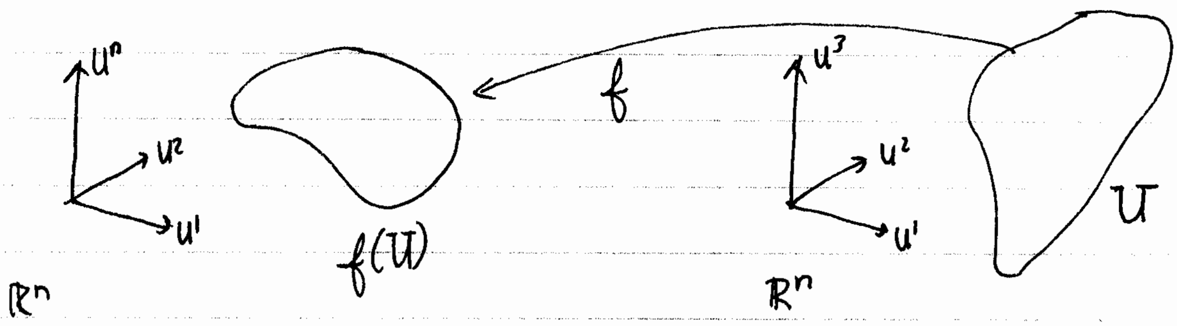
Integration of differential forms

First recall what you learned or almost learned in multivariable calculus about changing variables in multivariable integration



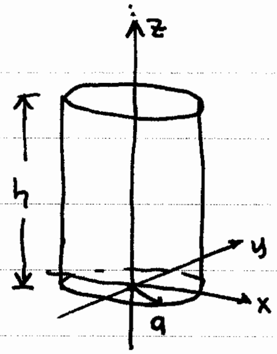
Suppose $f: \mathbb{R}^n \to \mathbb{R}^n$ is an invertible map from a closed region U onto its image $f(U)$. Let $\{u^i\}$ be the standard Cartesian coordinates on $\mathbb{R}^n$ and $f^i = u^i \circ f$ the component functions of this map. The cylindrical and spherical coordinate parametrization maps from the coordinate space into $\mathbb{R}^3$ are good examples of this

$$\begin{aligned} x &= \rho\cos\varphi \\ y &= \rho\sin\varphi \\ z &= z \end{aligned} \quad \leftrightarrow \quad \begin{aligned} f^1(u) &= u^1 \cos u^2 \\ f^2(u) &= u^1 \cos u^2 \\ f^3(u) &= u^3 \end{aligned}$$

Then the integral of some real valued function f on the image $f(U)$ can be expressed as an integral over U by

$$\int \cdots \int_{f(U)} f \, du^1 du^2 \ldots du^n = \iint_U f \circ f \underbrace{\left| \det\left(\frac{\partial f^i}{\partial u^j}\right)\right|}_{\text{correction factor}} du^1 du^2 \ldots du^n$$

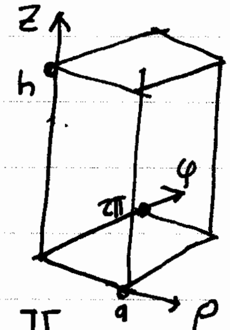
Besides re-expressing the function f in terms of the new variables to obtain $f \circ f$, a correction factor takes into account the change in the volume element against which the function is being integrated.



$f(U)$

$$\iiint_{f(U)} [\underbrace{x^2-y^2}_{f}]\, dx\,dy\,dz$$

$$= \iiint_{\substack{0\le\rho\le a\\ 0\le\varphi\le 2\pi\\ 0\le z\le h}} [\underbrace{\rho^2(\cos^2\varphi)}_{f\circ f}]\ \underbrace{\rho}_{\left|\det\left(\frac{\partial f^i}{\partial u^j}\right)\right|}\ d\rho\, d\varphi\, dz$$

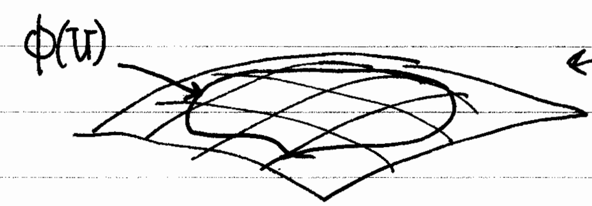


U

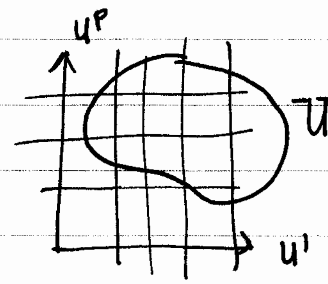
Changing to cylindrical coordinates to evaluate an integral over a cylinder is a familiar example of this. The absolute value in the correction factor guarantees that the integral of a positive function on $f(U)$ results in integrating a positive function on U.

This is all we need to know to describe integration of a p-form field on a p-surface in an n-dimensional space.

<u>parametrized p-surface</u>



ϕ



n-dim space with coords $\{x^i\}$

$\mathbb{R}^p$ "parameter space"

Suppose we have a 1-1 map ϕ from a closed region U of $\mathbb{R}^p$ into $\mathbb{R}^n$ or some n-dimensional space with local coordinates $\{x^i\}$. Let $\phi^i = x^i\circ\phi$ be those coordinates expressed as functions of the "parameters" $(u^1,\dots,u^p)$. Let $\alpha, \beta, \dots = 1,2,\dots,p$ denote the indices for the parameters.

For example, the above parameter map f for cylindrical coordinates represents $\mathbb{R}^3$ as a parametrized 3-surface or "3-space." Or fixing the radial coordinate

$$\begin{aligned} x &= a\cos\varphi \\ y &= a\sin\varphi \\ z &= z \end{aligned} \quad \leftrightarrow \quad \begin{aligned} \phi^1(u) &= a\cos u^1 \\ \phi^2(u) &= a\sin u^1 \\ \phi^3(u) &= u^2 \end{aligned}$$

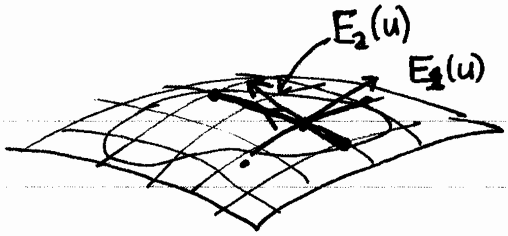
leads to a parametrized 2-surface representing a cylinder of radius a, or fixing z as well leads to a parametrized 1-surface or curve

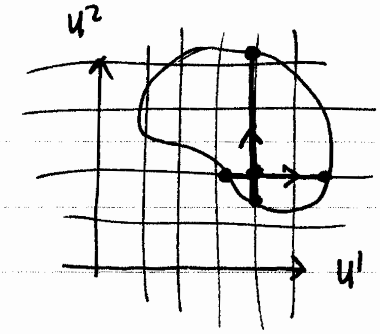
$$\begin{aligned} x &= a\cos\varphi \\ y &= a\sin\varphi \\ z &= z_0 \end{aligned} \quad \leftrightarrow \quad \begin{aligned} \phi^1(u) &= a\cos u^1 \\ \phi^2(u) &= a\sin u^1 \\ \phi^3(u) &= z_0 \end{aligned}$$

In general, fixing any $n-p$ coordinates in the parametrization map associated with local coordinates on $\mathbb{R}^n$ leads to a parametrized p-surface in which the p parameters correspond to p of these coordinates. $\{\theta, \varphi\}$ of spherical coordinates parametrize the coordinate spheres, for example, in $\mathbb{R}^3$.

For a given parametrized p-surface, fixing all the parameters but one yields a parametrized curve which is the image of the u^α coordinate line on $\mathbb{R}^p$ under the parametrization map ϕ. Varying the other parameters moves this curve around. One can do this for each of the parameters in turn. This corresponds to mapping to coordinate grid of $\mathbb{R}^p$ onto the image surface in the n-dimensional space. (See diagram on previous page.)

$$E_\alpha(u) \equiv \frac{\partial \phi^i(u)}{\partial u^\alpha} \frac{\partial}{\partial x^i}\bigg|_{\phi(u)}$$ is the tangent vector to the parametrized curve corresponding to the u^α coordinate line in $\mathbb{R}^p$.





The tangent p-plane to the parametrized p-surface is spanned by the p-vectors $\{E_\alpha(u)\}$, assuming that they are linearly independent so that its dimension is actually p. This is a condition we must place on the parametrized p-surface. If $\{E_\alpha(u)\}$ are linearly independent, then the p-vector

$$E_1(u)\wedge\ldots\wedge E_p(u) \quad \text{with components} \quad [E_1(u)\wedge\ldots\wedge E_p(u)]^{i_1\cdots i_p} = p!\; E_1(u)^{[i_1}\cdots E_p(u)^{i_p]}$$

must be nonzero everywhere. This determines the orientation of the tangent p-plane. This condition reduces to the requirement that the tangent vector to a parametrized curve not vanish anywhere, which should ring a bell from defining line integrals in multivariable calculus.

Pulling back differential forms

Suppose $T = \frac{1}{p!} T_{i_1\cdots i_p}\, dx^{i_1\cdots i_p} = T_{i_1\cdots i_p}\, dx^{|i_1\cdots i_p|}$

is a p-form field on our n-dimensional space, usually called a "differential p-form" or a "differential form" if the "degree" p is not made explicit. Expressing this differential form in terms of the parameters on the parametrized p-surface leads to a differential p-form on the parameter space

$$\phi^* T \equiv \frac{1}{p!} \underbrace{T_{i_1\cdots i_p}\circ\phi}_{\substack{\text{express components}\\ \text{in terms of}\\ \text{parameters}}} \quad \underbrace{d(\overbrace{x^{i_1}\circ\phi}^{\phi^{i_1}})\wedge\ldots\wedge d(x^{i_p}\circ\phi)}_{\substack{\text{express coordinate}\\ \text{differentials in terms}\\ \text{of parameters}}}$$

$$\left(\frac{\partial\phi^{i_1}}{\partial u^{\alpha_1}}du^{\alpha_1}\right)\wedge\ldots\wedge\left(\frac{\partial\phi^{i_p}}{\partial u^{\alpha_p}}du^{\alpha_p}\right)$$

$$\left.\frac{\partial\phi^{i_1}}{\partial u^{\alpha_1}}\cdots\frac{\partial\phi^{i_p}}{\partial u^{\alpha_p}} \underbrace{du^{\alpha_1\cdots\alpha_p}}_{\epsilon^{\alpha_1\cdots\alpha_p}\,du^{1\cdots p}}\right\}\ \text{p-form on } \mathbb{R}^p$$

only antisymmetric part contributes to this contraction

$$= \frac{1}{p!}\underbrace{T_{i_1\cdots i_p}}_{\text{antisym}}\circ\phi \ \underbrace{\frac{\partial\phi^{[i_1}}{\partial u^{\alpha_1}}\cdots\frac{\partial\phi^{i_p]}}{\partial u^{\alpha_p}}\epsilon^{\alpha_1\cdots\alpha_p}}_{p!\,\frac{\partial\phi^{[i_1}}{\partial u^1}\cdots\frac{\partial\phi^{i_p]}}{\partial u^p}} \ du^{1\cdots p}$$

The antisymmetrization on the indices $[i_1\cdots i_p]$ makes the antisymmetrization over $\alpha_1\cdots\alpha_p$ unnecessary so this contraction has $p!$ equal terms

$$= T_{i_1\cdots i_p}\circ\phi\ \frac{\partial\phi^{[i_1}}{\partial u^1}\cdots\frac{\partial\phi^{i_p]}}{\partial u^p}\ du^{1\cdots p}$$

$$= \underbrace{\frac{1}{p!}T_{i_1\cdots i_p}\circ\phi\ [E_1(u)\wedge\ldots\wedge E_p(u)]^{i_1\cdots i_p}}\ du^{1\cdots p}$$

This function is the natural contraction of the p-covector T on the p-vector $E_1(u)\wedge\ldots\wedge E_p(u)$ at each point of the parametrized p-surface. It is a function on $\mathbb{R}^p$

Suppose we DEFINE the integral of a p-form on $\mathbb{R}^p$ to be

$$\int_U f\,du^{1\cdots p} = \int_U f\,du^1\wedge\ldots\wedge du^p \equiv \int\cdots\int_U f\,du^1du^2\ldots du^p$$

the ordinary multivariable integral of its $1\cdots p$ component function.

We can then define the integral of a p-form on a parametrized p-surface by

$$\int_{\Phi(U)} T \equiv \int_U \Phi^* T = \int \cdots \int_U T_{|i_1 \cdots i_p|} [E_1(u) \wedge \cdots \wedge E_p(u)]^{i_1 \cdots i_p} du^1 du^2 \cdots du^p$$

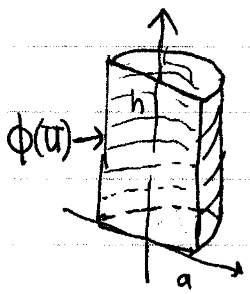
To summarize, we substitute the parametrization into T, expand it out to get a coefficient function times $du^{1\cdots p}$ and we just integrate that coefficient function on U in the ordinary sense. The coefficient function is the natural contraction of the p-form T with the p-vector $E_1(u) \wedge \cdots \wedge E_p(u)$ of the parametrization, divided by $p!$ to avoid overcounting.

Note that $\Phi : \mathbb{R}^p \to$ n-dim space maps from the parameter space into the space of interest, but Φ^* maps differential forms on that space back to differential forms on the parameter space. Φ goes forward and Φ^* "pulls back" differential forms from the image of Φ back to its domain, exactly like composition of a function f on the image leads to a function $f \circ \Phi \equiv \Phi^* f$ (pullback of a 0-form) on the domain.

This same "pullback" operation works with any covariant tensor field and simply amounts to substitution of the parametrization map into the expression for the tensor field. For example, the parameter map associated with cylindrical (or spherical) coordinates on $\mathbb{R}^3$ pulls back the Euclidean metric on $\mathbb{R}^3$ to the coordinate expression for the metric on the coordinate space.

$$= (y\,dy - x\,dx)\wedge dz$$

Suppose we consider our old friend $*X^\flat = *(y\,dx + x\,dy) = y\,dy\wedge dz + x\,dz\wedge dx \equiv T$ and integrate it over the parametrized half cylinder



$$x = a\cos u^1,\quad y = a\sin u^1,\quad z = u^2$$

$$U: \begin{cases} 0 \le u^1 \le \pi \\ 0 \le u^2 \le h \end{cases}$$

$$\int_{\phi(U)} T = \int_U \left[(a\sin u^1)\,d(a\sin u^1) - (a\cos u^1)\,d(a\cos u^1)\right]\wedge d[u^2]$$

$$= \int_U \underbrace{[a^2\sin u^1\cos u^1 + a^2\sin u^1\cos u^1]}_{2a^2\sin u^1\cos u^1 = a^2\sin 2u^1}\,du^1\wedge du^2$$

$$= \iint_U a^2\sin 2u^1\,du^1du^2 = \int_0^h\int_0^\pi a^2\sin 2u^1\,du^1\,du^2$$

$$= a^2\underbrace{\int_0^h du^2}_{h}\underbrace{\int_0^\pi \sin 2u^1\,du^1}_{-\frac{1}{2}\cos 2u^1\big|_0^\pi} = -\frac{a^2}{2}h\,[\cos 2\pi - \cos 0] = 0.$$

In this context the 2-form $\phi^*T = a^2\sin 2u^1\,du^1\wedge du^2$ is equivalent to re-expressing T in cylindrical coordinates and setting ρ ~~and $d\rho$~~ to a and $d\rho$ to 0:

$$T = (y\,dy - x\,dx)\wedge dz = [\rho\sin\varphi\,\underbrace{d(\rho\sin\varphi)}_{\rho(\cos\varphi\,d\varphi) + \sin\varphi\,d\rho} - \rho\cos\varphi\,\underbrace{d(\rho\cos\varphi)}_{(-\rho\sin\varphi\,d\varphi + \cos\varphi\,d\rho)}]\wedge dz$$

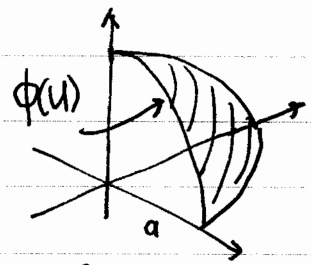
$$= [2\rho^2\sin\varphi\cos\varphi\,d\varphi + \rho(\sin^2\varphi - \cos^2\varphi)\,d\rho]\wedge dz$$

$$= \rho^2\sin 2\varphi\,d\varphi\wedge dz - \rho\cos 2\phi\,d\rho\wedge dz$$

$$\underbrace{T_{\substack{\rho=a\\ d\rho=0}} = a^2\sin 2\varphi\,d\varphi\wedge dz}_{\text{"restriction of } T \text{ to surface } \rho=a\text{"}} \leftrightarrow \phi^*T = a^2\sin 2u^1\,du^1\wedge du^2$$

$$\varphi \to u^1,\quad z \to u^2$$

exercise. Repeat the above discussion for $T = (y\,dy - x\,dx)\wedge dz$ on the part of a sphere $r = a$ in the first octant using the spherical coordinates $\{\theta, \varphi\}$ as the parameters $\{u^1, u^2\}$.



$x^2+y^2+z^2=a^2$

$$x = a\sin u^1 \cos u^2$$
$$y = a\sin u^1 \sin u^2$$
$$z = a\cos u^1$$

$$U: \begin{cases} 0 \le u^1 \le \frac{\pi}{2} \\ 0 \le u^2 \le \frac{\pi}{2} \end{cases}$$

First evaluate $\int_{\phi(U)} T$. Then evaluate T in spherical coordinates and restrict it to the sphere by setting $r = a$, $dr = 0$. Compare with your result for $\phi^* T$.

exercise.

In the cylindrical problem, the tangent vectors to the parameter grid are

$$(E_1{}^i(u)) = (-a\sin u^1, a\cos u^1, 0) \qquad E_1(u) = -a\sin u^1 \frac{\partial}{\partial x}\Big|_{\phi(u)} + a\cos u^1 \frac{\partial}{\partial y}\Big|_{\phi(u)}$$

$$(E_2{}^i(u)) = (0, 0, 1) \qquad E_2(u) = \frac{\partial}{\partial z}\Big|_{\phi(u)}$$

$$E_1(u)\wedge E_2(u) = +a\sin u^1 \frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x}\Big|_{\phi(u)} + a\cos u^1 \frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\Big|_{\phi(u)}$$

The natural contraction with T is then

$$\frac{1}{2} T_{ij}\circ\phi\, [E_1(u)\wedge E_2(u)]^{ij} = \underbrace{T_{23}\circ\phi}_{\substack{y\circ\phi \\ = a\sin u^1}} \underbrace{[E_1(u)\wedge E_2(u)]^{23}}_{a\cos u^1} + \underbrace{T_{31}\circ\phi}_{\substack{x \\ = a\cos u^1}} \underbrace{[E_1(u)\wedge E_2(u)]^{31}}_{a\sin u^1} + \underbrace{T_{12}\circ\phi}_{0} \underbrace{[E_1(u)\wedge E_2(u)]^{12}}_{0}$$

$$= 2a^2\sin u^1\cos u^1 = a^2\sin 2u^1$$

Calculate $E_1(u)\wedge E_2(u)$ for the previous exercise and its contraction with T in exactly this same way.

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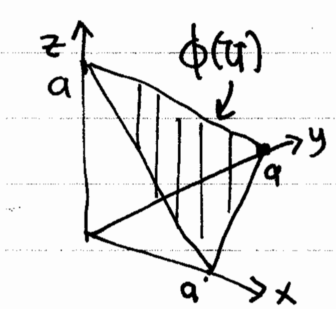
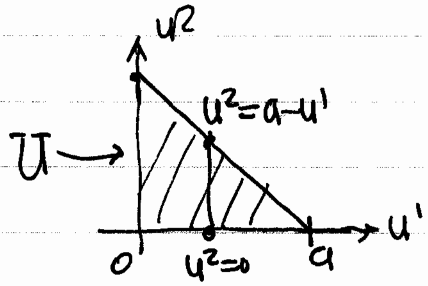
exercise



$x + y + z = a$

$x = u^1$
$y = u^2$
$z = a - u^1 - u^2$

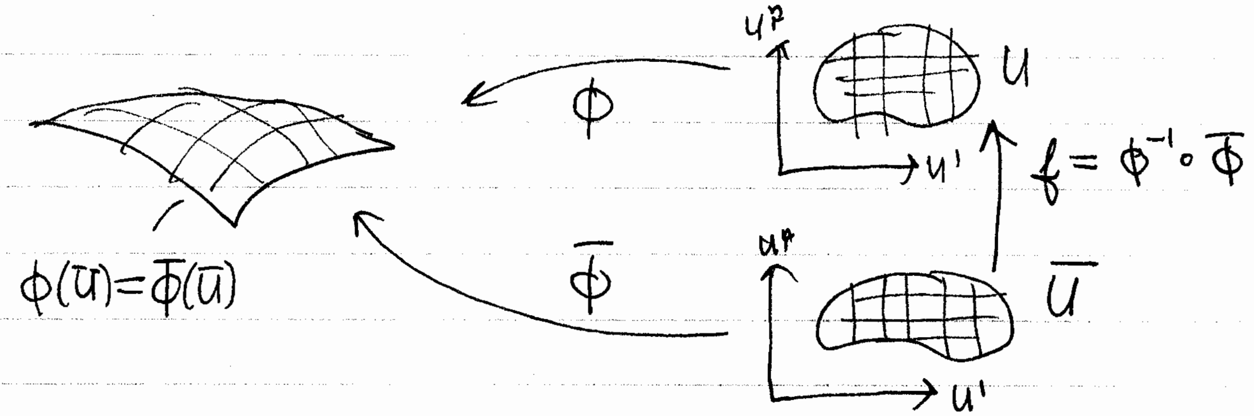
$U:\quad 0 \le u^1 \le a$
$\quad\;\; 0 \le u^2 \le a - u^1$



$$\underbrace{\int_0^a \int_0^{u^1} [\quad] \, du^2 \, du^1}_{\text{suggested iteration.}}$$

Repeat both exercises of the preceding page for this simpler plane surface.

Perhaps you were wondering why this section began with the change of variable discussion for multivariable integration and ~~did~~ not use it. Now is its time,



Suppose we have 2 different such parametrizations of the same p-surface. What do we need in order that the integral of a p-form T on it not depend on the parametrization? Well, $f = \phi^{-1}\circ\bar\Phi$ is a map from $\mathbb{R}^p$ to $\mathbb{R}^p$ which corresponds to the relationship between the parameters which specify the same points on our p-surface

The values for each parametrized surface as defined above are

$$\int_{\phi(U)} T = \int\cdots\int_{U} \frac{1}{p!} T_{i_1\cdots i_p}\circ\phi\, [E_1(u)\wedge\cdots\wedge E_p(u)]^{i_1\cdots i_p}\, du^1 du^2 \cdots du^p$$

$$\int_{\bar\Phi(U)} T = \int\cdots\int_{\bar U} \frac{1}{p!} T_{i_1\cdots i_p}\circ\bar\Phi\, [\bar E_1(\bar u)\wedge\cdots\wedge\bar E_p(\bar u)]^{i_1\cdots i_p}\, \underbrace{d\bar u^1 d\bar u^2\cdots d\bar u^p}_{\substack{\text{dummy variables so} \\ \text{can use any symbol}}}$$

But the function $f:\mathbb{R}^p\to\mathbb{R}^p$ just represents a change of variable in this ordinary integral $u^\alpha = f^\alpha(\bar u)$.

But $\bar\Phi = \phi\circ f$, $\bar\Phi^i(\bar u) = \phi^i(f(u))$

$$\bar E^i_\alpha(\bar u) = \frac{\partial\bar\Phi^i}{\partial\bar u^\alpha}(\bar u) = \frac{\partial\phi^i}{\partial u^\beta}(u)\frac{\partial f^\beta}{\partial\bar u^\alpha} = E^i_\beta(u(\bar u))\,\frac{\partial f^\beta}{\partial\bar u^\alpha}(\bar u)$$

$$\bar E_1(\bar u)\wedge\cdots\wedge\bar E_p(\bar u) = \left[E_{\alpha_1}(u(\bar u))\frac{\partial f^{\alpha_1}}{\partial\bar u^1}(\bar u)\right]\wedge\cdots\wedge\left[E_{\alpha_p}(u(\bar u))\frac{\partial f^{\alpha_p}}{\partial\bar u^p}(\bar u)\right]$$

$$= \underbrace{E_{\alpha_1}(u(\bar u))\wedge\cdots\wedge E_{\alpha_p}(u(\bar u))}_{\epsilon_{\alpha_1\cdots\alpha_p} E_1(u(\bar u))\wedge\cdots\wedge E_p(u(\bar u))}\;\frac{\partial f^{\alpha_1}}{\partial\bar u^1}(\bar u)\cdots\frac{\partial f^{\alpha_p}}{\partial\bar u^p}$$

$$= \det\left(\frac{\partial f^\alpha}{\partial \bar{u}^\beta}(\bar{u})\right) \; E_1(u(\bar{u}))\wedge\dots\wedge E_p(u(\bar{u}))$$

since $\epsilon_{\alpha_1\dots\alpha_p} \dfrac{\partial f^{\alpha_1}}{\partial \bar{u}^1}(\bar{u}) \dots \dfrac{\partial f^{\alpha_p}}{\partial \bar{u}^p}(\bar{u}) = \det\left(\dfrac{\partial f^\alpha}{\partial \bar{u}^\beta}(\bar{u})\right)$

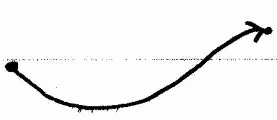
so $$\int_{\Phi(U)} T = \int\dots\int_{\bar{U}} \frac{1}{p!} T_{i_1\dots i_p}\circ\Phi\circ f \;\left[E_1(u)\wedge\dots\wedge E_p(u)\right]^{i_1\dots i_p}\circ f \; \det\left(\frac{\partial f^\alpha}{\partial \bar{u}^\beta}(\bar{u})\right) \; d\bar{u}^1 d\bar{u}^2 \dots d\bar{u}^p$$

But this is exactly $\int_{\Phi(U)} T$ under a change of variable except that the correction factor $\left|\det\left(\frac{\partial f^\alpha}{\partial \bar{u}^\beta}\right)\right|$ is right only if $\det\left(\frac{\partial f^\alpha}{\partial \bar{u}^\beta}\right) > 0$.

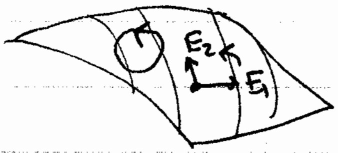
In other words, once we wish to define an integral of a p-form on a p-surface independent of the parametrization, we have to give it the additional mathematical structure of an "orientation."

The p-vectors $E_1(u)\wedge\dots\wedge E_p(u)$ for all parametrizations are proportional and nonzero since they determine the same tangent p-plane at each point of the p-surface. However, the nonzero proportionality factor can be positive or negative. Each parametrization determines its own orientation for the p-surface it parametrizes — namely any basis of the tangent p-plane has the same orientation if the proportionality factor relating the basis p-vector to the p-vector $E_1(u)\wedge\dots\wedge E_p(u)$ of the parametrization is positive.

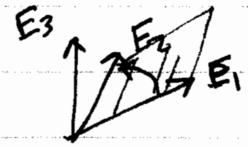
Thus the integral of a p-form on an oriented p-surface is well defined independent of the choice of parametrization.



For $p=1$, this just corresponds to a choice of direction for a curve segment.

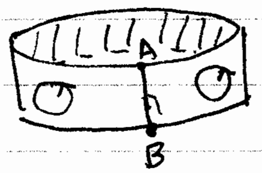


For $p=2$, this is a choice of a screwsense for a loop in the surface. This tells us the direction one must rotate the first basis vector to go towards the second basis vector if they are to have the chosen orientation, at each tangent space to the 2-surface.



For $p=3$, this is a choice of left or righthanded basis of each tangent space to the 3-surface. Curling fingers of righthand from E_1 to E_2 then requires E_3 to lie on the side of the plane of E_1 and E_2 determined by the thumb if the basis is to be righthanded. Otherwise it is lefthanded.

Of course this has to be able to be done consistently on the p-surface: the surface must be "orientable." This is not always possible.



Suppose we take a cylindrical strip and cut it as shown and twist it once so that A and B exchange places and then reattach the two ends smoothly. Now as we move our orientation indicating circle around the strip it comes out reversed after one loop. It is not possible to continuously assign an orientation to this "Mobius strip." It is not orientable, so one cannot define the integral of a 2-form on it.