(9) Coordinate Transformations

(10) Examples of computing components of metric & connection

Coordinate Transformations

From our discussion of the tangent space in the familiar case of $\mathbb{R}^3$, the following picture has been reached. On $\mathbb{R}^n$, with standard basis $\{e_\alpha\}$ and dual basis $\{\omega^\alpha\}$, one has the standard cartesian coordinate functions $\{x^\alpha\}$, which are nothing more than the dual basis 1-forms designated by a notation which doesn't require an underlying vector space but only a topological space, ie., a space on which open sets are defined.

The tangent space $T\mathbb{R}^n_{P_0}$ to $\mathbb{R}^n$ at a point P_0 is an n-dimensional vector space whose basis may be taken to be the standard cartesian coordinate partial derivatives at P_0

$$e_\alpha|_{P_0} = \partial/\partial x^\alpha|_{P_0} .$$

A tangent vector $X = X^\alpha e_\alpha|_{P_0}$ at P_0 is just a linear first order differential operator at P_0 whose value on a function is

$$Xf = X^\alpha e_\alpha|_{P_0} f = X^\alpha \partial f/\partial x^\alpha|_{P_0}$$
$$= df|_{P_0}(X),$$

which is defined to be the value of the differential df of the function at P_0 on the tangent vector X

$$df|_{P_0} = \partial f/\partial x^\alpha|_{P_0} dx^\alpha|_{P_0} ,$$

having defined the differentials at P_0 of the standard cartesian coordinates to be the basis dual to $\{e_\alpha|_{P_0}\}$

$$dx^\alpha|_{P_0}(e_\beta|_{P_0}) = \delta^\alpha{}_\beta .$$

The "coordinate components" of a tangent vector X at P_0 are obtained by allowing the tangent vector to differentiate the coordinate functions.

$$\mathbb{X}^\alpha = dx^\alpha|_{P_0}(\mathbb{X}) = \mathbb{X}x^\alpha.$$

Tangent vectors arise as the tangents to parametrized curves. Suppose $c(t)$ (an $\mathbb{R}^n$-valued function c on the real line with "variable" t) is a parametrized curve passing through the point P_0 at $t=0$, i.e., $c(0) = P_0$. Let $c'(t)$ be the tangent vector at $c(t)$, now thought of as a first order ~~partia~~ differential operator on functions at $c(t)$. If we evaluate the standard coordinate functions along the curve $c^\alpha(t) \equiv x^\alpha(c(t)) = (x^\alpha \circ c)(t)$, usually denoted by the sloppy notation $x^\alpha(t)$, one obtains the coordinate components of the tangent vector by differentiating these functions with respect to the parameter

$$c'^\alpha(t) = (c^\alpha(t))' = \frac{d}{dt}(c^\alpha(t)),$$

~~which by the chain rule is~~

$$\cancel{c'^\alpha(t) = \partial x^\alpha}$$

so

$$c'(t) = c'^\alpha(t) \left.\frac{\partial}{\partial x^\alpha}\right|_{c(t)} = \frac{dc^\alpha(t)}{dt} \left.\frac{\partial}{\partial x^\alpha}\right|_{c(t)}.$$

The derivative of a function by this tangent vector is

$$c'(t) f = \frac{dc^\alpha(t)}{dt} \left.\frac{\partial f}{\partial x^\alpha}\right|_{c(t)} = df(c'(t)).$$

One can think of the tangent space at a point as the space of tangent vectors to all ~~curves thro~~ (differentiable) parametrized curves through the point. The dual of the tangent space can be thought of as the differentials at the point of all (~~differentiable at the point~~) functions which are differentiable at the point.

$\underline{EX}$.
$$c(t) = (1-t,\ 2t^2+1) \in \mathbb{R}^2$$
$$c^1(t) = 1-t, \quad c^2(t) = 2t^2+1$$
$$c'^1(t) = -1, \quad c'^2(t) = 4t$$
$$c'(t) = -\frac{\partial}{\partial x^1}\Big|_{c(t)} + 4t\frac{\partial}{\partial x^2}\Big|_{c(t)}$$
$$c'(t)f = -\frac{\partial f}{\partial x^1}\Big|_{c(t)} + 4t\frac{\partial f}{\partial x^2}\Big|_{c(t)}.$$

All we needed to discuss all of this structure was the set of standard cartesian coordinate functions $\{x^\alpha\}$. on $\mathbb{R}^n$. The discussion can be repeated for other sets of coordinate functions on $\mathbb{R}^n$, leading to the topic of coordinate transformations on $\mathbb{R}^n$. These may be global coordinates like the cartesian coordinates or local coordinates which do not cover all of $\mathbb{R}^n$ but have associated coordinate singularities or edges.

Take any ordered set of n functions $\{x^{\alpha'}\}$ which obey the condition

$$\det\left(\frac{\partial x^{\alpha'}}{\partial x^\beta}\right) \neq 0 \qquad \text{(invertibility of Jacobian matrix } \left(\frac{\partial x^{\alpha'}}{\partial x^\beta}\right)\text{)}$$

in some open set $U \subset \mathbb{R}^n$ (the functions are said to be functionally independent on U). These functions can be taken as new coordinate functions on U.

$\underline{EX}$. Cylindrical coordinates on $\mathbb{R}^3$.

Define $\{x^{1'}, x^{2'}, x^{3'}\} = \{\rho, \varphi, z\}$ where

$$\rho = [(x^1)^2 + (x^2)^2]^{1/2}, \quad z = x^3$$

$$\varphi = \begin{cases} \text{1st quad:} & \tan^{-1} y/x \\ \text{2nd quad:} & \tan^{-1} y/x + \pi \\ \text{3rd quad:} & \tan^{-1} y/x + \pi \\ \text{4th quad:} & \tan^{-1} y/x + 2\pi \end{cases}; \quad \text{with } \varphi = \tfrac{\pi}{2} \text{ on positive } y\text{-axis and } \varphi = -\tfrac{\pi}{2} \text{ on negative } y\text{-axis}$$

$$U = \mathbb{R}^3 - \{(x^1, x^2, x^3) \mid x^2 = 0,\ x^1 \geq 0\} = \mathbb{R}^3 - \text{halfplane}$$

These define standard cylindrical coordinates

$$\rho \in [0,\infty) \;,\; \varphi \in [0,2\pi) \;,\; z \in (-\infty,\infty)$$

with a coordinate singularity at the ~~xz-plane for positive x~~ coordinate plane $x^2 = 0$ for ~~positive~~ nonnegative values of x^1.

Inverse transformation:
$x^1 = \rho\cos\varphi$
$x^2 = \rho\sin\varphi$
$x^3 = z$

EX. Spherical coordinates on $\mathbb{R}^3$

Define $\{x^{1'}, x^{2'}, x^{3'}\} = \{r, \theta, \varphi\}$ where

$$r = [(x^1)^2 + (x^2)^2 + (x^3)^2]^{1/2}$$

$\varphi =$ same as above

$$\theta = \begin{cases} \tan^{-1} x^3/r & , x^3 > 0 \\ \frac{\pi}{2} & , x^3 = 0 \\ \tan^{-1} x^3/r + \pi & , x^3 < 0 \end{cases}$$

Inverse transformation:
$x^1 = r\sin\theta\cos\varphi$
$x^2 = r\sin\theta\sin\varphi$
$x^3 = r\cos\theta$

~~$U = \mathbb{R}^3$~~ – U = same as above.

These define standard spherical coordinates

$$r \in (0,\infty) \;,\; \theta \in (0,\pi) \;,\; \varphi \in (0,2\pi)$$

with a coordinate singularity at the same place as cylindrical coordinates.

The chain rule gives the transformation induced on the bases of the ~~dual space~~ tangent space and its duals by the coordinate transformation

$$dx^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\beta}\, dx^\beta \equiv A^\alpha{}_\beta\, dx^\beta$$

$$\frac{\partial}{\partial x^\alpha} = \frac{\partial x^{\beta'}}{\partial x^\alpha}\frac{\partial}{\partial x^{\beta'}} \equiv A^\beta{}_\alpha \frac{\partial}{\partial x^{\beta'}}.$$

The matrix $\underline{A} = (A^\alpha{}_\beta)$ at each point is invertible by assumption

$$\frac{\partial}{\partial x^{\alpha'}} = \underbrace{A^{-1\,\beta}{}_{\alpha}}_{\frac{\partial x^\beta}{\partial x^{\alpha'}}}\frac{\partial}{\partial x^\beta}$$

and defines a change of basis on each tangent space from the standard cartesian coordinate derivative basis to the new basis. The components of tangent vectors and of the tangent covectors or 1-forms in the dual tangentspace or "cotangent space" (elements also called "cotangent vectors). then transform accordingly.

$$X = X^\alpha \frac{\partial}{\partial x^\alpha}\Big|_{P_0} = X^{\alpha'} \frac{\partial}{\partial x^{\alpha'}}\Big|_{P_0}$$

$$X^{\alpha'} = dx^{\alpha'}\big|_{P_0}(X) = \frac{\partial x^{\alpha'}}{\partial x^\beta}\Big|_{P_0} X^\beta \quad ;$$

$$\sigma = \sigma_\alpha \, dx^\alpha\big|_{P_0}$$

$$\sigma_{\alpha'} = \sigma\Big(\frac{\partial}{\partial x^{\alpha'}}\Big|_{P_0}\Big) = \sigma_\beta \frac{\partial x^\beta}{\partial x'^{\alpha}}\Big|_{P_0}$$

and all components in the tensor algebra over the tangent space transform similarly. For example, the Euclidean metric on $\mathbb{R}^n$

$$g = g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta \quad , \quad g_{\alpha\beta} = \delta_{\alpha\beta}$$

can be re-expressed as

$$g = g_{\alpha'\beta'}\, dx^{\alpha'} \otimes dx^{\beta'}$$

$$g_{\alpha'\beta'} = \frac{\partial x^\gamma}{\partial x^{\alpha'}} \frac{\partial x^\delta}{\partial x^{\beta'}} g_{\gamma\delta}$$

at each point of $\mathbb{R}^n$. It is a $\binom{0}{2}$-tensor field on $\mathbb{R}^n$.

Any smooth choice of $\binom{p}{q}$-tensor ~~form~~ over the tangent space at each point of $\mathbb{R}^n$, leads to a $\binom{p}{q}$-tensor field on $\mathbb{R}^n$.

For example, if X^α are n functions on $\mathbb{R}^n$ rather than constants then $X = X^\alpha \frac{\partial}{\partial x^\alpha}$ defines a vector field on $\mathbb{R}^n$, while $\sigma = \sigma_\alpha dx^\alpha$ defines a ~~differ~~ 1-form field or differential 1-form on $\mathbb{R}^n$.

Thus in classical discussions if we have a $\binom{p}{q}$-tensor

$$T = T^{\alpha\cdots}{}_{\beta\cdots} \frac{\partial}{\partial x^\alpha} \otimes \cdots \otimes dx^\beta \otimes \cdots \Big|_{P_0}$$

at P_0 then its components in the new coordinate basis are

$$T'^{\alpha\cdots}{}_{\beta\cdots} = \frac{\partial x^{\alpha'}}{\partial x^{\gamma}} \cdots \frac{\partial x^{\delta}}{\partial x'^{\beta}} \cdots T^{\gamma\cdots}{}_{\delta\cdots} \Big|_{P_0},$$

which is the "tensor transformation law".

Ex. In classical discussions one refers instead to the "line element" of the metric

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta.$$

One can easily compute on $\mathbb{R}^3$

$$ds^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2 \qquad \text{cylindrical}$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 \qquad \text{spherical}.$$

Check this.

The mental image or even explicit image for diagrams that we have for a tangent vector in $\mathbb{R}^n$ is still the old directed ~~arro~~ line segment arrow of its original definition. Thus we picture the basis $\{e_\alpha|_{P_0}\} = \{\partial/\partial x^\alpha|_{P_0}\}$ as unit vectors along the coordinate lines at P_0, with initial points at P_0. The coordinate transformation leads to a new coordinate basis at P_0 which we can still picture in this way. We picture $\partial/\partial x^{\alpha'}|_{P_0}$ as a vector tangent to the coordinate line of $x^{\alpha'}$ at P_0 in the direction of increasing values of that coordinate with a length ~~determine~~ $\cancel{g(\partial/\partial x^{\alpha'}, \partial/\partial x^{\alpha'})}$ $(g_{\alpha'\alpha'})^{1/2}$ evaluated at P_0. In terms of the new coordinates it is a tangent to the parametrized curve

$$C^{\alpha'}(t) = x_0^{\alpha'} + t, \quad C^{\beta'}(t) = x_0^{\beta'}, \quad \beta \neq \alpha.$$

There is no need to consider only coordinate bases of the tangent spaces. One can introduce a general "frame" by choosing any n ~~linear~~ vector fields which are linearly independent at each point of some open set $\mathcal{U}$, Such a "local frame", if U is not all of $\mathbb{R}^n$, is equivalent to specifying a nonsingular matrix-valued function on $\mathcal{U}$:

$$e_\alpha = e^\beta{}_\alpha \frac{\partial}{\partial x^\beta} = A^{-1\,\beta}{}_\alpha \frac{\partial}{\partial x^\beta}$$

$$\omega^\alpha = \omega^\alpha{}_\beta \, dx^\beta = A^\alpha{}_\beta \, dx^\beta .$$

A coordinate transformation induces a change of "coordinate frame" but partial derivatives still commute

$$\cancel{0 = [\partial_{\alpha'}, \partial_{\beta'}] = \left[\frac{\partial x^\gamma}{\partial x^{\alpha'}} \frac{\partial}{\partial x^\gamma}\right.},$$

$$e_\alpha = \frac{\partial}{\partial x^{\alpha'}} = \frac{\partial x^\beta}{\partial x^{\alpha'}} \frac{\partial}{\partial x^\beta} = e^\beta{}_\alpha \frac{\partial}{\partial x^\beta}$$

$$\begin{aligned}
0 &= \left[\frac{\partial}{\partial x^{\alpha'}}, \frac{\partial}{\partial x^{\beta'}}\right] = [e_\alpha, e_\beta] \\
&= \left[e^\gamma{}_\alpha \frac{\partial}{\partial x^\gamma}, e^\delta{}_\beta \frac{\partial}{\partial x^\delta}\right] \\
&= e^\gamma{}_\alpha \frac{\partial e^\delta{}_\beta}{\partial x^\gamma} \frac{\partial}{\partial x^\delta} - e^\delta{}_\beta \frac{\partial e^\gamma{}_\alpha}{\partial x^\delta} \frac{\partial}{\partial x^\gamma} \\
&= \left(e^\gamma{}_\alpha \frac{\partial e^\delta{}_\beta}{\partial x^\gamma} - e^\gamma{}_\beta \frac{\partial e^\delta{}_\alpha}{\partial x^\gamma}\right) \frac{\partial}{\partial x^\delta}
\end{aligned}$$

This defines the Lie bracket of the two vector fields e_α and e_β, another vector field, which must vanish for coordinate frame vector fields.

An orthogonal coordinate system on $\mathbb{R}^n$ is one for which the components of the Euclidean metric tensor are diagonal, i.e., the coordinate frame is orthogonal.

$$g'_{\alpha\beta} = 0 \,, \quad \alpha \neq \beta .$$

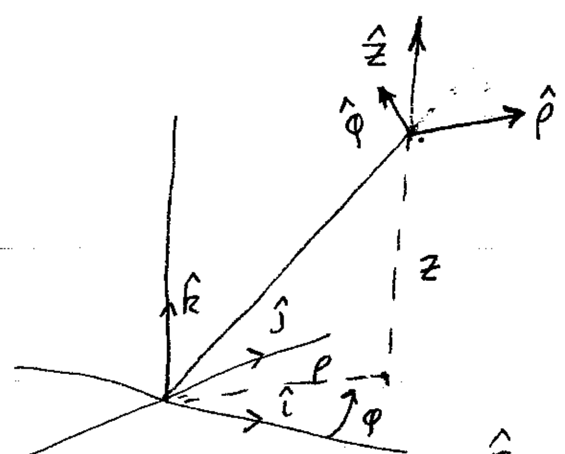
In this case it makes sense to introduce an orthonormal frame by normalizing the orthogonal coordinate frame vector fields

$$e_\alpha = (g'_{\alpha\alpha})^{-1/2} \frac{\partial}{\partial x^{\alpha'}} = \hat{x}^{\alpha'}$$

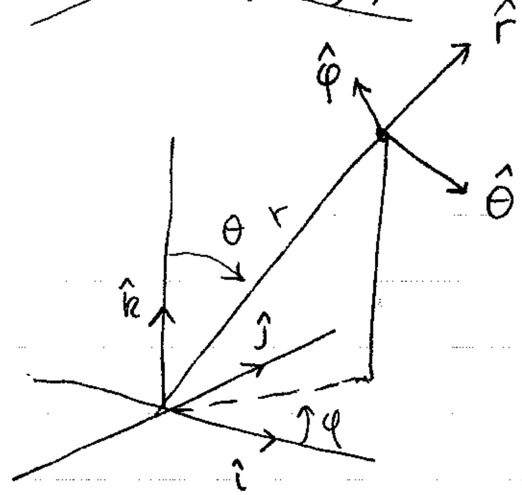
$$g = \delta_{\alpha\beta}\, \omega^\alpha \otimes \omega^\beta \,, \quad \delta_{\alpha\beta} = g(e_\alpha, e_\beta) .$$

Cylindrical and spherical coordinates on $\mathbb{R}^3$ are such orthogonal coordinates and we often picture the associated orthonormal frame

cylindrical



spherical



Such orthonormal frames are related to the original orthonormal cartesian coordinate frame by a position dependent rotation.

> EX. Evaluate this rotation matrix for cylindrical and spherical coordinates on $\mathbb{R}^3$. The columns of this matrix represent the tangent vectors $\{\hat{X}^1, \hat{X}^2, \hat{X}^3\}$ as difference vectors.

The most obvious question to ask now that we can use a position dependent basis for the tangent spaces compared to our "constant frame" associated with the standard cartesian coordinates on $\mathbb{R}^n$, is the following: how do we recognize a "constant vector field" if we work in a nonconstant frame? A constant vector field is clearly just a vector field whose cartesian coordinate frame components are constants. Next how can we define a gradient operator for vector fields which takes into account the variability of the frame vectors?

The answer is to simply define certain quantities in other coordinates by transforming from the cartesian coordinate frame where they are clear. So define the $\binom{1}{1}$-tensor field ∇X by the following expression in cartesian coordinates

$$\nabla X = \partial X^\alpha / \partial x^\beta \quad \partial/\partial x^\alpha \otimes dx^\beta ,$$

called the covariant derivative of the vector field $X = X^\alpha \partial/\partial x^\alpha$, and define its components in any other frame by the change of frame linear transformation. Note that a "constant vector field" has identically zero covariant derivative, and is said to be "covariantly constant."

Conventions

In component notation, the covariant derivative of a vector field adds a covariant index to the vector field symbol which is conventionally separated by a semicolon:

$$(\nabla Y)^\alpha{}_\beta \equiv Y^\alpha{}_{;\beta} = Y^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\beta\gamma} Y^\gamma$$

$$(\nabla_X Y)^\alpha = Y^\alpha{}_{;\beta} X^\beta = Y^\alpha{}_{,\beta} X^\beta + \Gamma^\alpha{}_{\beta\gamma} X^\beta Y^\gamma$$

while the ordinary derivative is denoted by a comma separator.

so transforming from cartesian coordinates

$$(\nabla X)^{\alpha'}{}_{\beta'} = \frac{\partial x^{\alpha'}}{\partial x^{\gamma}} \underbrace{\frac{\partial x^{\delta}}{\partial x^{\beta'}} \frac{\partial X^{\gamma}}{\partial x^{\delta}}}_{\frac{\partial X^{\gamma}}{\partial x^{\beta'}}\ \text{chain rule}} \quad \leftarrow (\nabla X)^{\gamma}{}_{\delta}$$

$$= \frac{\partial}{\partial x^{\beta'}} \Big(\underbrace{\frac{\partial x^{\alpha'}}{\partial x^{\gamma}} X^{\gamma}}_{X^{\alpha'}} \Big) - \frac{\partial^2 x^{\alpha'}}{\partial x^{\beta'} \partial x^{\gamma}} \underbrace{X^{\gamma}}_{\frac{\partial x^{\gamma}}{\partial x^{\prime\delta}} X^{\prime\delta}} \qquad \begin{pmatrix}\text{integration}\\ \text{by parts}\end{pmatrix}$$

$\leftarrow$ inverse transformation

$$= \partial X^{\alpha'} / \partial x^{\beta'} + \Gamma^{\alpha'}{}_{\beta'\delta'} X^{\prime\delta}$$

where

$$\Gamma^{\alpha'}{}_{\beta'\delta'} = - \frac{\partial^2 x^{\alpha'}}{\partial x^{\beta'} \partial x^{\gamma}} \frac{\partial x^{\gamma}}{\partial x^{\prime\delta}}$$

$$= \underbrace{- \frac{\partial}{\partial x^{\beta'}} \Big(\underbrace{\frac{\partial x^{\alpha'}}{\partial x^{\gamma}} \frac{\partial x^{\gamma}}{\partial x^{\prime\delta}}}_{\delta^{\alpha}{}_{\delta}} \Big)}_{0} + \frac{\partial x^{\alpha'}}{\partial x^{\gamma}} \frac{\partial^2 x^{\gamma}}{\partial x^{\beta'} \partial x^{\prime\delta}} \qquad \begin{pmatrix}\text{integration}\\ \text{by parts}\end{pmatrix}$$

has a simple interpretation. If $e_{\alpha'} = \frac{\partial}{\partial x^{\alpha'}}$ denotes the new coordinate frame vector field, then its covariant derivative

$$\nabla e_{\beta'} = \underbrace{\Big[\frac{\partial (\delta^{\gamma'}{}_{\beta'})}{\partial x^{\alpha'}} + \Gamma^{\gamma'}{}_{\alpha'\delta'} (\delta^{\delta'}{}_{\beta'}) \Big]}_{\Gamma^{\gamma'}{}_{\alpha'\beta'}} \frac{\partial}{\partial x^{\gamma'}} \otimes dx^{\alpha'}$$

$$\Gamma^{\gamma'}{}_{\alpha'\beta'} \frac{\partial}{\partial x^{\gamma'}} \otimes dx^{\alpha'}$$

These quantities define the covariant derivatives of the frame vectors themselves, unnecessary in the original cartesian coordinate discussion.

Why do we have a $\binom{1}{1}$-tensor for the covariant derivative of a vector field instead of another vector field? For the same reason we only have a ~~differential~~ $\binom{0}{1}$-tensor the differential df of a function until we have a vector field to evaluate it on and obtain the derivative of the ~~vector field~~ function along the vector field

$$df(X) = Xf = X^\alpha \frac{\partial f}{\partial x^\alpha}.$$

When we evaluate the additional vector argument of ∇Y on a vector field X

$$(\nabla Y)(\,, X) = X^{\beta'} \nabla_{\beta'} Y^{\alpha'} \frac{\partial}{\partial x^{\alpha'}}$$

$$= (X Y^{\alpha'} + \Gamma^{\alpha'}{}_{\beta'\gamma'} X^{\gamma'}) \frac{\partial}{\partial x^{\alpha'}}$$

$$\equiv \nabla_X Y$$

we get the covariant derivative of Y along X. In the original cartesian coordinate frame, this is just the derivative along X of the coordinate components.

Thus $\nabla_{e_{\alpha'}} e_{\beta'} = \Gamma^{\gamma'}{}_{\alpha'\beta'} \, e_{\gamma'}$.

~~If we extend the covariant derivative oper~~

The covariant derivative operator defined for vector fields by transforming the ordinary derivative from the cartesian coordinate frame can easily be shown to obey a scalar product product rule

$$\nabla_X (fY) = f \nabla_X Y + (Xf)\, Y,$$

and is linear in both X and Y over the real numbers.

In fact given an arbitrary frame $\{e_\alpha\}$, defining the "components of the connection"

$$\nabla_{e_\alpha} e_\beta = \Gamma^\gamma_{\alpha\beta} e_\gamma$$

one can then use the product rule and linearity to evaluate the covariant derivative in terms of them

$$\nabla_X Y = \nabla_{X^\alpha e_\alpha}(Y^\gamma e_\gamma) = X^\alpha\left(\frac{\partial}{\partial x^\alpha} Y^\gamma + \Gamma^\gamma{}_{\alpha\delta} Y^\delta\right) e_\gamma .$$

One can define a $\binom{1}{2}$-tensor $\Gamma^\alpha{}_{\beta\gamma}\, e_\alpha \otimes \omega^\beta \otimes \omega^\gamma$ in each frame but we still don't know the relationship of such tensors defined in different frames. In the original cartesian coordinate frame, the tensor is clearly identically zero. The transformation between these <u>different</u> tensors is easily evaluated using the product rule and linearity.

$$\Gamma^{\gamma'}{}_{\alpha'\beta'} = \underbrace{\omega^{\gamma'}}_{A^{\gamma'}{}_\sigma \omega^\sigma}\big(\nabla_{\underbrace{e_{\alpha'}}_{A^{-1\mu}{}_{\alpha'} e_\mu}} \underbrace{e_{\beta'}}_{A^{-1\nu}{}_{\beta'} e_\nu}\big)$$

$$= A^{\gamma'}{}_\sigma A^{-1\mu}{}_{\alpha'}\, \omega^\sigma\big(\underbrace{\nabla_{e_\mu}(A^{-1\nu}{}_{\beta'} e_\nu)}_{(e_\mu A^{-1\nu}{}_{\beta'}) e_\nu + A^{-1\nu}{}_{\beta'} \Gamma^\epsilon{}_{\mu\nu} e_\epsilon}\big)$$

$$= A^{\gamma'}{}_\sigma A^{-1\mu}{}_{\alpha'}\left(\Gamma^\sigma{}_{\mu\nu} A^{-1\nu}{}_{\beta'} + e_\mu A^{-1\sigma}{}_{\beta'}\right)$$

The first term corresponds to the transformation of a $\binom{1}{2}$-tensor but the second ~~bem~~ inhomogeneous term involves the derivatives of the linear frame transformation, leading to different tensors in different frames. The tensor is identically zero in all frames related to the original cartesian coordinate transformation by a ~~lea~~ constant linear transformation, but is nonzero for those involving non constant frame vectors.

[If the original frame is the cartesian coordinate frame, then $\Gamma^\sigma{}_{\mu\nu} = 0$ and this gives $\Gamma^{\gamma'}{}_{\alpha'\beta'} = A^{\gamma'}{}_\sigma A^{-1\mu}{}_{\alpha'} e_\mu A^{-1\sigma}{}_{\beta'}$. For a coordinate frame $e_{\alpha'} = \partial/\partial x^{\alpha'}$ this reduces to the original expression. $= A^{\gamma'}{}_\sigma e_{\alpha'} A^{-1\sigma}{}_{\beta'}$]

If the original frame $\{e_\alpha\}$ is the cartesian coordinate frame, then $\Gamma^\sigma{}_{\mu\nu} = \omega^\sigma(\nabla_{e_\mu} e_\nu) = 0$ and this reduces to

$$\Gamma^{\gamma'}{}_{\alpha'\beta'} = A^{\bullet\gamma'}{}_\sigma \underbrace{A^{-1\mu}{}_{\alpha'}\, e_\mu\, A^{-1\sigma}{}_{\beta'}} = A^{\bullet\gamma'}{}_\sigma\, e_{\alpha'} A^{-1\sigma}{}_{\beta'}$$

If $\{e_{\alpha'}\}$ is a coordinate frame, this reduces to the above expression when $A^{\alpha}{}_{\beta'} = \dfrac{\partial x^\alpha}{\partial x^{\beta'}}$, $A^{-1\beta'}{}_\alpha = \dfrac{\partial x^{\beta'}}{\partial x^\alpha}$.

One can also rewrite this expression by "integrating by parts"

$$\Gamma^{\gamma'}{}_{\alpha'\beta'} = \underbrace{e_{\alpha'}\big(\underbrace{A^{\bullet\gamma'}{}_\sigma A^{-1\sigma}{}_{\beta'}}_{\delta^{\gamma'}{}_{\beta'}}\big)}_{0} - \left(e_{\alpha'} A^{\gamma'}{}_\sigma\right) A^{-1\sigma}{}_{\beta'}$$

$$= -\left(e_{\alpha'} A^{\gamma'}{}_\sigma\right) A^{-1\sigma}{}_{\beta'}.$$

The quantities $\Gamma^{\alpha}{}_{\beta\gamma} = \omega^{\alpha}(\nabla_{e_\beta} e_\gamma)$ are called the components of the connection with respect to the ~~basis~~ frame $\{e_\alpha\}$. They are most naturally thought of as a matrix-valued 1-form

$$\omega = (\Gamma^{\alpha}{}_{\beta\gamma}\omega^{\beta}) \equiv (\omega^{\alpha}{}_{\beta}) \qquad \text{" connection 1-forms "}$$

whose value on a vector field is just the linear transformation of the tangent space which describes the covariant derivatives of the frame vectors $\omega^{\alpha}{}_{\beta}(X) = \Gamma^{\alpha}{}_{\gamma\beta} X^{\gamma}$.

The above transformation law for the connection components defines a "geometric object field", which is a way of specifying a tensor field for each choice of frame such that the components of these different tensor fields are related by a transformation determined entirely by the change of frame matrix and its derivatives.

Notice that in a coordinate frame $e_{\alpha'}$, then

$$\Gamma^{\alpha'}{}_{\beta'\delta'} = \frac{\partial X^{\alpha'}}{\partial X^{\gamma}} \frac{\partial^2 X^{\gamma}}{\partial X^{\beta'} \partial X'^{\delta}} = \Gamma^{\alpha'}{}_{\delta'\beta'}$$

holds by the symmetry of mixed partial derivatives. Such a connection is called "symmetric". In a noncoordinate frame, the previous page shows that this symmetry is broken since frame derivatives dont commute but define the Lie brackets of the frame vector fields.

So we know how to differentiate a vector field in a "covariant" manner, which simply represents the ordinary derivative of the cartesian coordinate components transformed to ~~the~~ an arbitrary frame. We can extend the covariant derivative to arbitrary rank tensor fields in two ways:

1) By transforming from cartesian components

$$\nabla_\gamma T^{\alpha\cdots}{}_{\beta\cdots} \equiv \partial T^{\alpha\cdots}{}_{\beta\cdots}/\partial x^\gamma \quad \cancel{\equiv (\nabla T)^{\alpha\cdots}{}_{\beta\cdots;\gamma}}$$

$$\equiv T^{\alpha\cdots}{}_{\beta\cdots;\gamma}$$

$$\equiv T^{\alpha\cdots}{}_{\beta\cdots,\gamma}$$

where we use the semicolon and comma notation (these are equivalent for the cartesian frame, so in a new frame $e_{\alpha'} = A^{-1\beta}{}_{\alpha'} e_\beta$, $\omega^{\alpha'} = A^{\alpha'}{}_\beta \omega^\beta$

$$\nabla_{\gamma'} T^{\alpha'\cdots}{}_{\beta'\cdots} = T^{\alpha'\cdots}{}_{\beta'\cdots;\gamma'}$$

$$\equiv A^{\alpha'}{}_\sigma \cdots A^{-1\mu}{}_{\beta'} \cdots A^{-1\nu}{}_{\gamma'}\, T^{\sigma\cdots}{}_{\mu\cdots;\nu}$$

$$\equiv A^{\alpha'}{}_\sigma \cdots A^{-1\mu}{}_{\beta'} \cdots A^{-1\nu}{}_{\gamma'}\, T^{\sigma\cdots}{}_{\mu\cdots,\nu}$$

and then "integrating by parts"

$$= (A^{\alpha'}{}_\sigma \cdots A^{-1\mu}{}_{\beta'} \cdots T^{\sigma\cdots}{}_{\mu\cdots})_{,\nu}\, A^{-1\nu}{}_{\gamma'}$$

$$- (A^{\alpha'}{}_\sigma \cdots A^{-1\mu}{}_{\beta'} \cdots)_{,\nu}\, T^{\sigma\cdots}{}_{\mu\cdots}\, A^{-1\nu}{}_{\gamma'}$$

$$= T^{\alpha'\cdots}{}_{\beta'\cdots,\gamma'} - \underbrace{(A^{\alpha'}{}_\sigma \cdots A^{-1\mu}{}_{\beta'}\cdots)_{,\gamma'}\, \underbrace{T^{\sigma\cdots}{}_{\mu\cdots}}_{A^{-1\sigma}{}_{\epsilon'}\cdots A^{\tau'}{}_\mu T^{\epsilon'\cdots}{}_{\tau'\cdots}}}_{\text{expand by product rule}}$$

$$\cancel{[\nabla(e_{\gamma'} A^{\alpha'}{}_\sigma) A^{-1\sigma}{}_{\epsilon'} + \cdots + (A^{\tau'}{}_\mu A^{-1\mu}{}_{\beta'})_{e_{\gamma'}} + \cdots]}$$

$$= T^{\alpha'\cdots}{}_{\beta'\cdots,\gamma'} - \Big[\overbrace{(e_{\gamma'} A^{\alpha'}{}_{\sigma}) A^{-1\sigma}{}_{\epsilon'}}^{-\Gamma^{\alpha'}{}_{\gamma'\epsilon'}} \delta^{\tau'}{}_{\beta'} + \cdots$$

$$+ \underbrace{A^{\tau'}{}_{\mu} (e_{\gamma'} A^{-1\mu}{}_{\beta'})}_{\Gamma^{\tau'}{}_{\gamma'\beta'}} \delta^{\alpha'}{}_{\epsilon'} \Big] T^{\epsilon'\cdots}{}_{\tau'\cdots}$$

~~But~~

$$\cancel{A^{\alpha'}{}_{\sigma} A^{-1\sigma}{}_{\epsilon'} = \delta^{\alpha'}{}_{\epsilon'}}$$

~~so differentiating this:~~

$$\cancel{(e_{\gamma'} A^{\alpha'}{}_{\sigma}) A^{-1\sigma}{}_{\epsilon'} + \underbrace{A^{\alpha'}{}_{\sigma} (e_{\gamma'} A^{-1\sigma}{}_{\epsilon'})}_{\Gamma^{\alpha'}{}_{\gamma'\epsilon'}} = 0}$$

~~so this coefficient is~~ $\cancel{-\Gamma^{\alpha'}{}_{\gamma'\epsilon'}}$ ~~and~~

$$= T^{\alpha'\cdots}{}_{\beta'\cdots,\gamma'} + \Gamma^{\alpha'}{}_{\gamma'\delta'} T^{\delta'\cdots}{}_{\beta'\cdots} + \cdots$$

$$- \Gamma^{\delta'}{}_{\gamma'\beta'} T^{\alpha'\cdots}{}_{\delta'\cdots}$$

Thus the covariant derivative has a correction term like the one for a vector field for each contravariant index (just the linear transformation applied to that index corresponding to the matrix of the matrix valued connection 1-form representing the covariant derivatives of the frame vector fields) and the opposite sign and transpose matrix correction terms for each covariant index (corresponding to the matrix representing the covariant derivatives of the frame dual 1-forms).

2) By recognizing that in the cartesian coordinate frame the covariant derivative (= ordinary derivative) obviously satisfies a product rule for tensor products of tensors and products of functions with tensors and is linear, and in any frame the covariant derivative of a function is just the ordinary derivative so for example

$$\nabla_X T = \nabla_X \left(T^{\alpha'\cdots}{}_{\beta'\cdots}\, e_{\alpha'} \otimes \cdots \otimes \omega^{\beta'} \otimes \cdots \right)$$

$\omega^{\gamma'}{}_{\alpha'}(X)\, e_{\gamma'}$ by definition

$$= \underbrace{\nabla_X T^{\alpha'\cdots}{}_{\beta'\cdots}}_{\substack{\text{ordinary derivative}\\ \text{of functions:}}}\, e_{\alpha'} \otimes \cdots \otimes \omega^{\beta'} \otimes \cdots + T^{\alpha'\cdots}{}_{\beta'}\, \overline{\nabla_X e_{\alpha'}} \otimes \cdots \otimes \omega^{\beta} \otimes \cdots + \cdots$$
$$+ T^{\alpha'\cdots}{}_{\beta'}\, e_{\alpha'} \otimes \cdots \otimes \underbrace{\nabla_X \omega^{\beta}}_{-\omega^{\beta'}{}_{\gamma'}(X)\,\omega^{\gamma'}} \otimes + \cdots$$

$X\, T^{\alpha'\cdots}{}_{\beta'\cdots}$

$$\boxed{\begin{aligned} &\nabla_X [\omega^{\beta}(e_\gamma)] = \nabla_X(\delta^{\beta}{}_{\gamma}) = 0 \\ &\underbrace{(\nabla_X \omega^{\beta})(e_\gamma) + \omega^{\beta}\underbrace{(\nabla_X e_\gamma)}_{\omega^{\beta}{}_{\gamma}(X)}} \\ &\nabla_X \omega^{\beta'} = -\omega^{\beta'}{}_{\gamma'}(X)\, \omega^{\gamma'} \end{aligned}}$$

s.d.e. q.e.d.

$$= X^{\delta'} \left(T^{\alpha'\cdots}{}_{\beta'\cdots,\delta'} + \Gamma^{\alpha'}{}_{\delta'\mu'}\, T^{\mu'\cdots}{}_{\beta'\cdots} + \cdots - \Gamma^{\mu'}{}_{\delta'\beta'}\, T^{\alpha'\cdots}{}_{\mu'\cdots} - \cdots \right)$$

these terms come from the covariant derivatives of the frame vectors

these from the covariant derivatives of the dual 1-forms

Any tensor field with constant components in the cartesian coordinate frame is "covariant constant" $\nabla T = 0$ or $T^{\alpha\cdots}{}_{\beta\cdots;\gamma} = 0$

In particular, the Euclidean metric is covariant constant

$$g = \delta_{\alpha\beta}\, dx^\alpha \otimes dx^\beta \equiv g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta$$

$$\nabla g = 0 \qquad g_{\alpha\beta;\gamma} = 0$$

Writing this out:

$$0 = g_{\alpha'\beta';\gamma'} = g_{\alpha'\beta',\gamma'} - \underbrace{\Gamma^{\delta'}{}_{\gamma'\alpha'}\, g_{\delta'\beta'}}_{\equiv \Gamma_{\beta'\gamma'\alpha'}} - \underbrace{\Gamma^{\delta'}{}_{\gamma'\beta'}\, g_{\alpha'\delta'}}_{\equiv \Gamma_{\alpha'\gamma'\beta'}}$$

by index lowering conventions

$$\underbrace{\Gamma_{\beta'\gamma'\alpha'} + \Gamma_{\alpha'\gamma'\beta'}}_{\equiv 2\, \Gamma_{(\alpha'|\gamma'|\beta')}} = g_{\alpha'\beta',\gamma'}$$

exclude γ' from symmetrization

It turns out that one can solve this relationship for $\Gamma_{\alpha'\beta'\gamma'}$ as a function of $g_{\alpha'\beta',\gamma'}$ by a trick in a coordinate frame where the last two indices are symmetric

$$\begin{aligned}
\Gamma_{\alpha'\gamma'\beta'} + \Gamma_{\beta'\gamma'\alpha'} &= g_{\alpha'\beta',\gamma'} \\
-\Gamma_{\beta'\alpha'\gamma'} - \Gamma_{\gamma'\alpha'\beta'} &= -g_{\beta'\gamma',\alpha'} \\
\Gamma_{\gamma'\beta'\alpha'} + \Gamma_{\alpha'\beta'\gamma'} &= g_{\gamma'\alpha',\beta'}
\end{aligned}$$

$$2\,\Gamma_{\alpha'\beta'\gamma'} = g_{\alpha'\beta',\gamma'} - g_{\beta'\gamma',\alpha'} + g_{\gamma'\alpha',\beta'}$$

$$\Gamma_{\alpha'\beta'\gamma'} = \frac{1}{2}\left(g_{\alpha'\beta',\gamma'} - g_{\beta'\gamma',\alpha'} + g_{\gamma'\alpha',\beta'}\right)$$

$$\boxed{\Gamma^{\alpha'}{}_{\beta'\gamma'} = g^{\alpha'\delta'}\,\Gamma_{\delta'\beta'\gamma'} = \frac{1}{2} g^{\alpha'\delta'}\left(g_{\delta'\beta',\gamma'} - g_{\beta'\gamma',\delta'} + g_{\gamma'\delta',\beta'}\right)}$$

So the components of the connection in a coordinate frame are entirely determined by the components of the metric and its derivatives. This connection is called "the metric connection" associated with the Euclidean metric on $\mathbb{R}^n$.

Since partial derivatives commute, the identity

$$\left(\frac{\partial^2}{\partial x^\alpha \partial x^\beta} - \frac{\partial^2}{\partial x^\beta \partial x^\alpha}\right) X^\gamma = 0$$

implies $$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) X^\gamma \equiv X^\gamma{}_{;\beta\alpha} - X^\gamma{}_{;\alpha\beta} = 0.$$

If we evaluate this in a general coordinate frame, this becomes an identity involving first derivatives of the connection and therefore second derivatives of the metric.

$$X$$

$$\nabla_{\beta'} X^{\gamma'} = X^{\gamma'}{}_{,\beta'} + \Gamma^{\gamma'}{}_{\beta'\delta'} X^{\delta'} \qquad X^{\mu'}{}_{,\beta'} + \Gamma^{\mu'}{}_{\beta'\nu'} X^{\nu'}$$

$$\nabla_{\alpha'}(\nabla_{\beta'} X^{\gamma'}) = \underbrace{(X^{\gamma'}{}_{;\beta'})_{,\alpha'}}_{X^{\gamma'}{}_{,\beta'\alpha'} + \Gamma^{\gamma'}{}_{\beta'\delta',\alpha'} X^{\delta'} + \Gamma^{\gamma'}{}_{\beta'\delta'} X^{\delta'}{}_{,\alpha'}} + \Gamma^{\gamma'}{}_{\alpha'\mu'} X^{\mu'}{}_{;\beta'} - \Gamma^{\mu'}{}_{\alpha'\beta'} X^{\gamma'}{}_{;\mu'}$$

$$0 = \nabla_{\alpha'} \nabla_{\beta'} X^{\gamma'} - \nabla_{\beta'} \nabla_{\alpha'} X^{\gamma'} =$$

$$\underbrace{X^{\gamma'}{}_{,\beta'\alpha'} - X^{\gamma'}{}_{,\alpha'\beta'}}_{\to 0} + \Gamma^{\gamma'}{}_{\beta'\delta'} X^{\delta'}{}_{,\alpha'} + \Gamma^{\gamma'}{}_{\alpha'\mu'} X^{\mu'}{}_{,\beta'} - \Gamma^{\mu'}{}_{\alpha'\beta'} X^{\gamma'}{}_{,\mu'}$$
$$- \Gamma^{\gamma'}{}_{\alpha'\delta'} X^{\delta'}{}_{,\beta'} - \Gamma^{\gamma'}{}_{\beta'\mu'} X^{\mu'}{}_{,\alpha'} + \Gamma^{\mu'}{}_{\beta'\alpha'} X^{\gamma'}{}_{,\mu'}$$

$$+ \underbrace{\left(\Gamma^{\gamma'}{}_{\beta'\delta',\alpha'} - \Gamma^{\gamma'}{}_{\alpha'\delta',\beta'} + \Gamma^{\gamma'}{}_{\alpha'\mu'} \Gamma^{\mu'}{}_{\beta'\nu'} - \Gamma^{\gamma'}{}_{\beta'\mu'} \Gamma^{\mu'}{}_{\alpha'\nu'}\right)}_{\equiv R^{\gamma'}{}_{\delta'\alpha'\beta'}} X^{\nu'}$$

This four index expression must vanish identically since this equality must hold for arbitrary X

— This defines the curvature tensor field of the connection, which must vanish identically for the flat Euclidean metric.