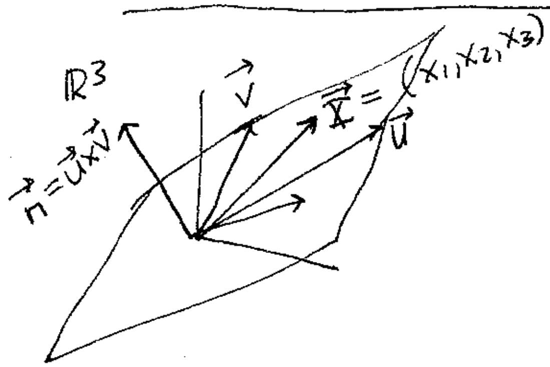
DUAL 1

Specifying ~~sub~~ vector subspaces (k-planes through origin)



How do we specify the plane of $\vec{u}$ and $\vec{v}$?

Introduce the normal vector $\vec{n} = \vec{u}\times\vec{v}$:

$$\vec{n}\cdot\vec{X} = n_1x_1 + n_2x_2 + n_3x_3 = 0$$

All pts perpendicular to the normal

$$\|\vec{n}\| = \text{area of parallelogram } (\vec{u},\vec{v})$$

Alternatively take the 2-vector

$$\vec{u}\wedge\vec{v} = (u_1e_1 + \cdots)\wedge(v_ke_k + \cdots) = \underbrace{(u_2v_3 - u_3v_2)}_{n_1} e_{23} + \underbrace{(u_1v_3 - u_3v_1)}_{-n_2} e_{13} + \underbrace{(u_1v_2 - u_2v_1)}_{n_3} e_{12}$$

All $\vec{X}$ in the plane of $\vec{u}$ and $\vec{v}$ satisfy

$$0 = \vec{u}\wedge\vec{v}\wedge\vec{X} = (n_1e_{23} - n_2e_{13} + n_3e_{12})\wedge(x_1e_1 + x_2e_2 + x_3e_3)$$

$$\begin{aligned} &= n_1x_1e_{23}\wedge e_1 && = n_1x_1\, e_{123} && = (n_1x_1 + n_2x_2 + n_3x_3)\, e_{123} \\ &\quad -n_2x_2e_{13}\wedge e_2 && \quad +n_2x_2\, e_{123} && \\ &\quad +n_3x_3e_{12}\wedge e_3 && \quad +n_3x_3\, e_{123} && \end{aligned}$$

or

$$0 = n_1x_1 + n_2x_2 + n_3x_3 = n^\flat(\vec{X})$$

so the kernel of the normal covector $n^\flat = n_a\omega^a = "\vec{n}\cdot"$ gives the plane.

If we put indices at correct positions then

$$\underset{\substack{\uparrow \\ \text{covector}}}{n_a} = \epsilon_{abc}u^bv^c = \epsilon_{abc}\underbrace{u^{[b}v^{c]}}_{\substack{\uparrow \\ \text{2-vector}}}$$

If we introduce a "natural duality" operation by

$$\circledast F_a = \tfrac{1}{2}\epsilon_{abc}F^{bc} \qquad \text{from 2-vectors to covectors.}$$

then

$$n^\flat = \circledast(\vec{u}\wedge\vec{v})$$

and

$$\vec{u}\wedge\vec{v}\wedge\vec{X} = n^\flat(\vec{X})\, e_{123}.$$

Whenever we have a metric g_{ab} we can define a length for tensors by contracting the fully covariant form of the tensor by the fully contravariant form. For example.

$$\cancel{F = F_{ab}\,\omega^a \otimes \omega^b \rightarrow g^{ac} g^{bd} F_{ab} F_{cd} = F_{ab} F^{ab}}$$

$$\cancel{\circledast F = \circledast F_a}$$

$$F = F^{ab}\, e_a \otimes e_b \rightarrow \quad F^{ab} F_{ab} = g_{ac}\, g_{bd}\, F^{ab} F^{cd} \quad \cancel{= 2!\, F^{[ab]} F}$$

$$\circledast F = \circledast F_a\, \omega^a \rightarrow \quad \circledast F_a\, {}^{*}F^a = g^{ab} \circledast F_a \circledast F_b .$$

For the Euclidean metric and a 2-vector F:

$$\underbrace{F^{ab} F_{ab}} = 2!\, F^{ab} F_{|ab|} = \circledast F_a\, {}^{*}F^a$$

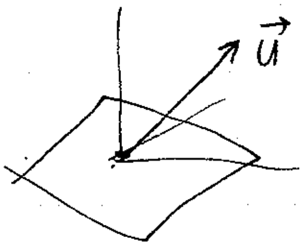
length of 2-vector related by factorial to length of dual.

In our example, the two-vector $\vec{u} \wedge \vec{v}$ which determines the plane also gives the length of the normal covector, which in turn is the ~~length~~ area of the parallelogram formed by $\vec{u}$ and $\vec{v}$.

Suppose we consider the line determined by $\vec{u}$.

Define $\quad (*\vec{u})_{ab} = \epsilon_{abc}u^c \quad , \quad *\vec{u} = u^1\omega^{23}+u^2\omega^{31}+u^3\omega^{12}.$

$\qquad$ 2-covector $\qquad\qquad$ 1-vector



Notice that contracting this 2-covector by $\vec{X}$ leads to

$$C_{\vec X}(*\vec{u}) = C_{\vec X}\begin{pmatrix} u^1(\omega^2\otimes\omega^3-\omega^3\otimes\omega^2) \\ +u^2(\omega^3\otimes\omega^1-\omega^1\otimes\omega^3) \\ +u^3(\omega^1\otimes\omega^2-\omega^2\otimes\omega^1)\end{pmatrix} = \begin{matrix} u^1(X^2\omega^3-X^3\omega^2) \\ +u^2(X^3\omega^1-X^1\omega^3) \\ +u^3(X^1\omega^2-X^2\omega^1)\end{matrix}$$

$$= (u^2X^3-u^3X^2)\omega^1 + (u^3X^1-u^1X^3)\omega^2 + (u^1X^2-u^2X^1)\omega^3$$

$$= (\vec{u}\times\vec{X})_a\,\omega^a \qquad = 0 \quad \text{for } \vec{X} \text{ along } \vec{u}.$$

So again the vector subspace is determined by the kernel of the dual of the k-vector spanning the subspace. (Before $C_X(*u\wedge v)=0$ gave the 2-d vector subspace associated with $u\wedge v$.

And the length of $*\vec{u}$ apart from a factorial gives the length of $\vec{u}$, ie. the 1-measure of the 1-parallelopiped formed by $\vec{u}$ alone.

~~In general~~

In general suppose we take p linearly independent vectors $\{X_{(1)},\ldots,X_{(p)}\}$ then $X_{(1)}\wedge\ldots\wedge X_{(p)}$ determines the vector subspace they span by the fact that the wedge product of any other linearly ind set will be proportional to this (differing by the determinant of the linear transformation between them), the $(n-p)$-covector $*(X_{(1)}\wedge\ldots\wedge X_{(p)})$ defined in a way generalizing above has the vector subspace as its kernel under contraction $\quad C_X\,*(X_{(1)}\wedge\ldots\wedge X_{(p)})=0 \leftrightarrow X\in\text{span}\{X_{(1)},\ldots,X_{(p)}\}$ and the length of either apart from factorials gives the p-measure of the p-parallelopiped they form.

DUAL 4

The duality operation takes p-vectors to (n-p)-covectors and viceversa so the dual of the dual takes p-vectors to p-vectors and p-covectors to p covectors:

$$(\circledast T)_{\alpha_{p+1}\ldots\alpha_n} = \frac{1}{p!}\, T^{\alpha_1\ldots\alpha_p}\, \epsilon_{\alpha_1\ldots\alpha_p\alpha_{p+1}\ldots\alpha_n}$$

$$(\circledast S)^{\alpha_{p+1}\ldots\alpha_n} = \frac{1}{p!}\, S_{\alpha_1\ldots\alpha_p}\, \epsilon^{\alpha_1\ldots\alpha_p\alpha_{p+1}\ldots\alpha_n}$$

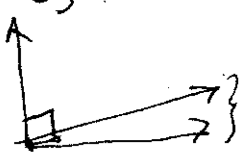
$$\circledast\circledast T = (-1)^{p(n-p)}$$ from the $\epsilon^{\cdot\cdot}\,\epsilon_{\cdot\cdot}$ contraction formula

DUAL5

Given a metric, coupled with natural duality in specifying vector subspaces by k-vectors or as the kernel of an $(n-k)$-covector dual to the k-vector, one can define orthogonal vector subspaces by raising the indices on the $(n-k)$-covector to obtain an $(n-k)$-vector associated with an $(n-k)$-dim vector subspace of vectors which are orthogonal to the original vector subspace. ~~Apart from~~ The k-covector which is its dual in turn is proportional to the k-covector obtained from lowering the indices on the original k-vector.

$$C_X(S^\#) = 0 \text{ for } X \in \text{subspace.}$$

$${}^*S^\# \sim C_X(S^\#)$$



$$X_{(1)} \wedge \dots \wedge X_{(k)} = S \longrightarrow C_X(\underbrace{{}^*S}_{\text{normal } (n-k)\text{-covector}}) = 0 \quad \text{for } X \in \text{subspace}$$

To eliminate proportionality factors & make the correspondence exact, one has to rethink the definition of the determinant.

For the standard basis of $\mathbb{R}^n$ we defined the determinant of n vectors:

$$\det(X_{(1)}, \dots, X_{(n)}) = \epsilon_{\alpha_1 \dots \alpha_n} X^{\alpha_1}_{(1)} \cdots X^{\alpha_n}_{(n)} = \underbrace{\epsilon_{\alpha_1 \dots \alpha_n} \omega^{\alpha_1} \otimes \dots \otimes \omega^{\alpha_n}}_{\omega^{1 \dots n} \,=\, \text{"det" as an } n\text{-covector}} (X_{(1)}, \dots, X_{(n)})$$

The symbol $\epsilon_{\alpha_1 \dots \alpha_n}$ are the components of this n-covector with respect to the standard basis, and in fact it is just the n-covector basis $\omega^{1 \dots n}$.

What happens to the components of this tensor if we change the basis?

$$e'_a = e^b{}_a e_b \qquad \omega'^a = A^{-1a}{}_b\, \omega^b \equiv \underbrace{B^a{}_b}_{}\omega^b$$

$$= \underbrace{A^b{}_a}_{} e_b \equiv B^{-1b}{}_a e_b$$

matrix whose colum vectors are components of new basis vectors wrt old basis.

matrix whose row ~~vect~~ vectors are the components of the new dual basis wrt old dual basis.

$$\omega^a = B^{-1a}{}_b\, \omega'^b$$

$$\omega^{1\cdots n} = \omega^1 \wedge \cdots \wedge \omega^n = \underbrace{B^{-1\,1}{}_{\alpha_1} \cdots B^{-1\,n}{}_{\alpha_n} \underbrace{\omega'^{\alpha_1} \wedge \cdots \wedge \omega'^{\alpha_n}}_{\epsilon^{\alpha_1 \cdots \alpha_n}\, \omega'^{1\cdots n}}}_{\det B^{-1}} = \underbrace{(\det B)^{-1} \epsilon_{\alpha_1 \cdots \alpha_n}}_{\text{new components of the tensor } \det = \omega^{1\cdots n}} \omega'^{\alpha_1} \otimes \cdots \otimes \omega'^{\alpha_n}$$

$$\omega^{1\cdots n} = \epsilon_{\alpha_1 \cdots \alpha_n} \omega^{\alpha_1} \otimes \cdots \otimes \omega^{\alpha_n}$$

In other words the alternating symbol $\epsilon_{\alpha_1 \cdots \alpha_n}$ breaks our notational convention in that ~~the~~ as a tensor its components are not numerically equal to the alternating symbol in other bases of $\mathbb{R}^n$ differing by a transformation with nonunit determinant.

Instead we can introduce a nontensor whose value in any basis is $\omega'^{1\cdots n} = \epsilon_{\alpha_1 \cdots \alpha_n} \omega'^{\alpha_1} \otimes \cdots \otimes \omega'^{\alpha_n}$ ~~and~~ so when evaluated on n ~~vector~~ vector arguments one gets a different result in different ~~frames~~ bases whose components transform by

$$\epsilon_{\alpha_1 \cdots \alpha_n} = (\det B) \underbrace{\epsilon_{\beta_1 \cdots \beta_n} B^{-1\beta_1}{}_{\alpha_1} \cdots B^{-1\beta_n}{}_{\alpha_n}}_{}$$

as if ~~transformed~~ as a tensor
but result $= \det B^{-1}\, \epsilon_{\alpha_1 \cdots \alpha_n}$
multiply by $\det B$ to get back original components.

This is nothing more than a basis-dependent tensor, equal to $\omega^{1\cdots n}$ in every choice of basis. It is called a TENSOR DENSITY of weight (-1) one since it transforms by the extra factor $(\det B^{-1})^W$ with $W=1$.

Suppose we have a metric $g = g_{ab}\,\omega^a \otimes \omega^b = g'_{ab}\,\omega'^a \otimes \omega'^b$

$$g'_{ab} = B^{-1c}{}_a B^{-1d}{}_b\, g_{cd} \qquad \text{~~or $g' = B^{-1T} g B^{-1}$ det~~}$$

$$\cancel{\det(g'_{ab})} = B^{-1c}{}_a\, g_{cd}\, B^{-1d}{}_b$$

or in matrix form $\underline{g}' = B^{-1T}\underline{g}\, B^{-1}$

$$\det \underline{g}' = \det B^{-1T} \det g \,\det B^{-1}$$
$$= (\det B^{-1})^2 \det \underline{g}$$

so $\det \underline{g}$ is a weight 2 scalar density (no indices) by the above definition, since it transforms by the extra factor $(\det B^{-1})^w$ with $w=2$.

Let $\quad \mathfrak{g} \equiv |\det \underline{g}|$,

then $\quad \mathfrak{g}' = (\det B^{-1})^2\, \mathfrak{g}$

$$\mathfrak{g}'^{\frac{1}{2}} = |\det B^{-1}|\, \mathfrak{g}^{1/2}$$
$$= \mathrm{sgn}(\det B)\,(\det B^{-1})\, \mathfrak{g}^{1/2}$$

So $\mathfrak{g}^{1/2}$ is an "oriented" scalar density of weight 1, "oriented" since it switches sign if $\det B$ is negative, which corresponds to a change of orientation for the basis.

Finally: $\eta'_{\alpha_1\ldots\alpha_n} \equiv \mathfrak{g}^{1/2\,\prime}\,\epsilon_{\alpha_1\ldots\alpha_n} = \mathrm{sgn}(\det B)\underbrace{\mathfrak{g}^{\frac{1}{2}}\epsilon_{\beta_1\ldots\beta_n}}_{\equiv \eta_{\alpha_1\ldots\alpha_n}} B^{-1\beta_1}{}_{\alpha_1}\cdots B^{-1\beta_1}{}_{\alpha_n}$

is just an oriented scalar since the weights cancel and hence defines a unique tensor for all ~~frames wi~~ bases with the same orientation.

If one picks an ON basis wrt this metric, so that

$$(g_{ab}) = \cancel{(\eta_{ab})} = \mathrm{diag}(1\ldots 1, -1, \ldots -1)$$

then $\mathfrak{g}^{1/2} = \cancel{\eta^{1/2}} = 1$ and this just produces the volume of an orthogonal unit parallelopiped when evaluated on the basis vectors & hence as in $\mathbb{R}^n$, produces the volume of a general parallelopiped when evaluated on arbitrary arguments.

As an oriented tensor it has the expression:

$$\eta = \eta_{\alpha_1\ldots\alpha_n}\,\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_n} = \frac{1}{n!}\,\eta_{\alpha_1\ldots\alpha_n}\,\omega^{\alpha_1\ldots\alpha_n}$$
$$= \eta_{1\ldots n}\,\omega^{1\ldots n} = g^{\frac{1}{2}}\,\omega^{1\ldots n}$$

and it determines the volume of the n-parallelopiped formed by $\{X_{(1)},\ldots,X_{(n)}\}$

$$\mathrm{Vol}\,(X_{(1)},\ldots,X_{(n)}) = |\,\eta\,(X_{(1)},\ldots,X_{(n)})\,|$$

with respect to the geometry of the metric g. The sign of this quantity gives the orientation of the set of vectors as long as the basis has the orientation specified for the space. η is called the unit-alternating tensor for the metric g or the volume n-covector.

Another "numerical tensor" defined by the same set of compnents in any basis is: $\delta^\alpha{}_\beta$,

i.e. $E = \delta^\alpha{}_\beta\, e_\alpha\otimes\omega^\beta$. It's value is $E(\sigma,X) = \delta^\alpha{}_\beta\,\sigma_\alpha X^\beta = \sigma_\alpha X^\alpha = \sigma(X)$, just the natural evaluation of the ~~tensor~~ covector argument by the vector argument. In other words, evaluation of a covector on a vector may be interpreted as defining a $\binom{1}{1}$-tensor, whose compnents are just the Kronecker delta symbol. This is often called the Identity tensor since its $\binom{1}{1}$-tensor interpretation as a linear map of the vector space into itself (or the dual space into itself) is just the identity transformation. The identity

$$\delta^\alpha{}_\beta = B^\alpha{}_\gamma\,B^{-1\delta}{}_\beta\,\delta^\gamma{}_\delta \;\left(\Leftarrow\; B^\alpha{}_\gamma B^{-1\gamma}{}_\beta = \delta^\alpha{}_\beta\ \checkmark\right)$$

says that indeed this tensor has the same compnents in any basis.

What about the generalized Kronecker delta symbols?

$$\delta^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_p} \;:\quad \delta^{(p)} = \delta^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_p}\, e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_p}\otimes\omega^{\beta_1}\otimes\cdots\otimes\omega^{\beta_p}$$
$$= e_{\beta_1\ldots\beta_p}\otimes\omega^{[\beta_1}\otimes\cdots\otimes\omega^{\beta_p]} = \frac{1}{p!}\, e_{\beta_1\ldots\beta_p}\otimes\omega^{\beta_1\ldots\beta_p}$$

Yes, since $\delta^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_p} = p!\,\delta^{\alpha_1}{}_{[\beta_1}\cdots\delta^{\alpha_p}{}_{\beta_p]} = p!\,\delta^{[\alpha_1}{}_{\beta_1}\cdots\delta^{\alpha_p]}{}_{\beta_p}$ is just an antisymmetrized tensor product of p factors of the identity tensor.

Finally, we can replace the tensor density $\epsilon_{\alpha_1\dots\alpha_n}$ by the tensor $\eta_{\alpha_1\dots\alpha_n}$ when we have a metric and obtain a metric duality operation which does not depend on the choice of basis, as does the natural dual:

$$(*T)_{\alpha_{p+1}\dots\alpha_n} = \frac{1}{p!} T^{\alpha_1\dots\alpha_p} \eta_{\alpha_1\dots\alpha_p\alpha_{p+1}\dots\alpha_n}$$
$$(*S)^{\alpha_{p+1}\dots\alpha_n} = \frac{1}{p!} S_{\alpha_1\dots\alpha_p} \eta^{\alpha_1\dots\alpha_p\alpha_{p+1}\dots\alpha_n}$$
(provisional definition)

But now that we have a metric at our disposal, we can raise or lower the indices to obtain a tensor of the same valence as the one we started. It turns out that this is more useful:

$$(*T)_{\alpha_{p+1}\dots\alpha_n} = \frac{1}{p!} \underbrace{T^{\alpha_1\dots\alpha_p}}_{g^{\alpha_1\beta_1}\dots g^{\alpha_p\beta_p}T_{\beta_1\dots\beta_p}} \eta_{\alpha_1\dots\alpha_p\alpha_{p+1}\dots\alpha_n}$$
(T a ~~T a $\binom{0}{p}$~~ p-covector)

This takes p-covectors to n-p covectors and p-vectors to n-p vectors when index positions are ~~are~~ interchanged.

Suppose we raise indices on the unit-alternating n-covector

$$\eta^{\alpha_1\dots\alpha_n} = g^{\alpha_1\beta_1}\dots g^{\alpha_n\beta_n} \underbrace{\eta_{\beta_1\dots\beta_n}}_{g^{1/2}\epsilon_{\beta_1\dots\beta_n}} = \underbrace{g^{1/2}(\det \underline{g}^{-1})}_{=\mathrm{sgn}(\det \underline{g})\, g^{-1/2}} \epsilon^{\alpha_1\dots\alpha_n}$$

$$= \mathrm{sgn}(\det \underline{g})\, g^{-1/2}\, \epsilon^{\alpha_1\dots\alpha_n}$$

Thus for metrics with an odd number of minus signs in the ~~sig~~ diagonal values of the metric in an orthonormal basis, the definition of the unit alternating n-vector differs by a sign from what might wish to define it analogous to the unit alternating n-covector.

EX. $\mathbb{R}^3$ with Euclidean metric: $\eta = \omega^{123}$.

$*\omega^1 = \omega^{23}$, $*\omega^2 = \omega^{31}$, $*\omega^3 = \omega^{12}$

$*\omega^{23} = \omega^1$, $*\omega^{31} = \omega^2$, $*\omega^{12} = \omega^3$

$*\omega^{123} = 1$.

AND: $*(\vec{u}\wedge\vec{v}) = \vec{u}\times\vec{v}$, $*(\vec{u}\wedge\vec{v}\wedge\vec{w}) = \vec{u}\cdot(\vec{v}\times\vec{w})$

$*(*\vec{u}\wedge\vec{v}) = \vec{u}\cdot\vec{v}$

Notice that for an n-vector or an n-covector, the dual is just a real number:

$$* T = \frac{1}{n!} T^{\alpha_1 \dots \alpha_n} \eta_{\alpha_1 \dots \alpha_n} = \frac{1}{n!} T_{\alpha_1 \dots \alpha_n} \eta^{\alpha_1 \dots \alpha_n}$$

That means if we want to define $**$ for n-vectors or covectors we have to distinguish between a covariant $*$ operator and a contravariant $*$ operator. When we are dealing with ~~an~~ n-vectors $**$ will produce ~~and we~~ an n-vector and $*1$ ~~an~~ will be an n-vector, & similarly when we are dealing with n-covectors.

For example

$$* \eta = \frac{1}{n!} \eta^{\alpha_1 \dots \alpha_n} \eta_{\alpha_1 \dots \alpha_n} = \frac{1}{n!} (\operatorname{sgn} \det \underline{g}) \, \epsilon^{\alpha_1 \dots \alpha_n} \epsilon_{\alpha_1 \dots \alpha_n}$$

$$= \frac{1}{n!} (\operatorname{sgn} \det \underline{g}) \underbrace{\delta^{\alpha_1 \dots \alpha_n}_{\alpha_1 \dots \alpha_n}}_{n!} = \operatorname{sgn} \det \underline{g}$$

while $(*1)_{\alpha_1 \dots \alpha_n} = \frac{1}{0!} 1 \, \eta_{\alpha_1 \dots \alpha_n}$, i.e. $*1 = \eta$.

and hence $**1 = \operatorname{sgn} \det \underline{g}$ or $** \eta = (\operatorname{sgn} \det \underline{g}) \eta$.

In general one can show $** S = (\operatorname{sgn} \det \underline{g}) (-1)^{p(n-p)} S$ for a ~~p-index multive~~ p-covector or p-vector

For $\mathbb{R}^3$ with the Euclidean metric, this sign is positive for all values of p.

Exercise: Use the formula for the contraction of two ϵ-symbols & the definition of η to derive this formula.

DUAL 11

Finally what is the interpretation of the dual? Suppose $\{X_{(1)}, \dots, X_{(p)}\}$ determine a p-dimensional subspace. The p-vector $X_{(1)} \wedge \dots \wedge X_{(p)}$ determines this subspace. The (n-p)-vector $*(X_{(1)} \wedge \dots \wedge X_{(p)})$ determines the subspace orthogonal to the first. The (n-p)-covector $(*(X_{(1)} \wedge \dots \wedge X_{(p)}))^\flat$ obtained by lowering indices is the (n-p)-covector which determines the first space as its kernel. Similarly $(X_{(1)} \wedge \dots \wedge X_{(p)})^\flat$ is the p-covector which determines the orthogonal space as its kernel.

Dual 12

One last detail:

If $T = \frac{1}{p!} T_{\alpha_1\ldots\alpha_p}\,\omega^{\alpha_1\ldots\alpha_p}$ is a covector

then $*T_{\alpha_{p+1}\ldots\alpha_n} = \frac{1}{p!}\, T^{\alpha_1\ldots\alpha_p}\,\eta_{\alpha_1\ldots\alpha_p\alpha_{p+1}\ldots\alpha_n}$

so $*T = \frac{1}{(n-p)!}\, *T_{\alpha_{p+1}\ldots\alpha_n}\,\omega^{\alpha_{p+1}\ldots\alpha_n} = \frac{1}{p!(n-p)!}\, T^{\beta_1\ldots\beta_p}\,\eta_{\beta_1\ldots\beta_p\alpha_{p+1}\ldots\alpha_n}\,\omega^{\alpha_{p+1}\ldots\alpha_n}$

hence

$$\underbrace{\underbrace{T}_{p}\wedge\underbrace{*T}_{n-p}}_{n} = \frac{1}{(p!)^2(n-p)!}\, T_{\alpha_1\ldots\alpha_p} T^{\beta_1\ldots\beta_p}\, \underbrace{\eta_{\beta_1\ldots\beta_p\alpha_{p+1}\ldots\alpha_n}}_{g^{1/2}\epsilon_{\beta_1\ldots\beta_p\alpha_{p+1}\ldots\alpha_n}}\, \underbrace{\omega^{\alpha_1\ldots\alpha_p}\wedge\omega^{\alpha_{p+1}\ldots\alpha_n}}_{\omega^{\alpha_1\ldots\alpha_n} = \epsilon^{\alpha_1\ldots\alpha_n}\,\omega^{1\ldots n}}$$

has to be a multiple of n-covector η

η

$$\epsilon^{\alpha_1\ldots\alpha_p\alpha_{p+1}\ldots\alpha_n}\,\epsilon_{\beta_1\ldots\beta_p\alpha_{p+1}\ldots\alpha_n}$$

$$\downarrow$$

$$\delta^{\alpha_1\ldots\alpha_p\alpha_{p+1}\ldots\alpha_n}_{\beta_1\ldots\beta_p\alpha_{p+1}\ldots\alpha_n} = (n-p)!\,\delta^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_p}$$

$$= \frac{1}{(p!)^2}\,\underbrace{\delta^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_p}\,T_{\alpha_1\ldots\alpha_p}}_{p!\,T_{\beta_1\ldots\beta_p}}\,T^{\beta_1\ldots\beta_p}\,\eta = \frac{1}{p!}\,T_{\beta_1\ldots\beta_p}T^{\beta_1\ldots\beta_p}\,\eta$$

$$= \underbrace{T_{|\beta_1\ldots\beta_p|}\,T^{\beta_1\ldots\beta_p}}_{\equiv\ \langle T,T\rangle}\,\eta$$

$\equiv \langle T, T\rangle$ — defines an inner product — for p-covectors or p-vectors.

In the Euclidean case this is just the sum of the squares of the independent components of the p-covector

This is very useful to quickly compute the duals of the basis covectors. These all have $\underline{1}$ independent component of value 1. If $\{e_\alpha\}$ is an orthonormal basis, then $\eta = \omega^{1\ldots n}$ and

$$\omega^{\alpha_1\ldots\alpha_p}\wedge\underline{*\omega^{\alpha_1\ldots\alpha_p}} = \omega^{1\ldots n}$$

→ complementary indices — choose sign

EX. $n=4$, Euclidean metric

$$\omega^{12}\wedge\underbrace{(*\omega^{12})}_{\omega^{34}} = \omega^{1234}$$

$$\omega^{23}\wedge *\omega^{23} = \omega^{1234}$$

$$\omega^{14}? = \omega^{1234}\ \checkmark$$

DUAL 13

$$\omega^{32} \wedge \underbrace{{}^*\omega^{32}}_{\sim \omega^{14}} = \omega^{1234}$$

$+\omega^{1324}$

$-\omega^{1234}$ $\longrightarrow$ so ${}^*\omega^{32} = -\omega^{14}$.

EX $\quad n=3$; Euclidean metric

$$T = \vec{u} \wedge \vec{v} = \underbrace{(u^1 v^2 - u^2 v^1)}_{(\vec{u}\times\vec{v})^1} e_{12} + \underbrace{(u^3 v^1 - u^1 v^3)}_{(u\times v)^2} e_{31} + \underbrace{(u^1 v^2 - u^2 v^1)}_{(u\times v)^3} e_{12}$$

$$e_{12} \wedge \underbrace{{}^*e_{12}}_{\therefore e_3} = e_{123}, \ldots \text{cyclic symmetry}, \ {}^*e_{23} = e_1, \quad {}^*e_{31} = e_2.$$

$$ {}^*T = (u\times v)^1 e_1 + (\vec{u}\times\vec{v})^2 e_2 + (\vec{u}\times\vec{v})^3 e_3$$

$$\langle {}^*T, {}^*T \rangle = \cancel{({}^*T)_1{}^2 + ({}^*T)_2{}^2 + ({}^*T)_3{}^2}\ ({}^*T)^{1\,2} + ({}^*T)^{2\,2} + ({}^*T)^{3\,2}$$
$$= \|\vec{u}\times\vec{v}\|^2$$

$$\langle T, T \rangle = \text{same.} = (\text{area of parallelogram of } \vec{u} \text{ and } \vec{v})^2.$$

In general if $\{X_{(1)}, \ldots, X_{(p)}\}$ form a p-parallellogram, then

set $T = X_{(1)} \wedge \ldots \wedge X_{(p)}$

$$|\langle T, T \rangle| = \mathrm{vol}^2(X_{(1)}, \ldots, X_{(p)}).$$

The inner product for p-vectors represents the p-volume of the parallelopiped they form.

(absolute value signs are needed if the metric has a nontrivial signature so inner products can have negative values)

Dual 14

<u>Problem</u> $n=4$, Lorentz metric of Minkowski space.

$(g_{\alpha\beta}) = \mathrm{diag}(-1,1,1,1) = (g^{\alpha\beta}) \qquad \alpha,\beta = \underset{\uparrow\ \text{time direction}}{0},1,2,3$

Define ω^{0123} to have the positive orientation, i.e., the ordering $\{0,1,2,3\}$

Then $g=1$, $\eta = \omega^{0123}$.

Now use the formula $T\wedge{}^*T = \underbrace{\langle T,T\rangle}\eta$

$$T_{|\alpha_1\ldots\alpha_p|}T^{\alpha_1\ldots\alpha_p} = T_{|\alpha_1\ldots\alpha_p|}\underbrace{g^{\alpha_1\beta_1}\cdots g^{\alpha_p\beta_p}}T_{\beta_1\ldots\beta_p}$$

now have to be careful of minus signs.

to evaluate the duals of the ordered bases for each space of p-forms.

$p=1 \quad \omega^1, \omega^2, \omega^3$

$p=2 \quad \omega^{23}, \omega^{13}, \omega^{12}, \omega^{14}, \omega^{24}, \omega^{34}$

$p=3 \quad \omega^{234}, \omega^{134}, \omega^{124}, \omega^{123}$

by first computing the sign of their self innerproducts.

<u>Problem</u> In $\mathbb{R}^4$ use the wedge and dual operations to find a vector orthogonal to the subspace spanned by the three vectors

$X_{(1)} = (1,0,-1,0)$

$X_{(2)} = (2,2,0\ 1)$

$X_{(3)} = (0,1,-3,0)$

what is the 3-volume of the parallelopiped they form?

For Wed

Read dual 12–13, do problems on 14.

Read new notes summarizing steps from linear algebra to tensor algebra.

Read Chapter 5, pages 43–49 on tangent spaces to $\mathbb{R}^n$.