Suppose we have a parametrized p-surface Σ with a boundary, part of which corresponds to constant values of the first parameter. The p-vector \( E_i(u) \), \( i = 1, \ldots, p \), determines the inner orientation of \( \Sigma \) at each point, said to be positively oriented. At the boundary, to be denoted by \( \partial \Sigma \), half of the tangent p-plane to \( \Sigma \) with hang off the p-surface — in fact the tangent (p-1)-plane to \( \partial \Sigma \) at these boundary points will cut the tangent p-plane to \( \Sigma \) into two halves. Half of these nonzero vectors will point inward towards interior points of \( \Sigma \), while half will point outward, except for those vectors in the tangent (p-1)-plane subspace which are tangent to \( \partial \Sigma \).

If \( u^i \leq b \) near part of the boundary \( \partial \Sigma \), so that \( E_i(u) \) points outward, then the remaining parameters \( \{ u^2, \ldots, u^p \} \) give a parametrization of \( \partial \Sigma \) whose associated orientation, namely that of \( E_2(u) \wedge \ldots \wedge E_p(u) \), is called the induced orientation of \( \partial \Sigma \), determined by the orientation of \( \Sigma \) (namely \( E_1(u) \wedge \ldots \wedge E_p(u) \)).

If \( u^1 \leq a \) instead, so that \( E_1(u) \) points inward, then \( \{ u^2, \ldots, u^p \} \) are said to give an orientation for \( \partial \Sigma \) (namely \( E_2(u) \wedge \ldots \wedge E_p(u) \)) which is opposite to the induced orientation.

Another way of stating this is that if \( \{ E_i \}_{i=1}^{p+1} \) is any set of vector fields which provide a positively oriented basis for the tangent p-planes to \( \Sigma \) such that on \( \partial \Sigma \), \( E_1 \) points outward while \( E_2 \wedge \ldots \wedge E_p \).
describes the \((p-1)\)-dimensional subspace tangent to \(\partial \Sigma\), then
\[E_2 \wedge \ldots \wedge E_p\]
is positively oriented with respect to the induced orientation of \(\partial \Sigma\). In the above parametrization definition,

\[-E_1(u)\text{ points outward when } 0 \leq u \leq 1\] describes the boundary, and

\[\begin{align*}
[-E_1(u) \wedge & \ldots \wedge E_p(u)] = \\
\text{outer induced orientation} & \quad \text{orientation for } \Sigma
\end{align*}\]

When \(u \leq b\), then

\[\begin{align*}
[E_1(u) \wedge & \ldots \wedge E_p(u)] = \\
\text{outer induced orientation} & \quad \text{orientation for } \Sigma
\end{align*}\]

We can even extend this to the case \(p = 1\) of a curve segment \(\Sigma\) with its two 0-dimensional endpoints \(\partial \Sigma\), on which a 0-vector (function) orientation can be induced:

\[\begin{align*}
\Sigma & \quad [E_1(u) \wedge [-1]] = E_1(u) \\
\partial \Sigma^+ & \quad \text{induced orientation} \quad \text{for } \partial \Sigma^+ \\
\partial \Sigma^- & \quad \text{induced orientation} \quad \text{for } \partial \Sigma^+
\end{align*}\]

This assigns a plus sign to the terminal point and a minus sign to the initial point of the directed curve segment.

Note that when \(p = n\), one can always use the orientation of the whole space on \(\Sigma\) for its orientation.
Note that for $n=3$, $p=2$ or $p=3$, we can also describe the inner orientation of a surface by a choice of a vector off the surface picking out one side or the other and linking it to the inner orientation by the right-hand rule. This is called an outer orientation for the surface, and is the way we were introduced to the orientation of a surface in multivariable calculus.

**Example**

Let $\Sigma$ be the region $r_1 \leq r \leq r_2$ in $\mathbb{R}^3$, $u = r$.

$x = r_2 \sin \theta \cos \psi$
$y = r_2 \sin \theta \sin \psi$
$z = r_2 \cos \theta$

$0 \leq \theta \leq \pi$
$0 \leq \psi \leq 2\pi$

$E_i(u)$ points out of $\Sigma$

$\{r, \theta, \psi\}$ orient the outer spherical boundary (equivalent to the choice of outer normal).

$x = r \sin \theta \cos \psi$
$y = r \sin \theta \sin \psi$
$z = r \cos \theta$

$0 \leq \theta \leq \pi$
$0 \leq \psi \leq 2\pi$

$E_i(u)$ points into $\Sigma$

$\{r, \theta, \psi\}$ orient the inner spherical boundary (equivalent to the choice of inner normal).
Example: A ball of radius \( R \) in \( \mathbb{R}^4 \): \( x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq R^2 \)

At the North pole, \( \partial \Omega \) points out of \( \Omega \).

The tangent plane \( x_4 = R \) is tangent to \( \partial \Omega \).

\[
\frac{\partial}{\partial x_4} \wedge \left(-\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}\right) = + \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}
\]

inner orientation of \( \partial \Omega \)

orientation of \( \mathbb{R}^4 \)

taken as orientation for \( \Omega \)

The induced orientation of the boundary \( \partial \Omega = S^3 \) at the North pole is the opposite of the subspace \( \mathbb{R}^3 \subset \mathbb{R}^4 \) \( (x_4 = \text{constant}) \) with its natural orientation, \( \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \).

STOKES THEOREM (no metric required):

\[
\oint_{\partial B} \mathbf{T} = \int_B d\mathbf{T}
\]

\( p \)-form \( \Rightarrow \) oriented \((p+1)\)-surface with boundary \( \partial B \), with induced orientation.

The proof of this is not worth our attention at this late date, considering that our time has expired.

If one has a metric around, one can rewrite this theorem using the metric so that one can picture what it means a little better.
Let $\mathbf{I}^b$ be our 1-form. Then using $z^z = 1$ on $\mathbb{R}^3$:

$$
\int_{\partial B} \mathbf{I}^b = \int_B d\mathbf{I}^b \\
\equiv \int_B \left( \sum_{ijk} \mathbf{I}^k \frac{\partial z^i}{\partial x^j} \right) \wedge \frac{d\mathbf{x}^j}{\sqrt{\det g}} \\
\equiv \int_B (\text{curl } \mathbf{I}) \cdot \frac{d\mathbf{x}^j}{\sqrt{\det g}}
$$

vector differential of surface area on $B$

$$
\int_B \mathbf{I}^b \cdot d\mathbf{s} = \int_B (\text{curl } \mathbf{I}) \cdot d\mathbf{s}
$$

A little more work shows

$$
\int_{\partial B} \mathbf{I}^b \cdot \mathbf{T} \cdot d\mathbf{s} = \int_B (\text{curl } \mathbf{I}) \cdot \mathbf{N} \cdot d\mathbf{s}
$$

differential of surface area

unit tangent with induced orientation

differential of arclength

right-hand rule related unit normal

Stokes' Theorem on $\mathbb{R}^3$
Let \( *I^<b> \) be our 2-form. Then

\[
\int_B \mathbf{x} \cdot dS = \int_B \mathbf{x} \wedge dS = \int_B \mathbf{x} \wedge \nabla \mathbf{x} \cdot dS = \int_B \mathbf{x} \cdot \nabla \mathbf{x} \cdot dS
\]

or

\[
\int_B \mathbf{x} \cdot \mathbf{N} \cdot dS = \int_B \mathbf{x} \cdot (\nabla \mathbf{x}) \cdot dV
\]
Okay, time for parting words.

One semester is so short a time. There are still many basic notions remaining, among the most important: groups of transformations and their associated derivative operator—the Lie derivative. This is also important for the metric geometry we have explored—to describe symmetries of the geometry.

The language I have partially introduced you to is basic to the description of finite-dimensional continuous physical systems (and some co-dimension too). It is interesting in its own right as pure mathematics, and a very powerful tool for describing many aspects of how our world works. I hope you have enjoyed seeing some of this structure a fraction as much as I have enjoyed the opportunity to rethink some of these ideas.
P.S. Some worked examples of Stokes' Theorem for $\mathbb{R}^3$

\[ \int_S \mathbf{I}^\bullet = \sum \int_{S_i} \mathbf{dI}^\bullet \]

$\mathbf{I}^\bullet$ a 1-form.

$\Sigma$: $x^2 + y^2 + z^2 = a^2$ 
oriented by upper normal

$\partial \Sigma$: $x^2 + y^2 = a^2$ 
induced orientation

Parametrizations with correct orientation:

\( \Phi_\Sigma \):

Set \( r = a \) in spherical coord
parametrization map

\[
\begin{align*}
\theta = \varphi & : x = a \sin \theta \cos \varphi \\
y = a \sin \theta \sin \varphi \\
z = a \cos \theta
\end{align*}
\]

\( 0 \leq \theta \leq \pi \)
\( 0 \leq \varphi \leq 2\pi \)

\( \theta, \varphi \) orient \( \Sigma \) with the correct orientation related to the upward normal by the right-hand rule

On the bounding circle (equator of sphere)

\[ e_\theta \times [ e_\varphi ] = e_\theta \wedge e_\varphi \]

Outer normal of \( \Sigma \)

Induced orientation of \( \partial \Sigma \)

\( \partial \Sigma \): Set \( r = a, \) \( \theta = \pi \) in spherical coord
parametrization map

\[
\begin{align*}
\theta = \pi & : x = a \cos \theta \\
y = a \sin \theta \\
z = 0
\end{align*}
\]

\( 0 \leq \theta \leq 2\pi \)

\( \varphi \) provides the correct orientation for \( \partial \Sigma \).

Now we need a 1-form to use to verify this version of Stokes' theorem.

Let's take our old friend \( \mathbf{I}^\bullet = y \mathbf{d}x + x \mathbf{d}y \)

\( d\mathbf{I}^\bullet = dy \mathbf{d}x + dx \mathbf{d}y = 0 \)

(Recall \( \mathbf{I}^\bullet = d(xy) \))

So \( d\mathbf{I}^\bullet = d^2(xy) = 0 \)

So the right side of Stokes' Theorem is identically zero.
For the left side,
\[
\phi_E^\ast (\mathbf{I}^b) = (\sin \theta) \frac{d}{d \theta} \left( \cos \phi \right) + (\cos \theta) \frac{d}{d \phi} \left( \sin \phi \right)  \\
= - \frac{2 \sin \theta \cos \phi d \phi}{a^2 \cos^2 \theta} + \frac{2 \cos \theta d \theta}{a^2 \sin \theta}  \\
\int_{\phi}^\phi\int_{\theta}^\theta a^2 \sin \theta \cos \phi d \phi d \theta = - \frac{2 \sin \theta \cos \phi |_{\phi}^{\phi}}{a^2 \cos^2 \theta}  \\
= 0  \\
\checkmark
\]

Okay, let's try something more interesting by switching a sign.

Take instead \[ \mathbf{I}^b = - y dx + x dy \]
\[
\phi_E^\ast (\mathbf{I}^b) = (\sin \theta) \frac{d}{d \phi} \left( \cos \phi \right) + (\cos \theta) \frac{d}{d \phi} \left( \sin \phi \right)  \\
= - \frac{2 \sin \theta \cos \phi d \phi}{a^2 \cos^2 \theta} + \frac{2 \cos \theta d \theta}{a^2 \sin \theta}  \\
\int_{\phi}^\phi\int_{\theta}^\theta a^2 \sin \theta \cos \phi d \phi d \theta = 2 \pi a^2  \\
\checkmark
\]

\[
d\mathbf{I}^b = - dy dx + dx dy = 2 dx dy
\]
\[
\phi_E^\ast (\mathbf{I}^b) = 2 \int_{\phi}^{\phi} \int_{\theta}^{\theta} a^2 \theta d \theta d \phi  \\
= 2 a^2 \left[ \cos \theta \sin \phi + \sin \theta \cos \phi \right] \left[ \cos \phi \sin \theta + \sin \phi \cos \theta \right]  \\
= 2 a^2 \left[ \cos \theta \cos \phi \sin \phi \cos \theta - \cos \theta \sin \phi \cos \phi \sin \theta \right] d \phi d \theta  \\
= 2 a^2 \sin \theta \cos \phi \sin \phi \cos \theta d \phi d \theta  \\
\int_{\phi}^{\phi}\int_{\theta}^{\theta} a^2 \sin \theta \cos \phi d \phi d \theta = 2 \pi a^2  \\
\checkmark
\]

\[
\int (\text{curl } \mathbf{I}) \cdot \mathbf{n} \, dS
\]
The multivariable calculus approach to some problem

\[ \mathbf{I} = (-y, x, 0) \]
\[ \text{curl} \mathbf{I} = (\frac{\partial}{\partial y} - \frac{\partial}{\partial z}, \frac{\partial}{\partial z} - \frac{\partial}{\partial x}, \frac{\partial}{\partial x} - \frac{\partial}{\partial y}) = (0, 0, 0) \]

\[ y^2 + z^2 = a^2 \rightarrow \mathbf{r}(x, y, z) = (ax, ay, az) \rightarrow \text{normalize to get unit normal} \]
\[ \hat{n} = \frac{r}{|r|} = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}} \]
\[ \text{curl} \mathbf{I} \cdot \hat{n} = \frac{2z}{r} = 2 \cos \theta \quad \text{in spherical coordinates.} \]

The surface area differential is \( a \sin \theta \, d\theta \, d\phi \) (recall this)

\[ \int_{S} \text{curl} \mathbf{I} \cdot \hat{n} \, ds = \int_{0}^{2\pi} \int_{0}^{\pi} (a \sin \theta) (a \sin \theta) \, d\theta \, d\phi = 2\pi a^2 \quad \text{as before.} \]

\[ \int_{S} \mathbf{I} \cdot ds = \int_{S} \mathbf{I} \cdot \hat{t} \, ds \]

\[ x = a \cos \phi \quad y = a \sin \phi \quad z = 0 \]

\[ \mathbf{r}'(\phi) = (-y, x, 0) \quad \mathbf{r}'(\phi) = (\sin \phi, \cos \phi, 0) \]

\[ \frac{\hat{t}}{\sqrt{x^2 + y^2}} = \frac{(x, y, 0)}{\sqrt{x^2 + y^2}} \]

\[ \int_{S} \mathbf{I} \cdot \hat{t} \, ds = \int_{0}^{2\pi} \int_{0}^{\pi} \mathbf{I} \cdot \hat{t} \, d\theta \, d\phi = 2\pi a^2. \]

\[ \text{or, just plugging in parameters} \]

\[ \int_{S} \mathbf{I} \cdot \mathbf{r}'(\phi) \, d\phi = \int_{0}^{2\pi} \left[ a \sin \phi \cos \phi \right] \, d\phi = 2\pi a^2 \]

The nonmetric version is clearly simpler, but the metric version gives us a physical picture of what we are integrating.

154
How about a Gauss's law problem? Let $\mathcal{E}$ be the interior of the upper hemisphere of radius $a$ at the origin, with the usual $\mathbb{R}^3$ orientation. $\partial \mathcal{E}$ has two parts: the upper hemisphere and the disk of radius $a$ in the $xy$ plane. In each case we can use a spherical coordinate parametrization:

$\mathcal{E}^+$:
- $x = r \sin \theta \cos \phi$
- $y = r \sin \theta \sin \phi$
- $z = r \cos \theta$

$0 \leq r \leq a$

$0 \leq \theta \leq \frac{\pi}{2}$

$0 \leq \phi \leq 2\pi$

$\mathcal{E}^-$:
- $x = r \cos \phi$
- $y = r \sin \phi$
- $z = 0$

$0 \leq r \leq a$

$0 \leq \phi \leq 2\pi$

$\mathcal{E}$ oriented according to $\mathbb{R}^3$.

Now we need a 2-form to integrate on $\partial \mathcal{E}$.

Take $\mathbf{x}^\times$ where $\mathbf{x} = x dydz + ydzdx + zdxdy$.

On $\mathcal{E}^+$:

$\mathbf{x}^\times = (x, y, z, r) = r^2 \sin \theta d\theta d\phi$

length $r$.

radial direction.

Integration:

$\int_{\partial \mathcal{E}} \mathbf{x}^\times = \int_{\mathcal{E}} (x dydz + ydzdx + zdxdy)$

$= \int_{\mathcal{E}^+} (x dydz + ydzdx + zdxdy)$

$= \int_{\mathcal{E}^-} (x dydz + ydzdx + zdxdy)$

$= \int_{\mathcal{E}} (x dydz + ydzdx + zdxdy)$

Now, $\mathbf{x}^\times = r^2 \sin \theta d\theta d\phi$.

$\int_{\partial \mathcal{E}} \mathbf{x}^\times = \int_{\mathcal{E}^+} r^2 \sin \theta d\theta d\phi$

and

$\int_{\partial \mathcal{E}} \mathbf{x}^\times = \int_{\mathcal{E}^-} r^2 \sin \theta d\theta d\phi$

Since

$\iiint_{\mathcal{E}} \mathbf{x}^\times = \int_{\partial \mathcal{E}} \mathbf{x}^\times$

is oriented $\mathbb{R}^3$.

$\int_{\mathcal{E}} r^2 \sin \theta d\theta d\phi = \frac{1}{2} \pi a^2$

$\int_{\partial \mathcal{E}} \mathbf{x}^\times = \int_{\mathcal{E}^+} r^2 \sin \theta d\theta d\phi + \int_{\mathcal{E}^-} r^2 \sin \theta d\theta d\phi$

$= \frac{1}{2} \pi a^2$

$\therefore \int_{\partial \mathcal{E}} \mathbf{x}^\times = \frac{1}{2} \pi a^2$

Okay, I quit.