{1}{\nu(\ )^{1/2}} \checkmark$$

11) ${}^{(2)}g = (\mu^2+\nu_0^2)\, d\mu \otimes d\mu + \mu^2\nu_0^2\, d\varphi\otimes d\varphi$, ${}^{(2)}\eta = \mu\nu_0(\mu^2+\nu_0^2)^{1/2}\, d\mu\wedge d\varphi$

Only the components $\bar\Gamma^i{}_{jk}$ on page 164 with no ν indices are relevant here:

${}^{(2)}\Gamma^\mu{}_{\mu\mu} = \frac{\mu}{\mu^2+\nu_0^2}$, ${}^{(2)}\Gamma^\varphi{}_{\mu\varphi} = \frac{1}{\mu} = {}^{(2)}\Gamma^\varphi{}_{\varphi\mu}$, ${}^{(2)}\Gamma^\mu{}_{\varphi\varphi} = -\frac{\mu\nu_0^2}{\mu^2+\nu_0^2}$.

To get ${}^{(2)}\underline{\omega}$, just delete the second row and column of $\underline{\bar\omega}_\mu d\mu + \underline{\bar\omega}_\varphi d\varphi$:

$$ {}^{(2)}\underline{\omega} = \begin{bmatrix} \frac{\mu}{\mu^2+\nu_0^2} & 0 \\ 0 & \mu^{-1} \end{bmatrix} d\mu + \begin{bmatrix} 0 & -\frac{\mu\nu_0^2}{\mu^2+\nu_0^2} \\ \mu^{-1} & 0 \end{bmatrix} d\varphi = \begin{bmatrix} \frac{\mu\, d\mu}{\mu^2+\nu_0^2} & -\frac{\mu\nu_0^2}{\mu^2+\nu_0^2} d\varphi \\ \mu^{-1} d\varphi & \mu^{-1} d\mu \end{bmatrix} $$

$$ d\left(\frac{\mu}{\mu^2+\nu_0^2}\right) = \frac{(\mu^2+\nu_0^2) - \mu(2\mu)}{(\mu^2+\nu_0^2)^2} = \frac{\nu_0^2-\mu^2}{(\mu^2+\nu_0^2)^2} \qquad d\mu^{-1} = -\mu^{-2} d\mu . $$

$$ d\,{}^{(2)}\underline{\omega} = \begin{bmatrix} 0 & \frac{\nu_0^2(\mu^2-\nu_0^2)}{(\mu^2+\nu_0^2)^2}\, d\mu\wedge d\varphi \\ -\mu^{-2} d\mu\wedge d\varphi & 0 \end{bmatrix} \qquad \left(d[f(\mu)\,d\mu] = \underset{=0}{f'(\mu)\, d\mu\wedge d\mu} \right) $$

$$ {}^{(2)}\underline{\omega}\wedge{}^{(2)}\underline{\omega} = \begin{bmatrix} \frac{\mu d\mu}{\mu^2+\nu_0^2} & -\frac{\mu\nu_0^2}{\mu^2+\nu_0^2} d\varphi \\ \mu^{-1} d\varphi & \mu^{-1} d\mu \end{bmatrix} \wedge \begin{bmatrix} \frac{\mu\, d\mu}{\mu^2+\nu_0^2} & -\frac{\mu\nu_0^2}{\mu^2+\nu_0^2} d\varphi \\ \mu^{-1} d\varphi & \mu^{-1} d\mu \end{bmatrix} $$

$$ = \begin{bmatrix} 0 & \underbrace{\left[-\frac{\mu^2\nu_0^2}{(\mu^2+\nu_0^2)^2} + \frac{\nu_0^2}{(\mu^2+\nu_0^2)}\right]}_{\frac{\nu_0^4}{(\mu^2+\nu_0^2)^2}} d\mu\wedge d\varphi \\ \left[-\frac{1}{\mu^2+\nu_0^2} + \frac{1}{\mu^2}\right] d\mu\wedge d\varphi & 0 \end{bmatrix} $$

$$ {}^{2}\underline{\Omega} = d{}^{(2)}\underline{\omega} + {}^{(2)}\underline{\omega}\wedge{}^{(2)}\underline{\omega} = \begin{bmatrix} 0 & \frac{\mu^2\nu_0^2}{(\mu^2+\nu_0^2)^2} d\mu\wedge d\varphi \\ -\frac{1}{\mu^2+\nu_0^2} d\mu\wedge d\varphi & \end{bmatrix} $$

$$ {}^{(2)}R^\varphi{}_{\mu\mu\varphi} = -\frac{1}{\mu^2+\nu_0^2} \qquad {}^{(2)}R^\mu{}_{\varphi\mu\varphi} = \frac{\mu^2\nu_0^2}{(\mu^2+\nu_0^2)^2} $$

$$ {}^{(2)}R_{\varphi\mu\mu\varphi} = -\frac{\mu^2\nu_0^2}{(\mu^2+\nu_0^2)} \qquad {}^{(2)}R_{\mu\varphi\mu\varphi} = \frac{\mu^2\nu_0^2}{(\mu^2+\nu_0^2)} = -{}^{(2)}R_{\varphi\mu\mu\varphi} . $$

$({}^{(2)}g_{\varphi\varphi}\,{}^{(2)}R^\varphi{}_{\mu\mu\varphi})$ $\qquad$ $({}^{(2)}g_{\mu\mu}\,{}^{(2)}R^\mu{}_{\varphi\mu\varphi})$

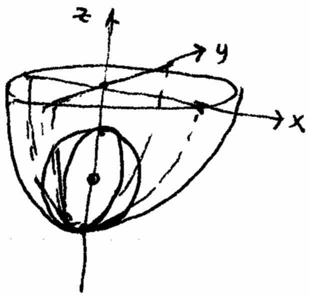
12) $$ {}^{(2)}R^{\hat\mu}{}_{\hat\varphi\hat\mu\hat\varphi} = (g_{\varphi\varphi})^{-1}\,{}^{(2)}R^\mu{}_{\varphi\mu\varphi} = \frac{1}{(\mu^2+\nu_0^2)^2} \xrightarrow{\text{at } \mu=0} \boxed{\frac{1}{\nu_0^4}} $$

12b) $z = \frac{1}{2\nu_0^2}\rho^2 - \frac{1}{2}\nu_0^2 \qquad \frac{dz}{d\rho} = \frac{\rho}{\nu_0^2} \qquad \frac{d^2z}{d\rho^2} = \frac{1}{\nu_0^2}$

$K = \dfrac{1/\nu_0^2}{[1+\rho^2/\nu_0^4]^{3/2}} \qquad K(\rho=0) = \frac{1}{\nu_0^2}.$

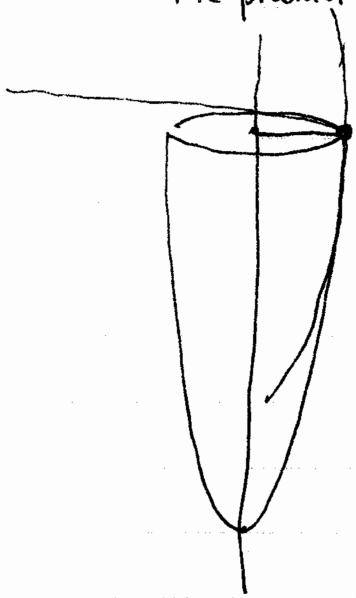
The relationship is $\quad {}^{(2)}R^{\hat{\mu}}{}_{\hat{\varphi}\hat{\mu}\hat{\varphi}}(\mu=0) = [K(\rho=0)]^2$

so $\quad {}^{(2)}R^{\alpha\beta}{}_{\gamma\delta} = {}^{(2)}K\,\delta^{\alpha\beta}_{\gamma\delta}, \quad {}^{(2)}K = \dfrac{1}{(\mu^2+\nu_0^2)^2}$



At the vertex any orthogonal pair of vertical planes through the z-axis are the same and lead to 2 orthogonal osculating circles of best fit to those parabolas with the same radius and center.

The "2-curvature" ${}^{(2)}K = R^{\hat{\mu}}{}_{\hat{\varphi}\hat{\mu}\hat{\varphi}}$ there is just the product of these two "1-curvatures" $K = \frac{1}{\nu_0^2}$. (See page 111.)



For very large μ ($\mu >> \nu_0$), (so $\frac{\rho^2}{\nu_0^4} = \frac{\mu^2\nu_0^2}{\nu_0^4} = \frac{\mu^2}{\nu_0^2} >> 1$) the horizontal cross-section (φ-coordinate circle of radius $\rho = \mu\nu_0$) is a circle whose connecting vector from its center to the point of tangency is almost along the normal direction. Together with the osculating circle of the parabola vertical cross-section, one obtains two nearly orthogonal circles of best fit:

$K_{circle} = \dfrac{1}{\rho} = \dfrac{1}{\mu\nu_0}$

$K_{parabola} = \dfrac{1}{\nu_0^2[1+\rho^2/\nu_0^4]^{3/2}} \approx \dfrac{1}{\nu_0^2\left(\frac{\rho^2}{\nu_0^4}\right)^{3/2}} = \dfrac{\nu_0^4}{\rho^3} = \dfrac{\nu_0^4}{(\mu\nu_0)^3} = \dfrac{\nu_0}{\mu^3}$

$K_{circle}\,K_{parabola} \approx \left(\dfrac{1}{\mu\nu_0}\right)\left(\dfrac{\nu_0}{\mu^3}\right) = \dfrac{1}{\mu^4}.$

${}^{(2)}K = \dfrac{1}{(\mu^2+\nu_0^2)^2} \approx \dfrac{1}{\mu^4}$

approximately equal.

$\underline{\mu \text{ lines}}$

13) $\mu = \lambda \quad \mu' = 1 \quad \mu'' = 0$

$\varphi = \varphi_0 \quad \varphi' = 0 \quad \varphi'' = 0$

$$\left.\begin{aligned} \frac{D^2\mu}{d\lambda^2} &= \underbrace{\mu''}_{0} + \underbrace{\Gamma^{\mu}{}_{\alpha\beta}\frac{d\bar{x}^\alpha}{d\lambda}\frac{d\bar{x}^\beta}{d\lambda}}_{\Gamma^{\mu}{}_{\mu\mu} = \frac{\mu}{\mu^2+\nu_0^2}} \\ \frac{D^2\varphi}{d\lambda^2} &= \underbrace{\varphi''}_{0} + \underbrace{\Gamma^{\varphi}{}_{\mu\mu}}_{0} = 0 \end{aligned}\right\} \qquad \frac{D^2\bar{x}^\alpha}{d\lambda^2} = \underbrace{\left(\frac{\mu}{\mu^2+\nu_0^2}\right)}_{} \frac{d\bar{x}^\alpha}{d\lambda}$$

proportional to tangent vector — so λ is nonaffine parametrization of geodesic.

$\underline{\varphi \text{ lines}}$

$\mu = \mu_0 \quad \mu' = 0 \quad \mu'' = 0$

$\varphi = \lambda \quad \varphi' = 1 \quad \varphi'' = 0$

$$\left.\begin{aligned} \frac{D^2\mu}{d\lambda^2} &= \underbrace{\mu''}_{0} + \underbrace{\Gamma^{\mu}{}_{\varphi\varphi}}_{-\frac{\mu\nu_0^2}{\mu^2+\nu^2}}\frac{d\varphi}{d\lambda}\frac{d\varphi}{d\lambda} = -\frac{\mu\nu_0^2}{\mu^2+\nu^2} \\ \frac{D^2\varphi}{d\lambda^2} &= \underbrace{\varphi''}_{0} + \underbrace{\Gamma^{\varphi}{}_{\varphi\varphi}}_{0}\frac{d\varphi}{d\lambda}\frac{d\varphi}{d\lambda} = 0 \end{aligned}\right\} \qquad \frac{D^2\bar{x}^\alpha}{d\lambda^2} \neq \cancel{-\frac{\mu\nu_0^2}{\mu^2+\nu_0^2}} \frac{d\bar{x}^\alpha}{d\lambda} \to \text{not geodesic.}$$

not proportional to

14) Evaluate $C^{\hat\varphi}{}_{\hat\mu\hat\varphi}$ of part 7) at $\nu = \nu_0$: $C^{\hat\varphi}{}_{\hat\mu\hat\varphi} = -\frac{1}{(\mu^2+\nu_0^2)^{1/2}\mu}$

$${}^{(2)}\Gamma^{\hat\varphi}{}_{\hat\varphi\hat\mu} = \tfrac12\Big(\underbrace{\cancel{C_{\hat\varphi\hat\varphi\hat\mu}}}_{-C_{\hat\varphi\hat\mu\hat\varphi}} - C_{\hat\varphi\hat\mu\hat\varphi} + \cancel{C_{\hat\mu\hat\varphi\hat\varphi}}\Big) = -C_{\hat\varphi\hat\mu\hat\varphi} = \frac{1}{\mu(\mu^2+\nu_0^2)^{1/2}} = -{}^{(2)}\Gamma^{\hat\mu}{}_{\hat\varphi\hat\varphi}$$

$${}^{(2)}\Gamma^{\hat\varphi}{}_{\hat\mu\hat\varphi} = \tfrac12\Big(C_{\hat\varphi\hat\mu\hat\varphi} - C_{\hat\mu\hat\varphi\hat\varphi} + \underbrace{C_{\hat\varphi\hat\varphi\hat\mu}}_{-C_{\hat\varphi\hat\mu\hat\varphi}}\Big) = 0 \text{ of course, antisymm in outer indices.}$$

$${}^{(2)}\underline{\hat\omega} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{1}{\mu(\mu^2+\nu_0^2)^{1/2}}(\mu\nu_0\, d\varphi) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\nu_0}{(\mu^2+\nu_0^2)^{1/2}} d\varphi \; ; \; {}^{(2)}\underline{\hat\omega} \wedge {}^{(2)}\underline{\hat\omega} = 0$$

$$d\,{}^{(2)}\underline{\hat\omega} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\left(-\frac12 \frac{2\nu_0\mu}{(\mu^2+\nu_0^2)^{3/2}}\right) d\mu\wedge d\varphi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{1}{(\mu^2+\nu_0^2)^2}\, \omega^{\hat\mu\hat\varphi}$$

$${}^{(2)}R^{\hat\mu}{}_{\hat\varphi\hat\mu\hat\varphi} = \frac{1}{(\mu^2+\nu_0^2)^2}\,.$$

[${}^{(2)}\underline{\hat\Omega} = d\,{}^{(2)}\underline{\hat\omega} + {}^{(2)}\underline{\hat\omega} \wedge {}^{(2)}\underline{\hat\omega}$ also valid for orthonormal frame — or in general — any frame]

14) b) $^{(2)}\nabla_{e_{\hat{\mu}}} e_{\hat{\mu}} = \underbrace{\Gamma^{\hat{\mu}}{}_{\hat{\mu}\hat{\mu}}}_{0} e_{\hat{\mu}} + \underbrace{\Gamma^{\hat{\varphi}}{}_{\hat{\mu}\hat{\mu}}}_{0} e_{\hat{\varphi}} = 0$

$^{(2)}\nabla_{e_{\hat{\mu}}} e_{\hat{\varphi}} = \underbrace{\Gamma^{\hat{\mu}}{}_{\hat{\mu}\hat{\varphi}}}_{0} e_{\hat{\mu}} + \underbrace{\Gamma^{\hat{\varphi}}{}_{\hat{\mu}\hat{\varphi}}}_{0} e_{\hat{\varphi}} = 0$

so they are parallel transported along $e_{\hat{\varphi}}$ which is the unit tangent to the μ coordinate lines. The first equality says $e_{\hat{\mu}}$ is autoparallel along μ & hence the curve must be a geodesic.

15) Yeah.

16) $\bar{X}^i = A^i{}_j(\bar{x})\, X^j(x(\bar{x}))$ recall $x^2+y^2+z^2 = r^2 = \left(\frac{\mu^2+\nu^2}{2}\right)^2$

$$\begin{bmatrix} X^\mu \\ X^\nu \\ X^\varphi \end{bmatrix} = \begin{bmatrix} \frac{\nu c}{(\ )} & \frac{\nu s}{(\ )} & \frac{\mu}{(\ )} \\ \frac{\mu c}{(\ )} & \frac{\mu s}{(\ )} & \frac{-\nu}{(\ )} \\ -\frac{1}{\mu\nu} s & \frac{1}{\mu\nu} c & 0 \end{bmatrix} \begin{bmatrix} -\mu\nu s \\ \mu\nu c \\ \left(\frac{\mu^2+\nu^2}{2}\right)^2 \end{bmatrix} = \begin{bmatrix} \frac{-\mu\nu^2 cs + \mu\nu^2 cs}{(\ )} + \frac{\mu}{\mu^2+\nu^2}\left(\frac{\mu^2+\nu^2}{2}\right)^2 \\ \frac{-\mu^2\nu cs + \mu^2\nu cs}{(\ )} - \frac{\nu}{\mu^2+\nu^2}\left(\frac{\mu^2+\nu^2}{2}\right)^2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\mu}{4}(\mu^2+\nu^2) \\ -\frac{\nu}{4}(\mu^2+\nu^2) \\ 1 \end{bmatrix}$$

$$\bar{X}_i = X_j(x(\bar{x}))\, A^{-1\,j}{}_i(\bar{x})$$

$$[X_\mu\ X_\nu\ X_\varphi] = \left[-\mu\nu s,\ \mu\nu c\ \ \left(\frac{\mu^2+\nu^2}{2}\right)^2\right] \begin{bmatrix} \nu c & \mu c & -\mu\nu s \\ \nu s & \mu s & \mu\nu c \\ \mu & -\nu & 0 \end{bmatrix}$$

$$= \left[-\mu\nu^2 cs + \mu\nu^2 cs + \frac{\mu}{4}(\mu^2+\nu^2)^2 \quad -\mu^2\nu cs + \mu^2\nu cs - \frac{\nu}{4}(\mu^2+\nu^2)^2 \quad \mu^2\nu^2\right]$$

$$= \left[\frac{\mu}{4}(\mu^2+\nu^2)^2 \quad -\frac{\nu}{4}(\mu^2+\nu^2)^2 \quad \mu^2\nu^2\right] = (g_{\mu\mu} X^\mu,\ g_{\nu\nu} X^\nu,\ g_{\varphi\varphi} X^\varphi) \checkmark$$

$X^\flat = \frac{(\mu^2+\nu^2)^2}{4}(\mu\, d\mu - \nu\, d\nu) + \mu^2\nu^2\, d\varphi$

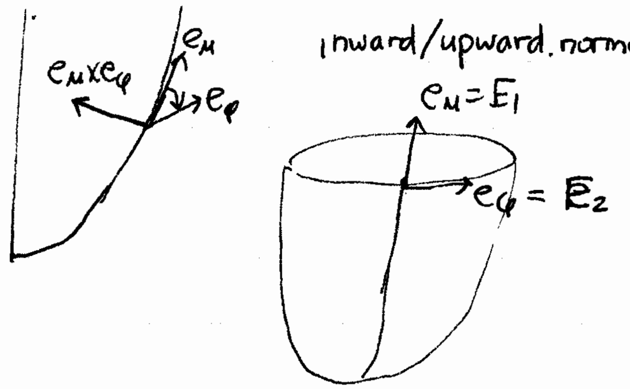
$$[\nabla_{e_\mu} X]^i = X^i{}_{;\mu} = X^i{}_{,\mu} + \Gamma^i{}_{\mu j} X^j$$

$$X^\mu{}_{;\mu} = \underbrace{X^\mu{}_{,\mu}}_{\frac{1}{4}(3\mu^2+\nu^2)} + \underbrace{\Gamma^\mu{}_{\mu\mu}}_{\frac{\mu}{\cancel{\mu^2+\nu^2}}}\underbrace{X^\mu}_{\frac{\mu(\cancel{\mu^2+\nu^2})}{4}} + \underbrace{\Gamma^\mu{}_{\mu\nu}}_{\frac{\nu}{\cancel{\mu^2+\nu^2}}}\underbrace{X^\nu}_{-\frac{\nu}{4}(\cancel{\mu^2+\nu^2})} = \frac{1}{4}(3\mu^2 + \cancel{\nu^2} + \mu^2 - \cancel{\nu^2}) = \mu^2$$

$$X^\nu{}_{;\mu} = \underbrace{X^\nu{}_{,\mu}}_{-\frac{\nu}{4}(2\mu)} + \underbrace{\Gamma^\nu{}_{\mu\mu}}_{\frac{-\nu}{(\cancel{\mu^2+\nu^2})}}\underbrace{X^\mu}_{\frac{\mu(\cancel{\mu^2+\nu^2})}{4}} + \underbrace{\Gamma^\nu{}_{\mu\nu}}_{\frac{\mu}{(\cancel{\mu^2+\nu^2})}}\underbrace{X^\nu}_{\left(-\frac{\nu}{4}(\cancel{\mu^2+\nu^2})\right)} = \frac{1}{4}(-2\mu\nu - \mu\nu - \mu\nu) = -\mu\nu$$

$$X^\varphi{}_{;\mu} = \underbrace{X^\varphi{}_{,\mu}}_{0} + \underbrace{\Gamma^\varphi{}_{\mu\mu}X^\mu}_{0} + \underbrace{\Gamma^\varphi{}_{\mu\nu}X^\nu}_{0} + \underbrace{\Gamma^\varphi{}_{\mu\varphi}}_{\frac{1}{\mu}}\underbrace{X^\varphi}_{1} = \frac{1}{\mu}$$

$$\nabla_{e_\mu} X = \mu^2 e_\mu - \mu\nu e_\nu + \mu^{-1} e_\varphi .$$

17) 

inward/upward normal.

E_1, E_2 oriented, E_1 outer so E_2 gives induced orientation counterclockwise from above.

$\partial\Sigma$: $\mu = \nu_0$, $\nu = \nu_0$, $\varphi = \lambda$ $\quad$ $\mu' = 0$, $\nu' = 0$, $\varphi' = 1$ $\quad$ $0 \le \lambda \le 2\pi$ is an oriented parametrization of $\partial\Sigma$.

$$\int_{\partial\Sigma} X^\flat = \int_{\partial\Sigma} \left(\frac{\mu^2+\nu^2}{2}\right)^2 (\mu\, d\mu - \nu\, d\nu) + \mu^2\nu^2 d\varphi$$

$$= \int_0^{2\pi} \nu_0^4 \left[0 - 0 + d\lambda \right] = 2\pi \nu_0^4$$

$$dX^\flat = \underbrace{\frac{2}{4}(\mu^2+\nu^2)(2\nu\, d\nu \wedge \mu\, d\mu) - \frac{2}{4}(\mu^2+\nu^2)(2\mu\, d\mu \wedge \nu\, d\nu)} + 2\mu\nu^2 d\mu \wedge d\varphi + 2\mu^2\nu\, d\nu \wedge d\varphi$$

$$\underbrace{-\mu\nu(\mu^2+\nu^2)\, d\mu \wedge d\nu - \mu\nu(\mu^2+\nu^2)\, d\mu \wedge d\nu}$$

$$\underbrace{-2\mu\nu(\mu^2+\nu^2)\, d\mu \wedge d\nu}$$

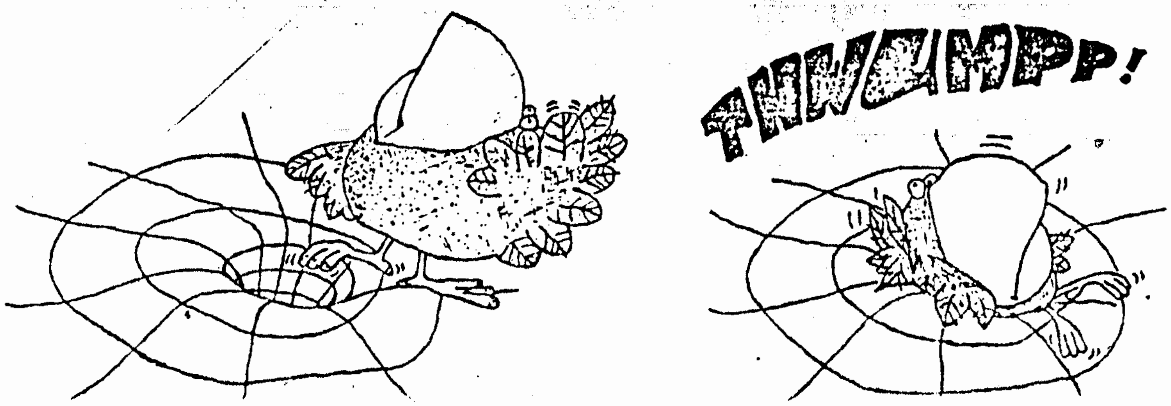
$\mu = u^1$, $\nu = \nu_0$, $\varphi = u^2$ $\quad$ $0 \le u^1 \le \nu_0$, $0 \le u^2 \le 2\pi$ } oriented parametrization of Σ

$$\int_\Sigma dX^\flat = \int_0^{2\pi}\int_0^{\nu_0} \left[0 + 2u^1\nu_0^2\, du^1 du^2 + 0 \right] = 2\pi\, \underline{u^{1\,2}}\, \nu_0^2 \Big|_0^{\nu_0}$$

$$= 2\pi \nu_0^4 \checkmark$$

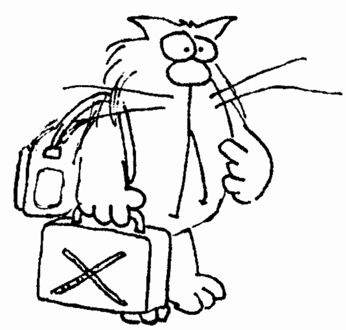
The End (for real).

A takehome final or a blackhole ...?



Didnt mean to suck you into this group project at the end — there just wasnt enough time.

I know you are all anxious to take off.



Please read over these notes.
Hope they made some impression.

— bob

(really, this is the last page)

(for now)