The Exterior Derivative d

Suppose we work in a coordinate frame on $\mathbb{R}^n$ or some n-dimensional space of interest. The for each p satisfying $0 \le p \le n$,
we have p-forms or "differential forms of degree p" which may be expressed in terms of the coordinate differentials as

$$T = \frac{1}{p!} T_{i_1 \cdots i_p} dx^{i_1 \cdots i_p} = \frac{1}{p!} T_{i_1 \cdots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

$$= T_{|i_1 \cdots i_p|} dx^{|i_1 \cdots i_p|} \quad \text{if we dont want to overcount.}$$

$$= T_{i_1 \cdots i_p} dx^{i_1} \otimes \cdots \otimes dx^{i_p} \quad \text{if we want to just think of it as a } \binom{0}{p}\text{-tensor field}$$

A function f is a 0-form and its differential $df = f_{,i} dx^i = \frac{\partial f}{\partial x^i} dx^i$ is a 1-form. Thus the differential d maps 0-forms to 1-forms, the extra covariant index associated with the derivative. If we start instead with a p-form, adding a derivative index to its component symbol will not yield an object which is antisymmetric in all of its indices unless we also take the antisymmetric part of this new object. We will then get a (p+1)-form. Apart from a normalization constant, this is how the <u>exterior derivative</u> d is defined as an extention of the operator d which takes the differential of a function.

The actual definition is simple for $1 \le p \le n$:

d: p-form $\rightarrow$ (p+1)-form

$$T = \frac{1}{p!} T_{i_1 \cdots i_p} dx^{i_1 \cdots i_p}$$

(1) $$dT \equiv \frac{1}{p!} dT_{i_1 \cdots i_p} \wedge dx^{i_1 \cdots i_p}$$

Take the differential of its component function and wedge it into the coordinate frame basis p-form to obtain a (p+1)-form

This is all we need in practice to evaluate dT for any p-form T, but we can develop shortcut formulas.

Its components are easily calculated by expanding the differential

$$dT = \frac{1}{p!} dT_{i_1 \cdots i_p} \wedge dx^{i_1 \cdots i_p} = \frac{1}{p!} T_{i_1 \cdots i_p, j} \underbrace{dx^j \wedge dx^{i_1 \cdots i_p}}_{dx^{j i_1 \cdots i_p}}$$

$$= \frac{1}{p!} T_{[i_1 \cdots i_p, j]} dx^{j i_1 \cdots i_p}$$ since only the antisymmetric part will contribute to the sum

$$\equiv \frac{1}{(p+1)!} [dT]_{j i_1 \cdots i_p} dx^{j i_1 \cdots i_p}$$ definition of the components of the $(p+1)$-form dT

Comparing the last two equalities we get

$$[dT]_{j i_1 \cdots i_p} = \underbrace{\frac{(p+1)!}{p!}}_{p+1} T_{[i_1 \cdots i_p, j]} = (p+1)\, T_{[i_1 \cdots i_p, j]}$$

So the exterior derivative of a p-form has its coordinate components equal to $(p+1)$ times the antisymmetric part of ~~their~~ derivatives, except the extra index is added at the beginning instead of at the end as in the covariant derivative. The notation $\partial_i f \equiv f_{,i}$ is better suited to this:

$$[dT]_{j i_1 \cdots i_p} = (p+1)\, \partial_{[j} T_{i_1 \cdots i_p]} .$$

The factor of $(p+1)$ is necessary to eliminate overcounting. Suppose we expand this expression:

$$[dT]_{j i_1 \cdots i_p} = \underbrace{\frac{(p+1)}{(p+1)!}}_{\frac{1}{p!}} \delta^{n\, m_1 \cdots m_p}_{j\, i_1 \cdots i_p} \partial_n T_{m_1 \cdots m_p} = \delta^{n\, m_1 \cdots m_p}_{j\, i_1 \cdots i_p} \partial_n T_{|m_1 \cdots m_p|}$$

It disappears once we avoid overcounting in the sum over the p antisymmetric indices. We can also write this as

$$[dT]_{i_1 \cdots i_{p+1}} = (p+1)\, \partial_{[i_1} T_{i_2 \cdots i_{p+1}]} = \delta^{j_1 j_2 \cdots j_{p+1}}_{i_1 i_2 \cdots i_{p+1}} \partial_{j_1} T_{|j_2 \cdots j_{p+1}|} .$$

(2)

But we can do better. In this antisymmetrization over $p+1$ indices, p of them are already antisymmetric, so the complete antisymmetrization collapses to something much simpler.

Recall that the generalized Kronecker delta may be defined as the determinant of a matrix of ordinary Kronecker deltas, which we can then expand along the first row using a cofactor expansion, the minors of which are by definition Kronecker deltas of one less order:

$$\delta^{j_1 j_2\cdots j_{p+1}}_{i_1 i_2\cdots i_{p+1}} = \begin{vmatrix} \delta^{j_1}{}_{i_1} & \delta^{j_1}{}_{i_2} & \cdots & \delta^{j_1}{}_{i_{p+1}} \\ \delta^{j_2}{}_{i_1} & \delta^{j_2}{}_{i_2} & \cdots & \delta^{j_2}{}_{i_{p+1}} \\ \vdots & & & \\ \delta^{j_{p+1}}{}_{i_1} & \delta^{j_{p+1}}{}_{i_2} & \cdots & \delta^{j_{p+1}}{}_{i_{p+1}} \end{vmatrix} = \begin{aligned} &\delta^{j_1}{}_{i_1}\,\delta^{j_2\cdots j_{p+1}}_{i_2\cdots i_{p+1}} \\ &-\delta^{j_1}{}_{i_2}\,\delta^{j_2\cdots j_{p+1}}_{i_1 i_3\cdots i_{p+1}} \\ &+\delta^{j_1}{}_{i_3}\,\delta^{j_2\cdots j_{p+1}}_{i_1 i_2 i_4\cdots i_{p+1}} \\ &-\cdots \\ &+(-1)^p\,\delta^{j_1}{}_{i_{p+1}}\,\delta^{j_2\cdots j_{p+1}}_{i_1\cdots i_p} \end{aligned}$$

$$= \sum_{k=1}^{p+1} (-1)^{k-1}\delta^{j_1}{}_{i_k}\,\delta^{j_2\cdots j_{p+1}}_{i_1\cdots i_{k-1} i_{k+1}\cdots i_{p+1}}$$

$$\equiv \sum_{k=1}^{p+1} (-1)^{k-1}\delta^{j_1}{}_{i_k}\,\delta^{j_2\cdots j_{p+1}}_{i_1\cdots \check{i}_k\cdots i_{p+1}}$$

where $\check{i}_k$ means this index is omitted from the index set (a convenient abbreviation).

Using this formula for the exterior derivative gives.

$$[dT]_{i_1\cdots i_{p+1}} = \delta^{j_1\cdots j_{p+1}}_{i_1\cdots i_{p+1}}\,\partial_{j_1} T_{|j_2\cdots j_{p+1}|} \qquad \text{do this sum yielding } \partial_{i_k}$$

$$= \sum_{k=1}^{p+1}(-1)^{k-1}\,\delta^{j_1}{}_{i_k}\,\underbrace{\delta^{j_2\cdots j_{p+1}}_{i_1\cdots\check{i}_k\cdots i_{p+1}}\,\partial_{j_1} T_{|j_2\cdots j_{p+1}|}}_{\text{do this sum}}$$

$$= \sum_{k=1}^{p+1}(-1)^{k-1}\,\partial_{i_k} T_{i_1\cdots\check{i}_k\cdots i_{p+1}}$$

(3) $$= \partial_{i_1} T_{i_2\cdots i_{p+1}} - \partial_{i_2} T_{i_1\check{i}_2 i_3\cdots i_{p+1}} - \partial_{i_3} T_{i_1 i_2\check{i}_3 i_4\cdots i_{p+1}} + \cdots + (-1)^p\,\partial_{i_p} T_{i_1\cdots i_{p-1}}\,.$$

So we have 3 formulas (1), (2), (3) that we can use to compute d.

$\boxed{\text{Reinforcement}}$

The case $p=1$: $\quad T = T_i dx^i$

$$dT = dT_i \wedge dx^i = \partial_j T_i \, dx^j \wedge dx^i = \partial_j T_i \, dx^{ji} = \partial_{[j} T_{i]} \, dx^{ji}$$

$$= \frac{1}{2}\{2\partial_{[j}T_{i]}\} dx^{ji} \equiv \frac{1}{2}[dT]_{ji} \, dx^{ji}$$

so $[dT]_{ji} = \underbrace{2\,\partial_{[j}T_{i]}}_{\text{This is formula (2) with } (p+1)=2} = \underbrace{\partial_j T_i - \partial_i T_j}_{\text{This is formula(3)}}.$

The case $p=2$: $\quad T = \frac{1}{2} T_{ij} \, dx^{ij}$

$$dT = \frac{1}{2} dT_{ij} \wedge dx^{ij} = \frac{1}{2}\partial_k T_{ij} \, dx^k \wedge dx^{ij} = \frac{1}{2}\partial_k T_{ij} \, dx^{kij} = \frac{1}{2}\partial_{[k}T_{ij]} \, dx^{kij}$$

$$= \frac{1}{3!}\{3\,\partial_{[k}T_{ij]}\} dx^{kij} \equiv \frac{1}{3!}[dT]_{kij} \, dx^{kij}$$

so $[dT]_{kij} = \underbrace{3\,\partial_{[k}T_{ij]}}_{\text{This is formula (2) with } p+1=3} = 3 \cdot \frac{1}{3!}\left\{ \begin{array}{l} \partial_k T_{ij} + \partial_i T_{jk} + \partial_j T_{ki} \\ \underbrace{-\partial_k T_{ji}}_{2\partial_k T_{ij}} \underbrace{-\partial_i T_{kj}}_{2\partial_i T_{jk}} \underbrace{-\partial_j T_{ik}}_{2\partial_j T_{ki}} \end{array} \right\}$

$$= \partial_k T_{ij} + \partial_i \underbrace{T_{jk}}_{-T_{kj}} + \partial_j T_{ki} = \underbrace{\partial_k T_{ij} - \partial_i T_{kj} + \partial_j T_{ki}}_{\text{formula (3)}}$$

In this case the cyclic sum is easier to remember than the alternating sign formula (3). [Why we use the index pairs 23, 31, 12 instead of 23, 13, 12.]

Actual examples

- Recall our friend $X^\flat = y\,dx + x\,dy = df\,, \quad f = xy.$

  Then $dX^\flat = \underbrace{dy\wedge dx}_{-dx\wedge dy} + dx\wedge dy = 0$ which shows that $d^2 f \equiv d(df) = 0.$

- $T = (x^2+y^2)\,dx + (x^2-y^2)\,dy$

  $$dT = (2x\,dx + \underbrace{2y\,dy}_{0})\wedge dx + (\underbrace{2x\,dx}_{0} - 2y\,dy)\wedge dy = 2y\,\underbrace{dy\wedge dx}_{-dx\wedge dy} + 2x\,dx\wedge dy$$

  $$= 2(x-y)\,dx\wedge dy$$

  $$d^2T \equiv d(dT) = 2(\underbrace{dx}_{0} - \underbrace{dy}_{0})\wedge dx\wedge dy = 0. \qquad \text{Again } d^2T = 0.$$

- $T = xyz\,dy\wedge dz + (x+y)\,dz\wedge dx + \sin(x+z)\,dx\wedge dy$

  $$dT = yz\,dx\wedge dy\wedge dz + \underbrace{dy\wedge dz\wedge dx}_{+dx\wedge dy\wedge dz} + \cos(x+z)\,\underbrace{dz\wedge dx\wedge dy}_{+dx\wedge dy\wedge dz} \quad \left(\text{omitting } 0 \text{ terms}\right)$$

  $$= [yz + 1 + \cos(x+z)]\,dx\wedge dy\wedge dz$$

  $d^2T = d(dT) = 0$ since nothing else can be wedged into $dx\wedge dy\wedge dz$ in three dimensions.

- We can also evaluate these exterior derivatives in other coordinate systems, like cylindrical coordinates. From page 90

  $$\mathbf{X}^\flat = \rho\sin 2\phi\,d\rho + \rho^2\cos 2\varphi\,d\varphi$$

  $$d\mathbf{X}^\flat = 2\rho\cos 2\varphi\,\underbrace{d\varphi\wedge d\rho}_{-d\rho\wedge d\varphi} + 2\rho\cos 2\varphi\,d\rho\wedge d\varphi = (2\rho\cos\varphi - 2\rho\cos\varphi)\,d\rho\wedge d\varphi = 0 \ \checkmark$$

<u>exercise</u> a) Transform T and dT of the previous page to cylindrical coordinates to obtain

$$T = (x^2+y^2)dx + (x^2-y^2)dy = \cdots = \rho^2[\cos\phi + \cos 2\phi\sin\phi]d\rho + \rho^3[-\sin\phi + \cos 2\phi\cos\phi]d\varphi$$

$$dT = 2(x-y)\,dx\wedge dy = \cdots = 2\rho^2(\cos\varphi - \sin\varphi)\,d\rho\wedge d\varphi$$

b) Now doing the exterior derivative in cylindrical coordinates, show that this result for dT is what you actually get (using trig identities!)

Two facts seem to be coming to light:

1) $d^2 T \equiv d(dT) = 0$ for any p-form T
2) our definition of dT in a particular coordinate system is actually independent of the coordinate system.

To show the first just do the exterior derivative twice:

$$dT = \frac{1}{p!}\, dT_{i_1\cdots i_p} \wedge dx^{i_1\cdots i_p} = \frac{1}{p!}\, T_{i_1\cdots i_p,j}\, dx^j \wedge dx^{i_1\cdots i_p}$$

$$d^2T = \frac{1}{p!}\, d(T_{i_1\cdots i_p,j}) \wedge dx^j \wedge dx^{i_1\cdots i_p}$$

$$= \frac{1}{p!}\, T_{i_1\cdots i_p,\underbrace{jk}}\, \underbrace{dx^k \wedge dx^j} \wedge dx^{i_1\cdots i_p} = \frac{1}{p!}\, T_{[i_1\cdots i_p,jk]}\, dx^{kji_1\cdots i_p}$$

But partial derivatives commute so antisymmetrizing over the index set containing the symmetric pair jk symbolizing the second partial derivatives whose order doesn't matter gives zero: $d^2T = 0$.

<u>True believers may skip the rest of the page</u>

Not so fast you say. I should have antisymmetrized after the first derivative and then again after the second derivative. How do I know that is equivalent to just antisymmetrizing after the second which is what I did? Well, let's be careful then:

$$[dT]_{i_1\cdots i_{p+1}} = \frac{1}{p!}\, \delta^{j_1\cdots j_{p+1}}_{i_1\cdots i_{p+1}}\, \partial_{j_1} T_{j_2\cdots j_{p+1}}$$

$$[d(dT)]_{i_1\cdots i_{p+2}} = \frac{1}{(p+1)!}\, \delta^{j_1 j_2\cdots j_{p+2}}_{i_1 i_2\cdots i_{p+2}}\, \partial_{j_1} \underbrace{[dT]_{j_2\cdots j_{p+2}}}_{\frac{1}{p!}\delta^{k_2\cdots k_{p+2}}_{j_2\cdots j_{p+2}}\partial_{k_2}T_{k_3\cdots k_{p+2}}}$$

$$= \frac{1}{(p+1)!}\, \frac{1}{p!}\, \delta^{k_2\cdots k_{p+2}}_{j_2\cdots j_{p+2}}\, \delta^{j_1 j_2\cdots j_{p+2}}_{i_1 i_2\cdots i_{p+2}}\, \partial_{j_1}\partial_{k_2} T_{k_3\cdots k_{p+2}}$$

These indices are antisymmetric. This is (p+1) times the antisymmetrizer for these already antisymmetric indices, so it just switches indices & adds factor of (p+1).

$$= \frac{1}{p!}\, \underbrace{\delta^{j_1 k_2\cdots k_{p+2}}_{i_1 i_2\cdots i_{p+2}}}_{-\delta^{k_2 j_1\cdots k_{p+2}}_{i_1 i_2\cdots i_{p+2}}}\, \underbrace{\partial_{j_1}\partial_{k_2}}_{\partial_{k_2}\partial_{j_1}} T_{k_3\cdots k_{p+2}} = (p+2)(p+1)\, \partial_{[j_1}\partial_{k_2} T_{k_3\cdots k_{p+2}]}$$

by antisymmetry   by commuting partials   now switch indices $k_2 j_1 \to j_1 k_2$ and get minus original expression

$$= -[d(dT)]_{i_1\cdots i_{p+2}} \quad \text{so} \quad d(dT) = 0.$$

Ugly but careful.

Now why is d independent of the coordinate system? An ugly way to show this is to simply transform its components and show that they obey the correct transformation law. We made a big deal out of the fact that the partial derivatives of tensor components do not transform "as a tensor" which led to the covariant derivative, but antisymmetrization kills the second derivative terms which arise from the derivatives of $A^i{}_j = \partial\bar{x}^i/\partial x^j$, restoring the correct transformation rule.

<u>True believers skip rest of page</u>

$$\bar{d}T = \frac{1}{p!}\, d\bar{T}_{i_1\cdots i_p} \wedge d\bar{x}^{i_1\cdots i_p} = \frac{1}{p!}\, d\left[A^{-1\,k_1}{}_{i_1}\cdots A^{-1\,k_p}{}_{i_p}\, T_{k_1\cdots k_p}\right]\wedge\left[A^{i_1}{}_{j_1}\cdots A^{i_p}{}_{j_p}\, dx^{j_1\cdots j_p}\right]$$

$$= \frac{1}{p!}\underbrace{\left[A^{-1\,k_1}{}_{i_1}\cdots A^{-1\,k_p}{}_{i_p}\right]\left[A^{i_1}{}_{j_1}\cdots A^{i_p}{}_{j_p}\right]}_{\delta^{k_1}{}_{j_1}\cdots\,\delta^{k_p}{}_{j_p}}\, dT_{k_1\cdots k_p}\wedge dx^{j_1\cdots j_p}$$

$$+ \frac{1}{p!}\left[dA^{-1\,k_1}{}_{i_1}\cdots A^{-1\,k_p}{}_{i_p} + \cdots + A^{-1\,k_1}{}_{i_1}\cdots dA^{-1\,k_p}{}_{i_p}\right]\overset{[A^{i_1}{}_{j_1}\cdots A^{i_p}{}_{j_p}]}{\wedge} T_{k_1\cdots k_p}\, dx^{j_1\cdots j_p}$$

$$= \underbrace{\frac{1}{p!}\, dT_{j_1\cdots j_p}\wedge dx^{j_1\cdots j_p}}_{dT} + \frac{1}{p!}\left[dA^{-1\,k_1}{}_{i_1}A^{i_1}{}_{j_1}\,\delta^{k_2}{}_{j_2}\cdots\delta^{k_p}{}_{j_p} + \cdots + \delta^{k_1}{}_{j_1}\cdots\delta^{k_{p-1}}{}_{j_{p-1}}\, dA^{-1\,k_p}{}_{i_p}A^{i_p}{}_{j_p}\right]\wedge T_{k_1\cdots k_p}\, dx^{j_1\cdots j_p}$$

$$= dT + \frac{1}{p!}\left[\underbrace{dA^{-1\,k_1}{}_{i_1}A^{i_1}{}_{j_1}}_{-A^{-1\,k_1}{}_{i_1}dA^{i_1}{}_{j_1}} T_{k_1 j_2\cdots j_p} + \cdots + \underbrace{dA^{-1\,k_p}{}_{i_p}A^{i_p}{}_{j_p}}_{-A^{-1\,k_p}{}_{i_p}dA^{i_p}{}_{j_p}} T_{j_1\cdots j_{p-1}k_p}\right]\wedge dx^{j_1\cdots j_p}$$

$$\left.\begin{aligned} A^{-1\,k}{}_i A^i{}_j &= \delta^k{}_j \\ dA^{-1\,k}{}_i A^i{}_j + A^{-1\,k}{}_i\, dA^i{}_j &= 0 \end{aligned}\right\}$$

but

$$A^i{}_j = \frac{\partial\bar{x}^i}{\partial x^j}$$

$$dA^i{}_j = \frac{\partial^2\bar{x}^i}{\partial x^\ell\partial x^j}\, dx^\ell = A^i{}_{j,\ell}\, dx^\ell$$

$$dA^i{}_j\wedge dx^j = \underbrace{A^i{}_{j,\ell}}_{\text{sym}}\,\underbrace{dx^\ell\wedge dx^j}_{\text{antis}} = 0$$

so each of these extra terms vanishes above, and $\bar{d}T = dT$ as claimed.

What properties does the exterior derivative have? Well, it is a derivative operator so it should obey sum and product rules:

$$T+S = \frac{1}{p!} T_{i_1\cdots i_p} dx^{i_1\cdots i_p} + \frac{1}{p!} S_{i_1\cdots i_p} dx^{i_1\cdots i_p} = \frac{1}{p!}\left(T_{i_1\cdots i_p} + S_{i_1\cdots i_p}\right) dx^{i_1\cdots i_p}$$

$$\begin{aligned} d(T+S) &= \frac{1}{p!} d\left(T_{i_1\cdots i_p} + S_{i_1\cdots i_p}\right)\wedge dx^{i_1\cdots i_p} \\ &= \frac{1}{p!}\left(dT_{i_1\cdots i_p} + dS_{i_1\cdots i_p}\right)\wedge dx^{i_1\cdots i_p} \qquad \text{ordinary differential sum rule} \\ &= \frac{1}{p!} dT_{i_1\cdots i_p}\wedge dx^{i_1\cdots i_p} + \frac{1}{p!} dS_{i_1\cdots i_p}\wedge dx^{i_1\cdots i_p} \\ &= dT + dS, \end{aligned}$$

so $\boxed{d(T+S) = dT + dS}$ (A)

Now if $T = \frac{1}{p!} T_{i_1\cdots i_p} dx^{i_1\cdots i_p}$ and $S = \frac{1}{q!} S_{j_1\cdots j_q} dx^{j_1\cdots j_q}$

Then $T\wedge S = \frac{1}{p!q!} T_{i_1\cdots i_p} S_{j_1\cdots j_q}\, dx^{i_1\cdots i_p}\wedge dx^{j_1\cdots j_q}$

$$d(T\wedge S) = \frac{1}{p!q!}\left[dT_{i_1\cdots i_p} S_{j_1\cdots j_q} + T_{i_1\cdots i_p} dS_{j_1\cdots j_q}\right]\wedge dx^{i_1\cdots i_p}\wedge dx^{j_1\cdots j_q}$$

$$= \left(\frac{1}{p!} dT_{i_1\cdots i_p}\wedge dx^{i_1\cdots i_p}\right)\wedge\left(\frac{1}{q!} S_{j_1\cdots j_q} dx^{j_1\cdots j_q}\right) \qquad = dT\wedge S$$

$$+ \frac{1}{p!} T_{i_1\cdots i_q} \frac{1}{q!} \underbrace{dS_{j_1\cdots j_q}\wedge dx^{i_1\cdots i_p}}_{\partial_k S_{j_1\cdots j_q}\, \underbrace{dx^k\wedge dx^{i_1\cdots i_p}}_{(-1)^p dx^{i_1\cdots i_p}\wedge dx^k}} \wedge dx^{j_1\cdots j_q} \qquad + (-1)^p T\wedge dS$$

$$\underbrace{\qquad}_{(-1)^p \quad dx^{i_1\cdots i_p}\wedge \partial_k S_{j_1\cdots j_q} dx^k}$$

$$dx^{i_1\cdots i_p}\wedge dS_{j_1\cdots j_q}$$

$$\underbrace{\qquad}_{(-1)^p\left(\frac{1}{p!} T_{i_1\cdots i_q} dx^{i_1\cdots i_p}\right)\wedge\left(\frac{1}{q!} dS_{j_1\cdots j_q}\wedge dx^{j_1\cdots j_q}\right)}$$

so $\boxed{d(\underset{p}{T}\wedge \underset{q}{S}) = dT\wedge S + (-1)^p T\wedge dS}$ (B)

and finally $\boxed{d^2 T \equiv d(dT) = 0}$ (C)

These three properties uniquely characterize the exterior derivative.
A final property extends the coordinate independence of this operator to any map between two spaces M and N.
Suppose $\quad \phi : M \longrightarrow N$
and suppose T is a p-form on N.
Then $\phi^* T$ is its pullback to M, also a p-form.
It turns out that we can do the exterior derivative before or after the pullback ~~without~~ and still get the same result

$$d(\phi^* T) = \phi^*(dT)$$

exterior derivative on N

exterior derivative on M.

A special case of this are the parameter maps associated with nonCartesian coordinate systems on $\mathbb{R}^n$. Expressing a p-form in terms of the new coordinates is equivalent to pulling it back to the coordinate space. Computing its exterior derivative in the new coordinates yields the same result as first taking the exterior derivative in Cartesian coordinates and then re-expressing the result in the new coordinates.

The exterior derivative and a metric

When we have a metric tensor field $g = g_{ij}\, dx^i \otimes dx^j$ on our space, we can use the lowering and raising maps $\flat$ and $\#$ to convert p-vector fields into p-forms and viceversa. These are inverse operations. We also have the metric duality map $*$ which converts p-vector fields and p-forms into (n-p)-vector fields and (n-p)-forms respectively and then back again, although $*$ is not its own inverse since $** = (-1)^{\text{integer}}$, which means that it differs from the inverse by a sign factor which depends on p, n and the signature of the metric (how many minus signs in an orthonormal frame). All of these operations may be used with the exterior derivative to make new differential operators.

First let Λ^p be the space of p-forms on our n-dimensional space and let $[\Lambda^p]^\#$ be the space of p-vector fields. Then the indexshifting and duality maps may be represented as follows

$$\begin{tikzcd}
\Lambda^{n-p} \arrow[r, shift left, "\#"] \arrow[d, shift right, "*"'] & {[\Lambda^{n-p}]^\#} \arrow[l, shift left, "\flat"] \arrow[d, shift right, "*"'] \\
\Lambda^p \arrow[r, shift left, "\#"] \arrow[u, shift right, "*"'] & {[\Lambda^p]^\#} \arrow[l, shift left, "\flat"] \arrow[u, shift right, "*"']
\end{tikzcd}$$

These operations commute, i.e., it doesn't matter if you first shift indices and then take the dual or first take the dual and then shift indices. For example, if $T \in \Lambda^p$ then

$$*[T^\#] = [*T]^\#$$

right then up $\quad$ up then right.

This means we can just write $*T^\#$ without specifying the order in which these two operations are done on T. If you need to be convinced:

$$[T^\#]^{i_1\cdots i_p} = T^{i_1\cdots i_p}$$
$$[*[T^\#]]^{i_{p+1}\cdots i_n} = \frac{1}{p!}\, T^{i_1\cdots i_p}\eta_{i_1\cdots i_p}{}^{i_{p+1}\cdots i_n} = \frac{1}{p!}\, T_{i_1\cdots i_p}\eta^{i_1\cdots i_p\, i_{p+1}\cdots i_n}$$
$$\downarrow$$
$$[*T]_{i_{p+1}\cdots i_n} = \frac{1}{p!}\, T_{i_1\cdots i_p}\eta^{i_1\cdots i_p}{}_{i_{p+1}\cdots i_n}$$
$$[[*T]^\#]^{i_{p+1}\cdots i_n} = \frac{1}{p!}\, T_{i_1\cdots i_p}\eta^{i_1\cdots i_p\, i_{p+1}\cdots i_n} = [*[T^\#]]^{i_{p+1}\cdots i_n}$$

so we just write $[[*T]^\#]^{i_{p+1}\cdots i_n} = [*T]^{i_{p+1}\cdots i_n} = [*[T^\#]]^{i_{p+1}\cdots i_n}$.

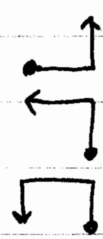
This is just to remind ourselves about stuff we did in part I.

Now how can we mix these operators with the exterior derivative? Suppose we just look at $*$ and d alone. We can make a picture like

$$\begin{tikzcd}
\cdots & \arrow[l] \Lambda^{n-p+1} \arrow[d, shift right, "*"'] & \arrow[l, "d"'] \Lambda^{n-p} \arrow[d, shift right, "*"'] & \arrow[l, "d"'] \Lambda^{n-p-1} \arrow[d, shift right, "*"'] & \arrow[l] \cdots \\
\arrow[r] & \Lambda^{p-1} \arrow[r, "d"'] \arrow[u, shift right, "*"'] & \Lambda^{p} \arrow[r, "d"'] \arrow[u, shift right, "*"'] & \Lambda^{p+1} \arrow[r] \arrow[u, shift right, "*"'] &
\end{tikzcd}$$

Yuch! ("Blech!" as ~~the psychiatrist~~ Lucy says when kissed by snoopy). Lets forget we saw that. If we start in Λ^p we can do things like

$$*d : \Lambda^p \to \Lambda^{n-p-1}$$
$$d* : \Lambda^p \to \Lambda^{n-p+1}$$
$$*d* : \Lambda^p \to \Lambda^{p-1}$$



(paths in above diagram starting at Λ^p, in case you haven't forgotten it.)

The last operator lowers the degree of the p-form by one, going in the opposite direction of d:

$$\Lambda^{p-1} \underset{d}{\overset{*d*}{\rightleftarrows}} \Lambda^p$$

We can also made second order operators, $d^2 \equiv 0$ is of no use but

$$d*d* : \Lambda^p \to \Lambda^p \quad \text{and}$$
$$*d*d : \Lambda^p \to \Lambda^p$$

are two interesting second order linear differentials which produce p-forms from p-forms. These turn out to be related to the Laplacian (for 0-forms) and its generalization to p-forms.

By including index shifting, all of these operators can be extended to p-vector fields. First lower the indices to obtain a differential form, do the various above operations, then raise the indices to go back to a q-vector field for some value of q. With a little patience, we could get explicit component formulas for any of these, just by composing the component formulas for the individual operations.

One useful formula, however, re-expresses the exterior derivative in terms of the covariant derivative. In a coordinate frame recall that $\Gamma^k{}_{[ij]} = 0$:

$$\nabla_{i_1} T_{i_2 \cdots i_{p+1}} = \partial_{i_1} T_{i_2 \cdots i_{p+1}} - \Gamma^j{}_{i_1 i_2} T_{j i_3 \cdots i_{p+1}} - \cdots - \Gamma^j{}_{i_1 i_{p+1}} T_{i_1 \cdots i_p j}$$

$$\nabla_{[i_1} T_{i_2 \cdots i_{p+1}]} = \partial_{[i_1} T_{i_2 \cdots i_{p+1}]} - \Gamma^j{}_{[i_1 i_2} T_{|j| i_3 \cdots i_{p+1}]} - \cdots - \Gamma^j{}_{[i_1 i_{p+1}} T_{i_1 \cdots i_p] j}$$

exclude j from antisymmetrization in each term.

$$= \partial_{[i_1} T_{i_2 \cdots i_{p+1}]}$$

since antisymmetrization of the covariant derivative lower components gives zero

so the ordinary derivative can be replaced by the covariant derivative in a coordinate frame

$$[dT]_{i_1 \cdots i_{p+1}} = (p+1)\, \partial_{[i_1} T_{i_2 \cdots i_{p+1}]} = (p+1)\, \nabla_{[i_1} T_{i_2 \cdots i_{p+1}]}$$

but since dT and ∇T are frame independent objects, this is true in any frame, i.e.

$$\boxed{[dT]_{i_1 \cdots i_{p+1}} = (p+1)\, \nabla_{[i_1} T_{i_2 \cdots i_{p+1}]}} \, .$$

Returning to the previous discussion, recall that for a function f we already introduced the gradient as

$$\operatorname{grad} f = (df)^{\#} \equiv \vec{\nabla} f$$

$$[\operatorname{grad} f]^i = g^{ij}\partial_j f = g^{ij}\nabla_j f \equiv \nabla^i f.$$

Suppose we have a vector field $\mathbb{X}$. Then

$$\underbrace{* \, \underbrace{d \, \underbrace{* \, \underbrace{\mathbb{X}^\flat}_{1\text{-form}}}_{(n-1)\text{-form}}}_{n\text{-form}}}_{0\text{-form} = \text{function}}$$

is a function. What is its component formula?

$$[*\mathbb{X}^\flat]_{i_2\cdots i_n} = X_i \eta^i{}_{i_2\cdots i_n} = X^i \eta_{i i_2\cdots i_n}$$

since η is covariant constant

$$[d*\mathbb{X}^\flat]_{i_1 i_2\cdots i_n} = n\, \nabla_{[i_1}(\eta_{|i| i_2\cdots i_n]}\mathbb{X}^i) = n(\nabla_{[i_1}\mathbb{X}^i)\,\eta_{|i| i_2\cdots i_n]}$$

no antisymmetrization on i

$$*d*\mathbb{X}^\flat = \frac{1}{n!}[n\,(\nabla_{[i_1}\mathbb{X}^i)\eta_{i i_2\cdots i_n]}]\eta^{i_1\cdots i_n}$$

$$= \underbrace{\frac{1}{(n-1)!}\eta_{i i_2\cdots i_n}\eta^{i_1 i_2\cdots i_n}}_{\operatorname{sgn}\det(g_{mn})\ \delta^{i_1}{}_i} \nabla_{i_1} X^i = \operatorname{sgn}\det(g_{mn}) \underbrace{\nabla_i \mathbb{X}^i}_{\operatorname{div}\mathbb{X}}$$

Thus we get the divergence of the vector field, apart from a possible sign when the metric has negative self-inner product values. For $\mathbb{R}^n$ with the Euclidean metric, this sign is $+1$, so one gets exactly the divergence.

Suppose $n=3$. Consider the operator $[*d\mathbb{X}^\flat]^\#$ for a vector field $\mathbb{X}$. It results in a vector field:

$$\underbrace{\underbrace{[* \underbrace{d\underbrace{\mathbb{X}^\flat}_{1\text{-form}}}_{2\text{-form}}]}_{1\text{-form}}{}^{\#}}_{\text{vector field}}$$

What is the component expression for it?

$$[d\mathbb{X}^\flat]_{ij} = \partial_i \mathbb{X}_j - \partial_j \mathbb{X}_i$$

$$[{*}d\mathbb{X}^\flat]_k = \tfrac{1}{2}\,[\partial_i \mathbb{X}_j - \partial_j \mathbb{X}_i]\,\eta^{ij}{}_k$$

definition of antisym part

antisym unnecessary since only antisym part contributes to contraction by antisymmetry of η

$(-1)^2 = +1$

$$[{*}d\mathbb{X}^\flat]^k = \tfrac{1}{2}\,[\partial_i \mathbb{X}_j - \partial_j \mathbb{X}_i]\,\eta^{ijk} = \partial_{[i}\mathbb{X}_{j]}\,\eta^{ijk} = \partial_i \mathbb{X}_j\,\eta^{ijk} = \eta^{kij}\partial_i \mathbb{X}_j$$

In Cartesian coordinates on $\mathbb{R}^3$, this has the expression

$$[{*}d\mathbb{X}^\flat]^k = \epsilon^{kij}\frac{\partial \mathbb{X}_j}{\partial x^i}$$

which is the expression for the curl of the vector field $\mathbb{X}$. Since $[{*}d\mathbb{X}^\flat]^\#$ is a vector field independent of the choice of coordinates, this is true period:

$$[{*}d\mathbb{X}^\flat]^\# = \operatorname{curl}\mathbb{X}.$$

In calculus we learned that several second order derivatives constructed from grad, curl, div on $\mathbb{R}^3$ vanish identically. These are just consequences that $d^2T \equiv 0$ for p-forms T with $0 \le p \le 3$.

Specifically:

$(p=0)$ $$\operatorname{curl}\operatorname{grad} f = [{*}d\,\underbrace{(\operatorname{grad} f)^\flat}_{df}]^\# = [{*}\underbrace{d^2 f}_{0}]^\# = 0.$$

$(p=1)$ $$\operatorname{div}\operatorname{curl}\mathbb{X} = {*}d\,\underbrace{{*}[\operatorname{curl}\mathbb{X}]^\flat}_{{*}d\mathbb{X}^\flat} = {*}d\,{*}{*}d\mathbb{X}^\flat = {*}\underbrace{d^2\mathbb{X}^\flat}_{0} = 0$$

$**T = T$ for Euclidean metric on $\mathbb{R}^3$ (see part I page 98).

What about curl curl?

$$\operatorname{curl}\operatorname{curl}\mathbb{X} = [{*}d\,\underbrace{[\operatorname{curl}\mathbb{X}]^\flat}_{{*}d\mathbb{X}^\flat}]^\# = [\underbrace{{*}d{*}d}\,\mathbb{X}^\flat]^\#$$

one of those second order operators we mentioned above.

or

$$\operatorname{grad}\operatorname{div}\mathbb{X} = [\,d\,\underbrace{[\operatorname{div}\mathbb{X}]}_{{*}d{*}\mathbb{X}^\flat}\,]^\# = [\underbrace{d{*}d{*}}\,\mathbb{X}^\flat]^\#$$

another one of them!

or

$$\text{div grad } f = {}^*d{}^*\underbrace{[\text{grad} f]^\flat}_{df} = \underbrace{{}^*d{}^*d}f$$

there it is again but acting on a function rather than a vector field.

What about the other one: $\underbrace{d^*d\underbrace{{}^*f}_{\text{3-form}}}_{\equiv 0} = 0$ identically zero.

While we're at it, what about $\boxed{\vec\nabla, \vec\nabla\cdot, \vec\nabla\times \text{ and all that?}}$

Well, on $\mathbb{R}^3$ we can define $\underline{X}\times\underline{Y} = {}^*(X\wedge Y)$ for two vector fields.

What is the formula? $[\underline{X}\times\underline{Y}]^i = \frac{1}{2}\underbrace{[\underline{X}\wedge Y]^{jk}}_{2X^{[j}Y^{k]}}\underbrace{\eta_{jk}{}^i}_{\eta^i{}_{jk}} = \eta^i{}_{jk}\,\underline{X}^{[j}Y^{k]}$

$= \eta^{ijk}\underline{X}_j Y_k$

shifting indices and dropping antisymmetric part since η is antisymmetric so it is redundant.

In the Cartesian coordinate frame this is just the usual formula.

$$[\underline{X}\times Y]^i = \epsilon_{ijk}X^jY^k.$$

We've already defined $\vec\nabla$ as the covariant derivative operator with the derivative index raised, so $\text{grad} f = \vec\nabla f$ and

$$\text{curl}\,\underline{X} = \eta^{ijk}\partial_j\underline{X}_k = \eta^{ijk}\nabla_j\underline{X}_k = [\vec\nabla\times\underline{X}]^i$$

can replace ordinary by covariant derivative in exterior derivative

$$\text{div}\,\underline{X} = \nabla_i\underline{X}^i = g_{ij}\nabla^i\underline{X}^j = \vec\nabla\cdot\underline{X}$$

so $\text{div curl}\,\underline{X} = \vec\nabla\cdot(\vec\nabla\times\underline{X})$, etc. These may be easily evaluated in any coordinate system now.

What about the various product rules for grad, curl, div?
Most of them are disguised versions of the product rule for d:

$$\underset{p\text{-form}}{d(T} \wedge \underset{q\text{-form}}{S)} = dT\wedge S + (-1)^p T\wedge dS \qquad 0\le p,q\le n.$$

Because of $T\wedge S = (-1)^{pq} S\wedge T$, it is enough to look at the cases $p\le q$, but also $p+q<3$ since the exterior derivative of a 3-form is identically zero and $(p+q)$-forms are zero for $p+q>3$. This leaves

$$(p,q) \in \{(0,0), (0,1), (0,2), (1,1)\}$$

<u>exercise</u> Using the definitions of grad, curl, div in terms of d, re-express the left hand side of the following identities and use the above product rule for d with the given values of (p,q) and then rewrite in terms of grad, div, curl (recall $f\wedge T = fT$ for a zero-form f):

$(0,0)$ $\quad \text{grad } fh = h \text{ grad} f + f \text{ grad } h$

$(0,1)$ $\quad \text{curl } (f\mathbf{X}) = (\text{grad} f)\times \mathbf{X} + f \text{ curl } \mathbf{X}$

$(0,2)$ $\quad \text{div } (f\mathbf{X}) = (\text{grad} f)\cdot \mathbf{X} + f \text{ div } \mathbf{X}$

$(1,1)$ $\quad \text{div } (\mathbf{X}\times \mathbf{Y}) = \mathbf{Y}\cdot \text{curl } \mathbf{X} - \mathbf{X}\cdot \text{curl } \mathbf{Y}$

example: $\text{div} f\mathbf{X} = *d*(f\mathbf{X}^\flat) = *d(f*\mathbf{X}^\flat) = *[df\wedge *\mathbf{X}^\flat + (-1)^0 f\, d*\mathbf{X}^\flat]$

$$= \underbrace{*[\underbrace{df\wedge *\mathbf{X}^\flat}_{1\text{-forms}}]}_{\substack{<df,\mathbf{X}^\flat> \\ =(df)_i \mathbf{X}^i \\ = (\text{grad} f)\cdot \mathbf{X}}} + f \underbrace{*d*\mathbf{X}^\flat}_{\text{div } \mathbf{X}} = (\text{grad} f)\cdot \mathbf{X} + f \text{ div } \mathbf{X}$$

$*(T\wedge *S) = <T,S>$ for two p-forms

since $T\wedge *S = <T,S>\eta$ and $*\eta = 1$

see page 97

another example:

$$\operatorname{div}(X\times Y) = {}^*d{}^*[{}^*(X\wedge Y)]^\flat = {}^*d{}^*[{}^*(X^\flat\wedge Y^\flat)]$$

$$= {}^*d(X^\flat\wedge Y^\flat) = {}^*[\underbrace{dX^\flat\wedge Y^\flat}_{(-1)^{2\cdot 1}Y^\flat\wedge dX^\flat} - \underbrace{X^\flat\wedge dY^\flat}]$$

since $T\wedge S = (-1)^{pq} S\wedge T$

$$= {}^*[Y^\flat\wedge dX^\flat] - {}^*[X^\flat\wedge dY^\flat]$$

$$= {}^*[\underbrace{Y^\flat}\wedge {}^*\underbrace{({}^*dX^\flat)}_{(\operatorname{curl} X)^\flat}] - {}^*[X^\flat\wedge{}^*\underbrace{({}^*dY^\flat)}_{(\operatorname{curl} Y)^\flat}]$$

since ${}^{**}T = T$ for $n=3$, Euclidean metric

1-forms

$$= \langle Y^\flat, (\operatorname{curl} X)^\flat\rangle - \langle X^\flat, (\operatorname{curl} Y)^\flat\rangle$$

$$= Y\cdot\operatorname{curl} X - X\cdot\operatorname{curl} Y$$

Okay, these last two were a bit challenging since once needed the identity ${}^*(T\wedge{}^*S) = \langle T, S\rangle$ from part I so I did them for you. The first two are completely straightforward.

Notice that these "vector analysis" identities which are usually proved by Cartesian coordinate component calculations like

$$\operatorname{div}(X\times Y) = \partial_i(\epsilon^{ijk}X_j Y_k) = \epsilon^{ijk}[(\partial_i X_j)Y_k + X_j\partial_i Y_k]$$

$$= (\epsilon^{kij}\partial_i X_j)Y_k - (\epsilon^{jik}\partial_i Y_k)X_j = (\operatorname{curl} X)\cdot Y - (\operatorname{curl} Y)\cdot X$$

have just been proven for any ~~metric~~ positive-definite metric on a 3-dimensional space in any coordinate system (since they are independent of the coordinates). Thus we can extend all of this $\mathbb{R}^3$ vector analysis immediately to the 3-sphere, for example.

This is the power of real mathematics as opposed to "just getting by" techniques that are usually used in applied sciences.

Not impressed?

Maxwell's equations for the electromagnetic field which brings you cable TV and your favorite radio station and all the rest of our modern life, involving the electric and magnetic vector fields E and B and the charge density function ρ and current density vector field J:

$$\operatorname{div} B = 0 \qquad\qquad \operatorname{div} E = 4\pi\rho$$

$$\operatorname{curl} E + \frac{\partial B}{\partial t} = 0 \qquad\qquad \operatorname{curl} B - \frac{\partial E}{\partial t} = 4\pi J$$

can be written in the simple form

$$dF = 0 \qquad\qquad {*d*F} = 4\pi \mathcal{J}$$

by defining the electromagnetic 2-form

$$F = (E_x dx + E_y dy + E_z dz) \wedge dt + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$$

$$\mathcal{J} = -\rho dt + J_x dx + J_y dy + J_z dz .$$

on spacetime. Many of the somewhat complicated manipulations done in physics courses become very simple in this language. We dont have time to go into that here, but I wanted you to get a glimpse of this idea.