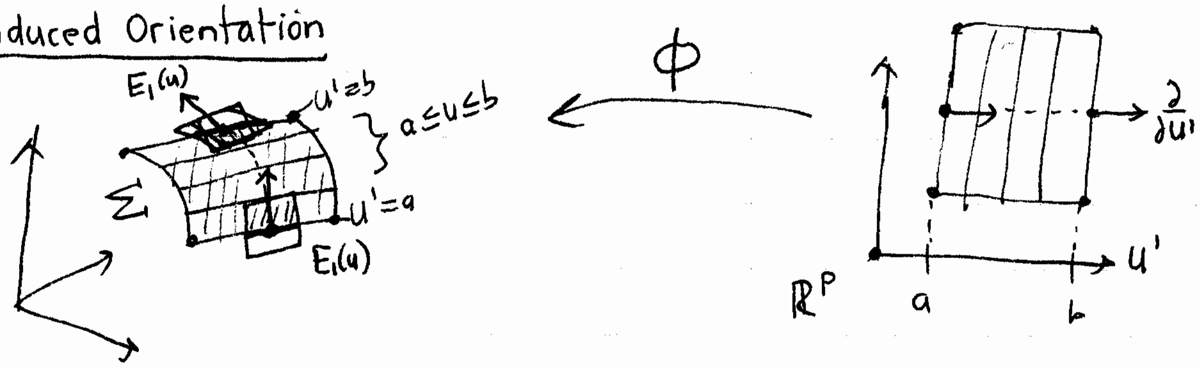
Induced Orientation



Suppose we have a parametrized p-surface Σ with a boundary, part of which corresponds to constant values of the first parameter. The p-vector $E_1(u)\wedge\ldots\wedge E_p(u)$ determines the inner orientation of Σ at each point, said to be positively oriented. At the boundary, to be denoted by $\partial\Sigma$, half of the tangent p-plane to Σ with hang off the p-surface — in fact the tangent (p-1)-plane to $\partial\Sigma$ at these boundary points will cut the tangent p-plane to Σ into two halves. Half of the nonzero vectors will point inward towards interior points of Σ, while half will point outward, except for those vectors in the tangent (p-1) plane subspace which are tangent to $\partial\Sigma$.

If $u^1 \le b$ near part of the boundary $\partial\Sigma$, so that $E_1(u)$ points outward, then the remaining parameters $\{u^2,\ldots,u^p\}$ give a parametrization of $\partial\Sigma$ whose associated orientation, namely that of $E_2(u)\wedge\ldots\wedge E_p(u)$, is called the induced orientation of $\partial\Sigma$, determined by the orientation of Σ (namely $E_1(u)\wedge\ldots\wedge E_p(u)$).

If $a \le u^1$ instead, so that $E_1(u)$ points inward, then $\{u^2,\ldots,u^p\}$ are said to give an orientation for $\partial\Sigma$ (namely $E_2(u)\wedge\ldots\wedge E_p(u)$) which is opposite to the induced orientation.

Another way of stating this is that if $\{E_\alpha\}_{\alpha=1,\ldots,p}$ is any set of vector fields which provide a positively oriented basis for the tangent p-planes to Σ such that on $\partial\Sigma$, E_1 points outward while $E_2\wedge\ldots\wedge E_p$

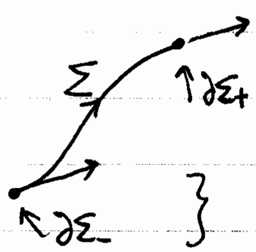
describes the $(p-1)$-dimensional subspace tangent to $\partial\Sigma$, then $E_2\wedge\ldots\wedge E_p$ is positively oriented with respect to the induced orientation of $\partial\Sigma$. In the above parametrization definition, $-E_1(u)$ points outward when $a \leq u^1$ describes the boundary, and

$$\underbrace{[-E_1(u)]}_{\text{outer}}\wedge\underbrace{[-E_2(u)\wedge\ldots\wedge E_p(u)]}_{\substack{\text{induced orientation}\\ \text{for } \partial\Sigma}} = \underbrace{E_1(u)\wedge\ldots\wedge E_p(u)}_{\text{orientation } \Sigma}$$

When $u^1 \leq b$, then

$$\underbrace{[E_1(u)]}_{\text{outer}}\wedge\underbrace{[E_2(u)\wedge\ldots\wedge E_p(u)]}_{\substack{\text{induced orientation}\\ \text{for } \partial\Sigma}} = \underbrace{E_1(u)\wedge\ldots\wedge E_p(u)}_{\text{orientation for } \Sigma}.$$

We can even extend this to the case $p=1$ of a curve segment Σ with its two 0-dimensional endpoints $\partial\Sigma$ on which a 0-vector (function) orientation can be induced:



$$\underbrace{[E_1(u)]}_{\text{outer}}\wedge\underbrace{[+1]}_{\substack{\text{induced}\\ \text{orientation}\\ \text{for } \partial\Sigma_+}} = E_1(u)$$

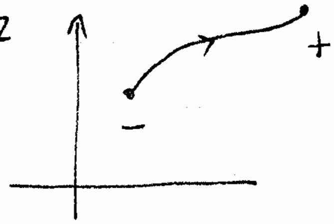
$$\underbrace{[-E_1(u)]}_{\text{outer}}\wedge\underbrace{[-1]}_{\substack{\text{induced}\\ \text{orientation}\\ \text{for } \partial\Sigma_+}} = E_1(u)$$

This assigns a plus sign to the terminal point and a minus sign to the initial point of the directed curve segment.

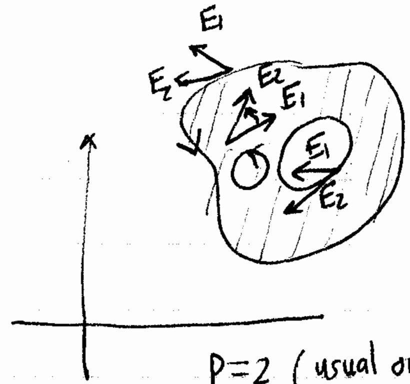
Note that when $p=n$, one can always use the orientation of the whole space on ~~the~~ Σ for its orientation.

examples

$\mathbb{R}^2$



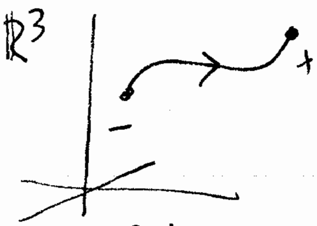
$p=1$



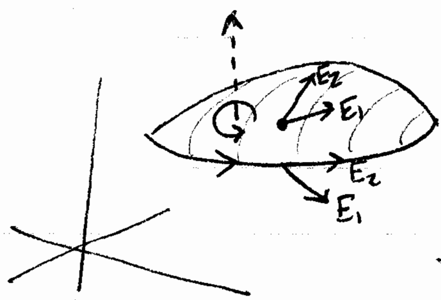
$p=2$ (usual orientation on $\mathbb{R}^2$)

Note that the induced orientation is the direction in which the loop flows if you bring it to the boundary.

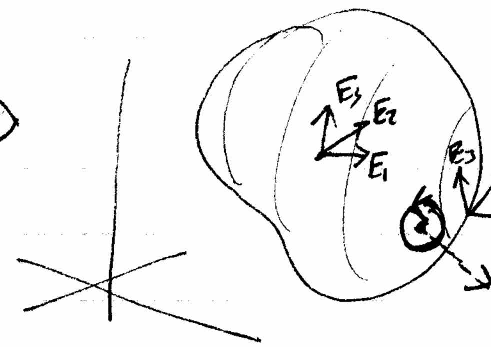
$\mathbb{R}^3$



$p=1$



$p=2$

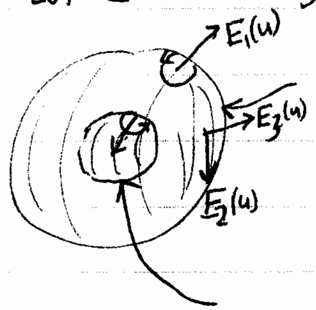


$p=3$

Note that for $n=3$, $p=2$ or $p=3$, we can also describe the inner orientation of a surface by a choice of a vector off the surface picking out one side or the other and linking it to the inner orientation by the right hand rule. This is called an outer orientation for the surface, and is the way we were introduced to the orientation of a surface in multivariable calculus.

example

Let Σ be the region $r_1 \le r \le r_2$ in $\mathbb{R}^3$. $u^1 = r$.



$x = r_2 \sin\theta\cos\varphi \qquad 0 \le \theta \le \pi$

$y = r_2 \sin\theta \sin\varphi \qquad 0 \le \varphi \le 2\pi$

$z = r_2 \cos\theta \qquad (u^2, u^3) = (\theta, \varphi)$

$E_1(u)$ points out of Σ; $\{\theta, \varphi\}$ orient the outer spherical boundary (equivalent to the choice of outer normal)

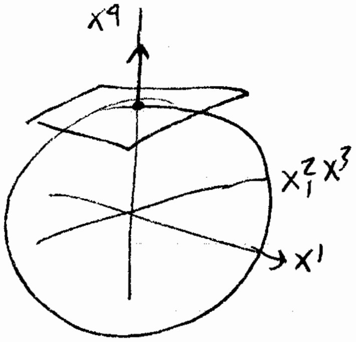
$x = r_2 \sin\theta \cos\varphi \qquad 0 \le \theta \le \pi$

$y = r_1 \sin\theta \sin\varphi \qquad 0 \le \varphi \le 2\pi$

$z = r_1 \cos\theta \qquad (u^2, u^3) = (\varphi, \theta)$

$E_1(u)$ points into Σ; $\{\varphi, \theta\}$ orient the inner spherical boundary (equivalent to the choice of inner normal).

example: Σ is a ball of radius R in $\mathbb{R}^4$: $(x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2 \le R^2$



$\mathbb{R}^4$, $p=4$

At the North pole $\partial/\partial x^4$ points out of Σ.
The tangent plane $x^4 = R$ is tangent to $\partial\Sigma$.

$$\underbrace{\frac{\partial}{\partial x^4}\wedge\Big(\underbrace{-\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\wedge\frac{\partial}{\partial x^3}}_{\text{inner orientation of }\partial\Sigma}\Big)} = \underbrace{+\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\wedge\frac{\partial}{\partial x^3}\wedge\frac{\partial}{\partial x^4}}_{\text{orientation of }\mathbb{R}^4\text{ taken as orientation for }\Sigma}$$

The induced orientation of the boundary $\partial\Sigma = S^3$ at the North pole is the opposite of the subspace $\mathbb{R}^3\subset\mathbb{R}^4$ ($x^4=$constant) with its natural orientation $\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\wedge\frac{\partial}{\partial x^3}$.

STOKES THEOREM (no metric required):

$$\int_{\partial B} T = \int_B dT$$

T: p-form

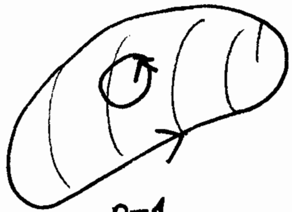
B: oriented $(p+1)$-surface with boundary ∂B with induced orientation

The proof of this is not worth our attention at this late date, considering that our time has expired.

If one has a metric around, one can rewrite this theorem using the metric so that one can picture what it means a little better.

example

$\mathbb{R}^3$



$p=1$

Let $X^\flat$ be our 1-form. Then using $** = 1$ on $\mathbb{R}^3$:

$$\int_{\partial B} X^\flat = \int_B dX^\flat = \int_B *(\underbrace{*dX^\flat})$$

$$\underbrace{\equiv [\mathrm{curl}\, X]^\flat}$$

$$(\mathrm{curl}\, X)^i \underbrace{\eta_{ijk}\, dx^{jk}/2}$$

$$\equiv \text{"} dS_i \text{"}$$ vector differential of surface area on B

$$= \int_B (\mathrm{curl}\, X)\cdot d\vec{S}$$

$$\int_{\partial B} X^\flat = \int_{\partial B} X_i\, dx^i = \int_{\partial B} \underbrace{X^i g_{ij}\, dx^j}_{X\cdot d\vec{s}}$$

$ds_i \equiv g_{ij}\,dx^j$,
$ds^i \equiv dx^i$
vector differential of arclength.

$$\boxed{\int_{\partial B} X\cdot d\vec{s} = \int_B (\mathrm{curl}\, X)\cdot d\vec{S}}$$

Usual Stokes's Theorem on $\mathbb{R}^3$

A little more work shows

$$\int_{\partial B} X\cdot \hat{T}\, ds = \int_B (\mathrm{curl}\, X)\cdot \hat{N}\, dS$$

$\hat{T}$: unit tangent with induced orientation

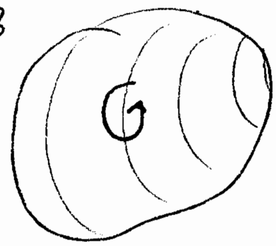
ds: differential of arclength

$\hat{N}$: righthand rule related unit normal

dS: differential of surface area.

example

$\mathbb{R}^3$



$p=2$

Let $*\mathbb{X}^\flat$ be our 2-form. Then

$$\int_{\partial B} *\mathbb{X}^\flat = \int_B d*\mathbb{X}^\flat = \int_B *(\underbrace{*d*\mathbb{X}^\flat}_{\operatorname{div}\mathbb{X}}) = \int_B \underbrace{*(\operatorname{div}\mathbb{X})}_{(\operatorname{div}\mathbb{X})\underbrace{\eta}_{\text{"}dV\text{"}}}$$

$$\int_{\partial B} *\mathbb{X}^\flat = \int_{\partial B} \tfrac{1}{2}\mathbb{X}^i \eta_{ijk}\, dx^{jk} = \int_{\partial D} \mathbb{X}^i dS_i = \int_{\partial B} \vec{\mathbb{X}}\cdot d\vec{S}$$

so

$$\boxed{\int_{\partial B} \vec{\mathbb{X}}\cdot d\vec{S} = \int_B (\operatorname{div}\mathbb{X})\, dV}$$

or

$$\int_{\partial B} \vec{\mathbb{X}}\cdot\hat{N}\, dS = \int_B (\operatorname{div}\mathbb{X})\, dV .$$

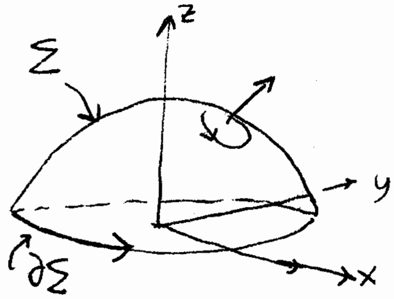
Okay, time for parting words.

One semester is so short a time. There are still many basic notions remaining, among the most important: group of transformations and their associated derivative operator — the Lie derivative. This is also important for the metric geometry we have explored — to describe symmetries of the geometry.

The language I have partially introduced you to is basic to the description of finite-dimensional continuous physical systems (and some co-dim ones too). It is interesting in its own right as pure mathematics, and a very powerful tool for describing many aspects of how our world works. I hope you have enjoyed seeing some of this structure a fraction as much as I have enjoyed the opportunity to rethink some of these ideas.

P.S. Some worked examples of Stoke's Theorem for $\mathbb{R}^3$

$$\int_{\partial\Sigma} X^\flat = \int_{\Sigma} dX^\flat \quad , \qquad X^\flat \text{ a 1-form.}$$



Σ: $x^2+y^2+z^2=a^2$
$z \geq 0$
oriented by upper normal

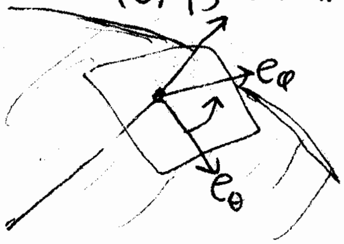
$\partial\Sigma$: $x^2+y^2=a^2$
$z=0$
induced orientation

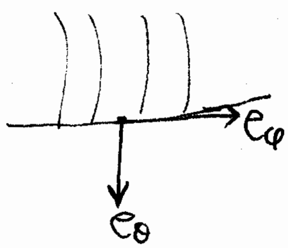
parametrizations with correct orientation: Φ_Σ:

Σ: set $r=a$ in spherical coord parametrization map

$$\begin{cases} x = a\sin\theta\cos\varphi \\ y = a\sin\theta\sin\varphi \\ z = a\cos\theta \end{cases} \qquad \begin{array}{l} 0 \leq \theta \leq \pi/2 \\ 0 \leq \varphi \leq 2\pi \end{array}$$

$\{\theta, \varphi\}$ orient Σ with the correct orientation related to the upward normal by the right hand rule





on the bounding circle (equator of sphere)

$$\underbrace{e_\theta}_{\text{outward}} \wedge \underbrace{[\, e_\varphi \,]}_{\text{induced orientation of } \partial\Sigma} = \underbrace{e_\theta \wedge e_\varphi}_{\text{inner orientation of } \Sigma}$$

$\partial\Sigma$: set $r=a$, $\theta = \frac{\pi}{2}$ in spherical coord parametrization map

φ provides the correct orientation for $\partial\Sigma$.

$$\Phi_{\partial\Sigma}: \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = 0 \end{cases} \qquad 0 \leq \varphi \leq 2\pi$$

Now we need a 1-form to use to verify this version of Stoke's theorem.
Let's take our old friend $X^\flat = y\,dx + x\,dy$

$$dX^\flat = \underbrace{dy\wedge dx}_{-dx\wedge dy} + dx\wedge dy = 0 \qquad \left(\begin{array}{l} \text{recall } X^\flat = d(xy) \\ \text{so } dX^\flat = d^2(xy) \equiv 0 \end{array}\right)$$

so the right side of Stoke's Theorem is identically zero.

For the left side,

$$\phi_{\partial\Sigma}{}^*(\mathbb{X}^\flat) = (a\sin\varphi)\, d(a\cos\varphi) + (a\cos\varphi)\, d(a\sin\varphi)$$

$$= -a^2\sin^2\varphi\, d\varphi + a^2\cos^2\varphi\, d\varphi = a^2\cos 2\varphi\, d\varphi.$$

$$\int_{\partial\Sigma} \mathbb{X}^\flat = \int_0^{2\pi} a^2\cos 2\varphi\, d\varphi = -\tfrac{1}{2}a^2\sin 2\varphi\Big|_0^{2\pi} = 0 \checkmark$$

Okay, let's try something more interesting by switching a sign.
Take instead $\mathbb{X}^\flat = -y\,dx + x\,dy$

$$\phi_{\partial\Sigma}{}^*(\mathbb{X}^\flat) = -(a\sin\varphi)\, d(a\cos\varphi) + (a\cos\varphi)\, d(a\sin\varphi)$$

$$= a^2\, d\varphi$$

$$\int_{\partial\Sigma} \mathbb{X}^\flat = \int_0^{2\pi} a^2 d\varphi = 2\pi a^2.$$

$$d\mathbb{X}^\flat = -dy\wedge dx + dx\wedge dy = 2\,dx\wedge dy$$

$$\phi_\Sigma{}^*(d\mathbb{X}^\flat) = 2\, d[a\sin\theta\cos\varphi]\wedge d[a\sin\theta\sin\varphi]$$

$$= 2a^2\,[\cos\theta\cos\varphi\, d\theta - \sin\theta\sin\varphi\, d\varphi]\wedge[\cos\theta\sin\varphi\, d\theta + \sin\theta\cos\varphi\, d\varphi]$$

$$= 2a^2\,[\sin\theta\cos\theta\cos^2\varphi\, d\theta\wedge d\varphi - \sin\theta\cos\theta\sin^2\varphi\, \underbrace{d\varphi\wedge d\theta}_{-d\theta\wedge d\varphi}]$$

$$= 2a^2\sin\theta\cos\theta\, d\theta\wedge d\varphi = a^2\sin 2\theta\, d\theta\wedge d\varphi$$

$$\int_\Sigma d\mathbb{X}^\flat = \int_0^{2\pi}\int_0^{\pi/2} a^2\sin 2\theta\, d\theta\, d\varphi = 2\pi a^2\underbrace{\left(-\tfrac{1}{2}\cos 2\theta\right)\Big|_0^{\pi/2}}_{\frac{1}{2}+\frac{1}{2}=1} = 2\pi a^2 \checkmark$$

$\sim$

$$\int (\operatorname{curl}\mathbb{X})\cdot\hat{n}\, dS$$

The multivariable calculus approach to same problem

$$\underline{X} = (-y, x, 0)$$

$$\text{curl}\,\underline{X} = \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x), \frac{\partial}{\partial z}(-y) - \frac{\partial}{\partial x}(0), \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right) = (0, 0, 2)$$

$x^2+y^2+z^2 = a^2 \rightarrow \vec{\nabla}(x^2+y^2+z^2) = (2x, 2y, 2z) \rightarrow$ normalize to get unit normal

$$\hat{n} = \frac{(x, y, z)}{r} \qquad r = \sqrt{x^2+y^2+z^2}$$

$\text{curl}\,\underline{X} \cdot \hat{n} = \frac{2z}{r} = 2\cos\theta$ in spherical coords.

The surface area differential is $a\sin\theta\, d\theta\, d\varphi$ (recall this)

so $$\int_{\Sigma} \text{curl}\,\underline{X} \cdot \hat{n}\, dS = \int_0^{2\pi}\int_0^{\pi} 2(\cos\theta)(a\sin\theta)\, d\theta\, d\varphi = 2\pi a^2 \text{ as before.}$$

$$\parallel$$

$$\int_{\partial\Sigma} \underline{X} \cdot d\vec{s} = \int_{\partial\Sigma} \underline{X} \cdot \hat{T}\, ds$$

$x = a\cos\varphi \qquad x' = -a\sin\varphi \qquad \vec{r}\,'(\varphi) = a(-\sin\varphi, \cos\varphi) = (-y, x, 0)$

$y = a\sin\varphi \qquad y' = a\cos\varphi$

$z = 0 \qquad z' = 0$

$$\hat{T} = \frac{(-y, x, 0)}{(x^2+y^2)^{1/2}} \qquad ds = a\,d\varphi \text{ on } \partial\Sigma$$

"geometric"

$$\underline{X} \cdot \hat{T} = \frac{y^2+x^2}{(x^2+y^2)^{1/2}} = \sqrt{x^2+y^2} = a \text{ on } \partial\Sigma$$

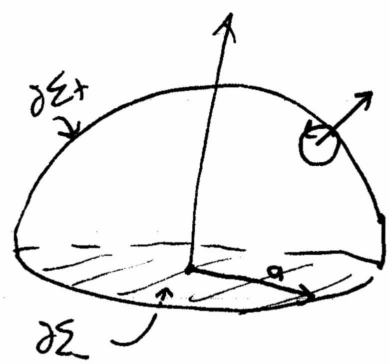
$$\int_{\partial\Sigma} \underline{X} \cdot \hat{T}\, ds = \int_0^{2\pi} a\,(a\,d\varphi) = 2\pi a^2.$$

or just plugging in parametrization: $$= \int \underline{X} \cdot r'(\varphi)\, d\varphi = \int_0^{2\pi} [(a\sin\varphi)(a\sin\varphi) + (a\cos\varphi)(a\cos\varphi)]\, d\varphi$$

$$= \int_0^{2\pi} a^2\, d\varphi = 2\pi a^2$$

The nonmetric version is clearly simpler, but the metric version gives us a physical picture of what we are integrating.

How about a Gauss's law problem? Let Σ be the interior of the upper hemisphere of radius a at the origin, with the usual $\mathbb{R}^3$ orientation. $\partial\Sigma$ has two parts: the upper hemisphere and the disk of radius a in the xy plane. In each case we can use a spherical coordinate parametrization



Σ: $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$ $\quad 0 \le r \le a$, $0 \le \theta \le \pi/2$, $0 \le \varphi \le 2\pi$ $\quad \{r, \theta, \varphi\}$ oriented

$\partial\Sigma_+$: $x = a\sin\theta\cos\varphi$, $y = a\sin\theta\sin\varphi$, $z = a\cos\theta$ $\quad 0 \le \theta \le \pi/2$, $0 \le \varphi \le 2\pi$ $\quad \{\theta, \varphi\}$ oriented

$\partial\Sigma_-$: $x = r\cos\varphi$, $y = r\sin\varphi$, $z = 0$ $\quad 0 \le r \le a$, $0 \le \varphi \le 2\pi$ $\quad \{\varphi, r\}$ oriented

since

$$\underbrace{e_\theta}_{\substack{\text{outward on } \partial\Sigma_- \\ (=-e_z) \text{ there}}} \wedge [\underbrace{e_\varphi \wedge e_r}_{\text{induced orientation.}}] = e_r \wedge e_\theta \wedge e_\varphi$$

So we need a 2-form to integrate on $\partial\Sigma$.

Take $\quad {}^*X^\flat$ where $X = \underbrace{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}}_{\text{radial direction,}} = r\frac{\partial}{\partial r}$

length r

$${}^*X^\flat = {}^*(x\,dx + y\,dy + z\,dz) = x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$$

$\parallel$ shortcut.

$${}^*(r\,dr) = r\,{}^*dr = r\,{}^*\omega^{\hat r} = r\,\omega^{\hat\theta\hat\varphi} = r\,(r\,d\theta)\wedge(r\sin\theta\,d\varphi) = r^3\sin\theta\,d\theta\wedge d\varphi$$

$$\int_{\partial\Sigma} {}^*X^\flat = \int_\Sigma d({}^*X^\flat) = \int_\Sigma 3\underbrace{dx\wedge dy\wedge dz}_{r^2\sin\theta\,dr\wedge d\theta\wedge d\varphi} = 3\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^a r^2\sin\theta\,dr\,d\theta\,d\varphi$$

$$= 3\cdot\left(\frac{2\pi a^3}{3}\right) = 2\pi a^3$$

$\parallel$

$$\int_{\partial\Sigma} r^3\sin\theta\,d\theta\wedge d\varphi = \int_{\partial\Sigma_+} r^3\sin\theta\,d\theta\wedge d\varphi = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} a^3\sin\theta\,d\theta\,d\varphi = 2\pi a^3 \quad \checkmark$$

$$+\int_{\partial\Sigma_-} r^3\sin\theta\,d\theta\wedge d\varphi + \iint 0$$

$\theta = \frac{\pi}{2}$

$d\theta = 0$ on $\partial\Sigma_-$

okay I quit.