EXERCISE : $e_{\alpha'} = A^{-1\beta}{}_{\alpha'} e_\beta$

$$\Gamma^{\gamma'}{}_{\alpha'\beta'} = A^{\gamma'}{}_\sigma \, e_{\alpha'} A^{-1\sigma}{}_{\beta'}$$

$$\omega^{\gamma'}{}_{\beta'} = \Gamma^{\gamma'}{}_{\alpha'\beta'} \, \omega^{\alpha'} = A^{\gamma'}{}_\sigma \underbrace{e_{\alpha'} A^{-1\sigma}{}_{\beta'} \, \omega^{\alpha'}}_{d A^{-1\sigma}{}_{\beta'}}$$

(just the differential of the function $A^{-1\sigma}{}_{\beta'}$)

$$\underline{\omega}' = \underline{A} \, d \underline{A}^{-1}$$

This is the matrix form of the formula for the connection components.

a) Use it to evaluate them for cylindrical coordinates using the coordinate frame

b) Repeat for the orthonormal frame obtained by normalizing the coordinate frame.

The spherical coordinate case is done explicitly in the following.

For spherical coordinates

$x^1 = r\sin\theta\cos\varphi \qquad \frac{\partial x^1}{\partial r} = \sin\theta\cos\varphi \qquad \frac{\partial x^1}{\partial\theta} = r\cos\theta\cos\varphi \qquad \frac{\partial x^1}{\partial\varphi} = -r\sin\theta\sin\varphi$

$x^2 = r\sin\theta\sin\varphi \qquad \frac{\partial x^2}{\partial r} = \sin\theta\sin\varphi \qquad \frac{\partial x^2}{\partial\theta} = r\cos\theta\sin\varphi \qquad \frac{\partial x^1}{\partial\varphi} = +r\sin\theta\cos\varphi$

$x^3 = r\cos\theta \qquad \frac{\partial x^3}{\partial r} = \cos\theta \qquad \frac{\partial x^3}{\partial\theta} = -r\sin\theta \qquad \frac{\partial x^1}{\partial\varphi} = 0$

$$\underline{A}^{-1} = (\vec{A}^{\alpha}{}_{\beta}) = \left(\frac{\partial x^\alpha}{\partial x'^\beta}\right) = \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

TRICK:
divide columns by lengths → orthog
→ transpose = inverse
→ divide rows by original 3 lengths

$$\underline{A} = (A^{\alpha}{}_{\beta}) = \left(\frac{\partial x'^\alpha}{\partial x^\beta}\right) = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ r^{-1}\cos\theta\cos\varphi & r^{-1}\cos\theta\sin\varphi & -r^{-1}\sin\theta \\ -r^{-1}\frac{\sin\varphi}{\sin\theta} & r^{-1}\frac{\cos\varphi}{\sin\theta} & 0 \end{pmatrix} \text{ (verify)}$$

$$\underline{dA}^{-1} = \begin{pmatrix} 0 & \cos\theta\cos\varphi & -\sin\theta\sin\varphi \\ 0 & \cos\theta\sin\varphi & \sin\theta\cos\varphi \\ 0 & -\sin\theta & 0 \end{pmatrix} dr$$

$$+ \begin{pmatrix} \cos\theta\cos\varphi & -r\sin\theta\cos\varphi & -r\cos\theta\sin\varphi \\ \cos\theta\sin\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ -\sin\theta & -r\cos\theta & 0 \end{pmatrix} d\theta$$

$$+ \begin{pmatrix} -\sin\theta\sin\varphi & -r\cos\theta\sin\varphi & -r\sin\theta\cos\varphi \\ \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ 0 & 0 & 0 \end{pmatrix} d\varphi$$

$$\underline{A}\,\underline{dA}^{-1} = A\,dA^{-1} = \begin{pmatrix} \cancel{r^{-1}\cos^2\varphi \quad -r^{-1}\sin^2\theta\sin\varphi\cos\varphi \quad -r^{-1}\sin\theta\cos\theta\cos\varphi} \\ \\ \end{pmatrix} dr$$

$$+ \begin{pmatrix} \\ \text{ugly} \\ \\ \end{pmatrix} d\theta$$

$$+ \begin{pmatrix} 0 \\ \\ \end{pmatrix} d\varphi$$

SPHERICAL COORDINATE CASE

The columns of $\underline{A}^{-1}$ are the cartesian components of the new frame vectors $e_r = \frac{\partial}{\partial r}$, $e_\theta = \frac{\partial}{\partial \theta}$, $e_\varphi = \frac{\partial}{\partial \varphi}$ so the Euclidean norm of the column vectors of $\underline{A}^{-1}$ give the norms of these new frame vectors

$$g_{rr} = e_r \cdot e_r = \left[(\sin\theta\cos\varphi)^2 + (\sin\theta\sin\varphi)^2 + (\cos\theta)^2 \right]^{1/2} = 1$$

$g(e_r, e_r) = \delta_{\alpha\beta} (e_r)^\alpha (e_r)^\beta$

$$g_{\theta\theta} = \cdots = r^2$$
$$g_{\varphi\varphi} = \cdots = r^2 \sin^2\theta$$

Dividing the columns of $\underline{A}^{-1}$ by their Euclidean lengths leads to the matrix $\underline{B}^{-1}$ which transforms the cartesian frame to the normalized spherical coordinate frame, an orthonormal frame

$$e_{\hat{\alpha}} = B^{-1\beta}{}_{\alpha}\, e_\beta = (g_{\alpha\alpha})^{-1/2} A^{-1\beta}{}_{\alpha}\, e_\beta$$

$$\underline{B}^{-1} = \begin{pmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$ = orthogonal matrix representing a rotation of the cartesian orthonormal frame

$$B = (B^{-1})^T = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix}$$ (since orthogonal)

The columns of $\underline{B}$ represent the new orthonormal components of the old cartesian frame vectors $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ with respect to the new spherical orthonormal frame $e_{\hat{r}} = \partial/\partial r$, $e_{\hat{\theta}} = r^{-1}\partial/\partial\theta$, $e_{\hat{\varphi}} = (r\sin\theta)^{-1}\partial/\partial\varphi$

For example: $\partial/\partial y = \sin\varphi\,(\sin\theta\, e_{\hat{r}} + \cos\theta\, e_{\hat{\theta}}) + \cos\varphi\, e_{\hat{\varphi}}$.

Now compute the matrix of connection 1-forms

$$\omega = \underline{B}\, d\underline{B}^{-1} =$$

$$= \underline{B}\left[\begin{pmatrix} \cos\theta\cos\varphi & -\sin\theta\cos\varphi & 0 \\ \cos\theta\sin\varphi & -\sin\theta\sin\varphi & 0 \\ -\sin\theta & -\cos\theta & 0 \end{pmatrix} d\theta + \begin{pmatrix} -\sin\theta\sin\varphi & -\cos\theta\sin\varphi & -\cos\varphi \\ \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ 0 & 0 & 0 \end{pmatrix} d\varphi \right]$$

$$= \begin{pmatrix} 0 & -d\theta & -\sin\theta\, d\varphi \\ +d\theta & 0 & -\cos\theta\, d\varphi \\ +\sin\theta\, d\varphi & +\cos\theta\, d\varphi & 0 \end{pmatrix} = \begin{pmatrix} 0 & -r^{-1}\omega^{\hat{\theta}} & -r^{-1}\omega^{\hat{\varphi}} \\ & 0 & -r^{-1}\cot\theta\,\omega^{\hat{\varphi}} \\ & & 0 \end{pmatrix}$$

The result is a simple antisymmetric matrix, where we have used the dual forms

$$\omega^{\hat r} = dr \quad , \omega^{\hat\theta} = r\,d\theta \quad , \omega^{\hat\varphi} = r\sin\theta\, d\varphi ,$$

in terms of which the metric is

$$g = \omega^{\hat r}\otimes\omega^{\hat r} + \omega^{\hat\theta}\otimes\omega^{\hat\theta} + \omega^{\hat\varphi}\otimes\omega^{\hat\varphi}.$$

Reading off the connection components from this matrix:

$$\cancel{\Gamma^{\hat r}{}_{\hat\varphi\hat\varphi} = -(r\,\text{[illegible]})^{-1} = -\Gamma^{\hat\varphi}{}_{\hat\varphi\hat r}}$$
$$\cancel{\Gamma^{\hat r}{}_{\hat\varphi\hat\varphi} = -r^{-1}\,\text{[illegible]} = -\Gamma^{\hat\varphi}{}_{\hat\varphi\hat r}}$$
$$\cancel{\Gamma^{\hat\theta}{}_{\hat\varphi\hat\varphi} = -r^{-1}\cot\theta = -\Gamma^{\hat\varphi}{}_{\hat\varphi\hat\theta}}$$

$$\Gamma^{\hat r}{}_{\hat\theta\hat\theta} = -r^{-1} = -\Gamma^{\hat\theta}{}_{\hat\theta\hat r}$$
$$\Gamma^{\hat r}{}_{\hat\varphi\hat\varphi} = -r^{-1} = -\Gamma^{\hat\varphi}{}_{\hat\varphi\hat r}$$
$$\Gamma^{\hat\theta}{}_{\hat\varphi\hat\varphi} = -r^{-1}\cot\theta = -\Gamma^{\hat\varphi}{}_{\hat\varphi\hat\theta}$$

Only 6 nonzero components instead of $3^3 = 27$ possible nonzero components and only 3 are independent because of this antisymmetry.

As a check on our work, let's verify that $X = \partial/\partial y$ is covariant constant

$$\begin{cases} X^{\hat r} = \sin\theta\sin\varphi \\ X^{\hat\theta} = \cos\theta\sin\varphi \\ X^{\hat\varphi} = \cos\varphi \end{cases}$$

$$[\nabla_{e_{\hat\alpha}} X]^{\hat\beta} = e_{\hat\alpha} X^{\hat\beta} + \Gamma^{\hat\beta}{}_{\hat\alpha\hat\gamma} X^{\hat\gamma}$$

$$[\nabla_{e_{\hat r}} X]^{\hat\beta} = \frac{\partial}{\partial r} X^{\hat\beta} + \Gamma^{\hat\beta}{}_{\hat r\hat\gamma} X^{\hat\gamma} = 0 + 0 \quad \text{since } \Gamma^{\hat\beta}{}_{\hat r\hat\gamma} = 0.$$

$$[\nabla_{e_{\hat\theta}} X]^{\hat r} = r^{-1}\underbrace{\frac{\partial}{\partial\theta} X^{\hat r}}_{+\cos\theta\sin\varphi} + \underbrace{\Gamma^{\hat r}{}_{\hat\theta\hat\theta}}_{-r^{-1}} \underbrace{X^{\hat\theta}}_{\cos\theta\sin\varphi} = 0$$

$$[\nabla_{e_{\hat\theta}} X]^{\hat\theta} = r^{-1}\underbrace{\frac{\partial}{\partial\theta} X^{\hat\theta}}_{-\sin\theta\sin\varphi} + \underbrace{\Gamma^{\hat\theta}{}_{\hat\theta\hat r}}_{r^{-1}} \underbrace{X^{\hat r}}_{\sin\theta\sin\varphi} = 0$$

$$[\nabla_{e_{\hat\theta}} X]^{\hat\varphi} = r^{-1}\underbrace{\frac{\partial}{\partial\theta} X^{\hat\varphi}}_{0} + \underbrace{\Gamma^{\hat\varphi}{}_{\hat\theta\hat\alpha}}_{= 0} \underbrace{X^{\hat\alpha}}_{\cancel{\sin\theta\sin\varphi}}$$

$$[\nabla_{e_{\hat\varphi}} X]^{\hat r} = (r\sin\theta)^{-1}\underbrace{\frac{\partial}{\partial\varphi} X^{\hat r}}_{\sin\theta\cos\varphi} + \underbrace{\Gamma^{\hat r}{}_{\hat\varphi\hat\varphi}}_{-r^{-1}} \underbrace{X^{\hat\varphi}}_{\cos\varphi} = 0$$

$$[\nabla_{e_{\hat\varphi}} X]^{\hat\theta} = (r\sin\theta)^{-1}\underbrace{\frac{\partial}{\partial\varphi} X^{\hat\theta}}_{\cos\theta\cos\varphi} + \underbrace{\Gamma^{\hat\theta}{}_{\hat\varphi\hat\varphi}}_{-r^{-1}\cot\theta} \underbrace{X^{\hat\varphi}}_{\cos\varphi} = 0$$

$$[\nabla_{e_{\hat\varphi}} X]^{\varphi} = (r\sin\theta)^{-1}\underbrace{\frac{\partial}{\partial\varphi} X^{\hat\varphi}}_{-\sin\varphi} + \underbrace{\Gamma^{\hat\varphi}{}_{\hat\varphi\hat r} X^{\hat r}}_{r^{-1}\sin\theta\sin\varphi} + \underbrace{\Gamma^{\hat\varphi}{}_{\hat\varphi\hat\theta} X^{\hat\theta}}_{r^{-1}\cot\theta\,\cos\theta\sin\varphi} = 0 \qquad \text{okay.}$$

Understanding the connection components using the covariant exterior derivative.

The spherical orthonormal frame example suggests another simpler way to understand the connection components which requires the definition of the covariant exterior derivative.

Transform the differentials of the cartesian coordinate frame components of a vector to the new frame:

$$e_{\alpha'} = A^{\beta}{}_{\alpha'}\, e_\beta \,, \qquad \omega^{\alpha'} = A^{\alpha}{}_{\beta}\, \omega^\beta$$

$$X^{\alpha'} = A^{\alpha'}{}_{\beta} X^\beta$$

$$X^{\alpha} = A^{-1\alpha}{}_{\beta'} X^{\beta'}$$

$$dX^\alpha = A^{-1\alpha}{}_{\beta'}\, dX^{\beta'} + dA^{-1\alpha}{}_{\beta} X^{\beta'}$$

$$A^{\gamma}{}_{\alpha} dX^\alpha = A^{\gamma}{}_{\alpha}\left(A^{-1\alpha}{}_{\beta} dX^{\beta} + dA^{-1\alpha}{}_{\beta} X^{\beta'}\right)$$

$$= dX^{\prime\gamma} + \underbrace{A^{\gamma}{}_{\alpha}\, dA^{-1\alpha}{}_{\beta}}_{\omega^{\gamma'}{}_{\beta'} = \Gamma^{\gamma}{}_{\alpha'\beta'}\omega^{\alpha'}} X^{\beta'}$$

$$\equiv DX^{\gamma'}$$

This is just the covariant derivative of X with its single ~~1~~-form argument evaluated on $\omega^{\gamma'}$:

$$= (\nabla X)(\omega^{\gamma'}, \ )$$

leaving one vector argument for the direction of the covariant derivative.

parallel matrix calculation.

$$\underline{X}' = \underline{A}\,\underline{X}$$

$$\underline{X} = \underline{A}^{-1}\underline{X}'$$

$$d\underline{X} = \underline{A}^{-1} d\underline{X}' + d\underline{A}^{-1}\underline{X}'$$

$$\underline{A}\, d\underline{X} = d\underline{X}' + \underbrace{\underline{A}\, d\underline{A}^{-1}}_{\underline{\omega}'}\underline{X}'$$

$$\equiv D\underline{X}'$$

The connection 1-form is necessary to take into account the differentials of the frame ~~dual 1-form~~ vector components which are now position dependent.

Now consider the identity for the transformation matrix to the spherical orthonormal basis:

$$\underline{B}\,\underline{B}^{-1} = \underline{1}$$

$$d\underline{B}\,\underline{B}^{-1} + \underline{B}\,d\underline{B}^{-1} = 0$$

$$\boxed{d\underline{B} + \underbrace{\underline{B}\,d\underline{B}^{-1}}_{\underline{\omega}}\,\underline{B} = 0} \rightarrow \boxed{\underbrace{d\underline{B} + \underline{\omega}\,\underline{B}}_{D\underline{B}} = 0}$$

The columns of $\underline{B}$ are the new components of the cartesian frame vectors with respect to the spherical orthonormal frame.
The columns of this matrix equation are just the new components of the covariant exterior derivatives of the components of the cartesian frame vectors. These are covariant constant, so the covariant exterior derivatives vanish.

Thus the connection 1-form matrix is just the linear transformation of the tangent space needed to correct for the derivatives of the frame vectors along the direction which is the argument of the 1-form.

Cylindrical coordinates

$$x^1 = \rho\cos\varphi \qquad \frac{\partial x^1}{\partial r} = \cos\varphi \quad \frac{\partial x^1}{\partial\varphi} = -\rho\sin\varphi \quad \frac{\partial x^1}{\partial z} = 0$$

$$x^2 = \rho\sin\varphi \qquad \frac{\partial x^2}{\partial r} = \sin\varphi \quad \frac{\partial x^2}{\partial\varphi} = \rho\cos\varphi \quad \frac{\partial x^2}{\partial z} = 0$$

$$x^3 = z \qquad \frac{\partial x^3}{\partial r} = 0 \quad \frac{\partial x^2}{\partial\varphi} = 0 \quad \frac{\partial x^3}{\partial z} = 1$$

$$e_{\alpha'} = \frac{\partial}{\partial x'^{\alpha}} = \underbrace{\frac{\partial x^{\beta}}{\partial x'^{\alpha}}}_{A^{-1\beta}{}_{\alpha}} \frac{\partial}{\partial x^{\beta}} \qquad A^{-1} = \begin{pmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$e_{\hat\alpha} = (g'_{\alpha\alpha})^{-1/2} e_{\alpha'} = (B^{-1})^{\beta}{}_{\alpha} \frac{\partial}{\partial x^{\beta}} \qquad B^{-1} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = (B^{-1})^{T} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{A} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\rho^{-1}\sin\varphi & \rho^{-1}\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{check } \underline{A}\,\underline{A}^{-1} = \underline{1}.$$

$$d\underline{A}^{-1} = \begin{pmatrix} 0 & -\sin\varphi & 0 \\ 0 & \cos\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} dr + \begin{pmatrix} -\sin\varphi & -\rho\cos\varphi & 0 \\ \cos\varphi & -\rho\sin\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} d\varphi$$

$$\underline{A}\,d\underline{A}^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} dr + \begin{pmatrix} 0 & -\rho & 0 \\ +\rho^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\varphi$$

$$\underline{B}\,d\underline{B}^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\sin\varphi & -\cos\varphi & 0 \\ \cos\varphi & -\sin\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} d\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\varphi$$

$$= \begin{pmatrix} 0 & -\rho^{-1} & 0 \\ \rho^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \underbrace{\rho\, d\varphi}_{\omega^{\hat\varphi}}$$

$$\Gamma'^{\alpha}{}_{\beta\gamma}\omega^{\beta} = (A\,dA^{-1})^{\alpha}{}_{\gamma} \quad \text{so}$$

$$\Gamma'^{\varphi}{}_{\rho\varphi} = \rho^{-1}, \quad \Gamma'^{\rho}{}_{\varphi\varphi} = -\rho, \quad \Gamma'^{\varphi}{}_{\varphi\rho} = +\rho^{-1}$$

(cyl coord components of connection)

$$\hat{\Gamma}^{\alpha}{}_{\beta\gamma}\omega^{\hat{\beta}} = (B\,dB^{-1})^{\alpha}{}_{\gamma} \quad \text{so}$$

$$\hat{\Gamma}^{\hat{\rho}}{}_{\hat{\varphi}\hat{\varphi}} = -\hat{\Gamma}^{\hat{\varphi}}{}_{\hat{\rho}\hat{\varphi}} = -\rho^{-1}.$$

(orthonormal cyl frame components of connection)

We also have another formula for the coordinate components of the connection involving derivatives of the metric components

$$\Gamma^{\alpha'}{}_{\beta'\gamma'} = \frac{1}{2} g^{\alpha'\delta'}\left(g_{\delta'\beta',\gamma'} - g_{\beta'\gamma',\delta'} + g_{\gamma'\delta',\beta'}\right) \equiv g^{\alpha'\delta'}\Gamma_{\delta'\beta'\gamma'}$$

$$(g_{\alpha'\beta'}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (g^{\alpha'\beta'}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Only nonzero derivative:

$$g_{\varphi\varphi} = \rho^2 \qquad g_{\varphi\varphi,\rho} = 2\rho$$

so only nonzero components of $\Gamma_{\delta'\beta'\gamma'}$:

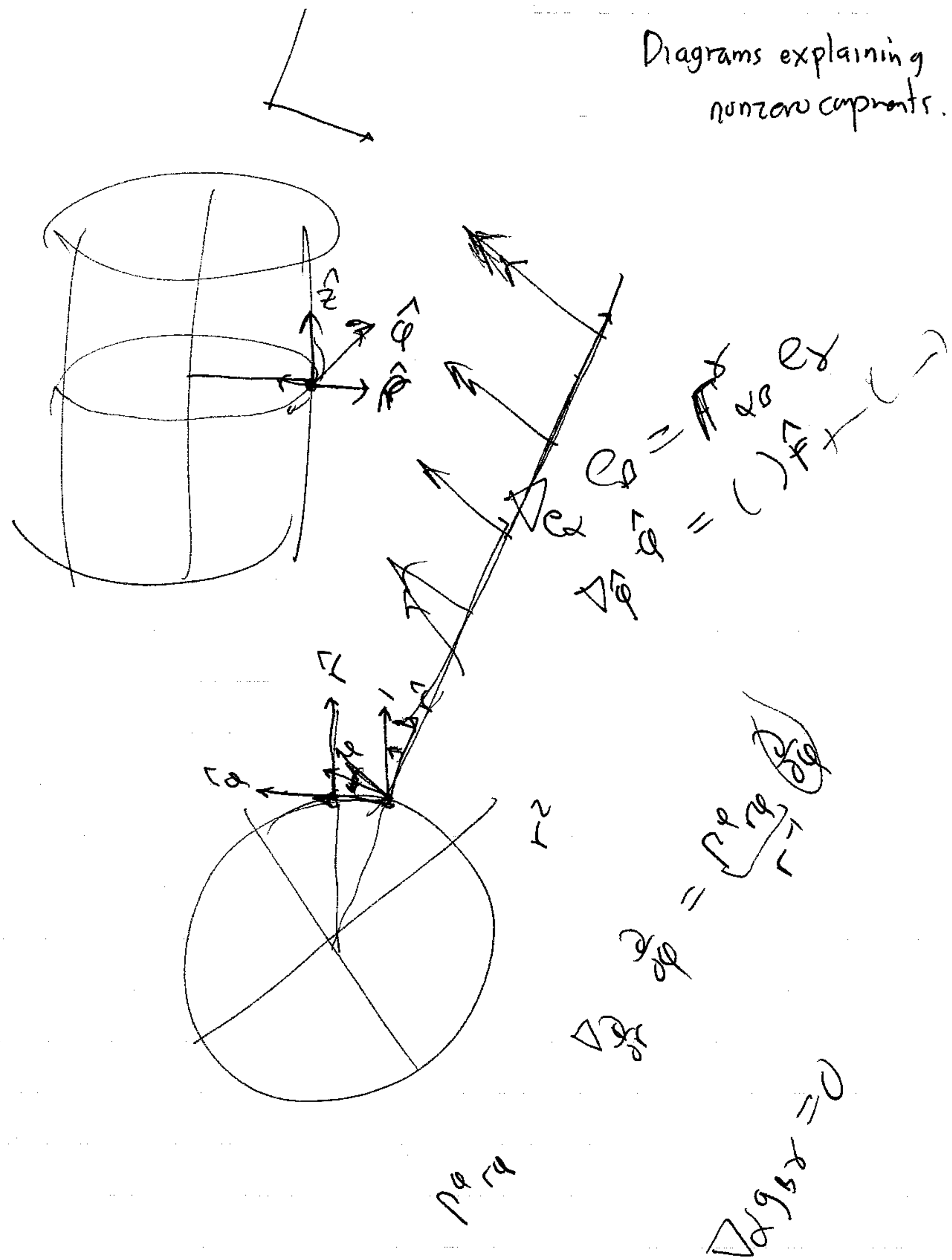
$$\Gamma'_{\rho\varphi\varphi} = \frac{1}{2}\left(\cancel{g_{\rho\varphi,\varphi}} - g_{\varphi\varphi,\rho} + \cancel{g_{\varphi\rho,\varphi}}\right) = -\rho$$

$$\Gamma'_{\varphi\varphi\rho} = \Gamma_{\varphi\rho\varphi} = \frac{1}{2}\left(g_{\varphi\varphi,\rho} - \cancel{g_{\varphi\rho,\varphi}} + \cancel{g_{\rho\varphi,\varphi}}\right) = \rho$$

$$\Gamma'^{\rho}{}_{\varphi\varphi} = g^{rr}\Gamma_{\rho\varphi\varphi} = -\rho$$

$$\Gamma'^{\varphi}{}_{\varphi\rho} = \Gamma'^{\varphi}{}_{\rho\varphi} = g^{\varphi\varphi}\Gamma'_{\varphi\varphi\rho} = \rho^{-1} \qquad \underline{\text{check}}!$$

Diagrams explaining nonzero components.



So what is the point?

First we relax the coordinate system on $\mathbb{R}^3$ to use more general coordinates $\{x'^{\alpha}\}$ leading to a new basis for each tangent space (and dual basis), in terms of which we can re-express any tensor field. The Euclidean metric

$$g = \delta_{\alpha\beta}\, dx^{\alpha} \otimes dx^{\beta} = g_{\alpha'\beta'}\, dx^{\alpha'} \otimes dx^{\beta'}$$

is a good example. Next, to reproduce the cartesian coordinate gradient operator, we transformed the operator to general coordinates and found correction terms (components of the connection acting as a linear transformation on each of the indices) that must be added to the new partial derivatives with respect to the coordinates. For orthogonal coordinates, one can obtain an orthonormal frame by normalizing the coordinate frame, simplifying the connection components and restoring the metric component matrix to the unit matrix.

The connection components we found could be determined in two ways. First by an explicit formula involving derivatives of the matrix from the cartesian coordinate frame to the new coordinate or noncoordinate frame. Second, for a coordinate frame, we showed that the covariant constancy of the metric itself determined the same connection components in terms of derivatives of the metric components. (A more general formula involving the Lie brackets of the frame vectors holds for a noncoordinate frame.)

Now we ask, how can we generalize this situation. The first method depends on the existence of the global cartesian coordinates whose frame vectors are covariant constant, and the geometry is always flat Euclidean geometry. The second takes as its starting point the expression for the metric in some coordinate system, obtaining the connection components by differentiation. If we relax the condition that the metric components arise by transformation from the standard cartesian expressions, then we can still discuss covariant differentiation. The only difference will be that the geometry will not (necessarily) be flat but exhibit "curvature", a concept we still must define.

For example, consider cylindrical coordinates where the Euclidean metric is $g = d\rho \otimes d\rho + \rho^2 d\varphi \otimes d\varphi + dz \otimes dz$.

The coordinate surfaces ~~are cylinders~~ $\rho = \rho_0 > 0$ are cylinders. Pick one (call it Σ) and consider its tangent spaces in the multivariable calculus sense, namely the tangents to all curves lying in this cylinder. We can use $\{\varphi, z\}$ as coordinates on this cylinder and evaluate lengths of vector fields ~~X~~

$$X = X^\varphi(\varphi, z) \frac{\partial}{\partial\varphi} + X^z(\varphi, z) \frac{\partial}{\partial z}$$

on the cylinder using the metric

$$g_\Sigma = \rho_0^2 \, d\varphi \otimes d\varphi + dz \otimes dz \quad \leftrightarrow (g_{\alpha\beta}) = \begin{pmatrix} \rho_0^2 & 0 \\ 0 & 1 \end{pmatrix}. \qquad \alpha, \beta = 2,3 \text{ only}$$

The inverse metric is

$$g^{-1}_\Sigma = \rho_0^{-2} \frac{\partial}{\partial\varphi} \otimes \frac{\partial}{\partial\varphi} + \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \quad \leftrightarrow (g^{\alpha\beta}) = \begin{pmatrix} \rho_0^{-2} & 0 \\ 0 & 1 \end{pmatrix}$$

From the 2-dimensional metric on the cylinder, we can compute the connection components by the formula.

$$\Gamma_{\alpha\beta\gamma} = \tfrac{1}{2}(g_{\alpha\beta,\gamma} - g_{\beta\gamma,\alpha} + g_{\gamma\alpha,\beta}) \qquad \alpha, \beta, \ldots = 2,3$$

$$\Gamma^\alpha{}_{\beta\gamma} = g^{\alpha\delta} \Gamma_{\delta\beta\gamma}$$

but the components of the metric are ~~zero~~ constant so the connection components are identically zero. Thus the geometry is still flat and in fact $y^1 = \rho_0 \varphi$ and $y^2 = z$ are orthonormal coordinates on the cylinder. But it has a "nontrivial" topology (i.e., not $\mathbb{R}^2$) since the direction of the coordinate y^1 is closed with total length $2\pi\rho_0$.

On the other hand, suppose we consider spherical coordinates where the metric is

$$g = dr\otimes dr + r^2\, d\theta\otimes d\theta + r^2\sin^2\theta\, d\varphi\otimes d\varphi$$

and take a coordinate surface $r = r_0 > 0$, which is a sphere. Here $\{\theta, \varphi\}$ are coordinates and a vector field on the sphere of a given radius (call it Σ) $X = X^\theta(\theta,\varphi)\frac{\partial}{\partial\theta} + X^\varphi(\theta,\varphi)\frac{\partial}{\partial\varphi}$ has its length determined by the metric

$$g_\Sigma = r_0^2\left(d\theta\otimes d\theta + \sin^2\theta\, d\varphi\otimes d\varphi\right) \leftrightarrow (g_{\alpha\beta}) = \begin{pmatrix} r_0^2 & 0 \\ 0 & r_0^2\sin^2\theta \end{pmatrix}$$

$$g_\Sigma^{-1} = r_0^{-2}\left(\frac{\partial}{\partial\theta}\otimes\frac{\partial}{\partial\theta} + \sin^{-2}\theta\, \frac{\partial}{\partial\varphi}\otimes\frac{\partial}{\partial\varphi}\right) \leftrightarrow (g^{\alpha\beta}) = \begin{pmatrix} r_0^{-2} & 0 \\ 0 & r_0^{-2}\sin^{-2}\theta \end{pmatrix}$$

Now we have a nonzero derivative $\quad g_{\varphi\varphi,\theta} = 2r_0^2\sin\theta\cos\theta$

and nonzero connection components

$$\Gamma_{\varphi\varphi\theta} = \tfrac{1}{2}(g_{\varphi\varphi,\theta} - \underbrace{g_{\varphi\theta,\varphi}}_{=0} + \underbrace{g_{\theta\varphi,\varphi}}_{=0}) = r_0^2\sin\theta\cos\theta \quad = \Gamma_{\varphi\theta\varphi}$$

$$\Gamma_{\theta\varphi\varphi} = \tfrac{1}{2}(\underbrace{g_{\theta\varphi,\varphi}}_{=0} - g_{\varphi\varphi,\theta} + \underbrace{g_{\varphi\theta,\varphi}}_{=0}) = -r_0^2\sin\theta\cos\theta$$

$$\Gamma^\varphi{}_{\varphi\theta} = g^{\varphi\varphi}\Gamma_{\varphi\varphi\theta} = \cot\theta \quad = \Gamma^\varphi{}_{\theta\varphi}$$

$$\Gamma^\theta{}_{\varphi\varphi} = g^{\theta\theta}\Gamma_{\theta\varphi\varphi} = -\sin\theta\cos\theta$$

We can now covariant differentiate vector fields on the sphere, and also evaluate the commutator of two covariant derivatives

$$[\nabla_\alpha,\nabla_\beta]X^\gamma \equiv \nabla_\alpha\nabla_\beta X^\gamma - \nabla_\beta\nabla_\alpha X^\gamma = R^\gamma{}_{\delta\alpha\beta}X^\delta$$

$$\cancel{R^\delta{}_{\alpha\beta\gamma} = \Gamma^\delta{}_{\beta\delta,\alpha} - \Gamma^\delta{}_{\alpha\delta,\beta} + \Gamma^\delta{}_{\alpha\mu}\Gamma^\mu{}_{\beta\nu} - \Gamma^\delta{}_{\beta\mu}\Gamma^\mu{}_{\alpha\nu}}$$

but now this 4 index object will not vanish! This is the manifestation of the curvature of the sphere. This object is actually a tensor called the curvature tensor.

Suppose we are even more courageous. Take the metric in spherical coordinates and arbitrarily change the coefficient functions

$$g_{(M)} = \left(1-\frac{2M}{r}\right)^{-1} dr\otimes dr + r^2(d\theta\otimes d\theta + \sin^2\theta\, d\varphi\otimes d\varphi)$$

$$g_{(k)} = (1+kr^2)^{-1} dr\otimes dr + r^2(d\theta\otimes d\theta + \sin^2\theta\, d\varphi\otimes d\varphi)$$

These are two 1-parameter families of new metrics on $\mathbb{R}^3$. We can compute the connection components and with more effort the curvature tensor components. The result will be nonzero. How do we interpret these metrics?

Well, the most striking thing about metric geometries are its so called geodesics. These maybe defined in two different ways. They are 1) curves whose tangent is covariant constant along the curve (direction of curve not changed as you move allong the curve: "autoparallel") 2) curves which at least locally minimize the distance between two points in the space.

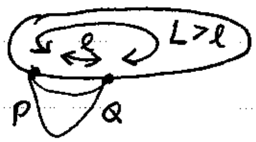
In Euclidean space these are just the straight lines. The tangent to a straight line is constant (constant cartesian components) and also minimizes the distance between any two points in the entire space.

On the 2-dimensional cylinder, which is just a piece of the flat plane rolled up into a cylinder, the straight lines on the plane roll into the geodesics on the cylinder. Those with constant z are actually closed curves, just circles. The others are helical curves except for those with constant φ which are still straight lines.

Notice that the tangents of all but these latter curves are no longer constant but rotate. However, if one computes the covariant derivative of the tangent with the connection coming from the metric on the cylinder, one finds that it is covariant constant with respect to this metric. (We'll get to this later) On the other hand

~~Example Take the curve $\phi = t$, $z = m\rho_0\phi$~~
~~Compute the tangent $X(t)$~~
~~Evaluate $\nabla_{X(t)}$~~

for a closed geodesic (z constant), there are two connecting geodesic curves between any two points on the circle (connect them in the shorter direction or in the longer direction), and one will be longer than many other nongeodesic connecting curves in the short direction.



So "auto parallel" is the characterization of geodesics.

On a sphere, the great circles are geodesics, all of which are closed.

So we have to develop the machinery to understand geodesics and the curvature and "parallel translation" which defines the autoparallel condition on geodesics.

SCHUTZ:

Read Chapter 6, Sections 1–13

Dont worry about mention of Lie dragging and the Lie derivative in 6.10, 6.11)

TOPIC: Covariant derivative, parallel transport, curvature tensor

Read Chapter 4, Sections 1–16

Differential Forms and the exterior derivative (we've done everything but the exterior derivative).

① Problem.

Evaluate the geodesic equations for the 2-sphere of radius r_0 using the $\{\theta, \varphi\}$ coordinatization given in the notes. (See Schutz 6.6).

Verify that the parametrized curves

$$\varphi = \varphi_0, \quad \theta = \frac{s}{r_0} \qquad s: 0 \to 2\pi$$

which are lines of ~~latitude~~ longitude (great circles thru north pole) ~~satisfy~~ satisfy the geodesic equations and are therefore geodesics.

Note that the tangent vector has unit length.

What does this tell us about the "affine parameter" s? (multivariable calculus).

② If motivated, calculate the single independent component of the curvature tensor $R^1{}_{212} = R^\theta{}_{\varphi\theta\varphi}$ for this case.

What is $R^{\hat\theta}{}_{\hat\varphi\hat\theta\hat\varphi}$? (Recall $e_{\hat\theta} = \frac{e_\theta}{(g_{\theta\theta})^{1/2}}$, $e_{\hat\varphi} = \frac{e_\varphi}{(g_{\varphi\varphi})^{1/2}}$)