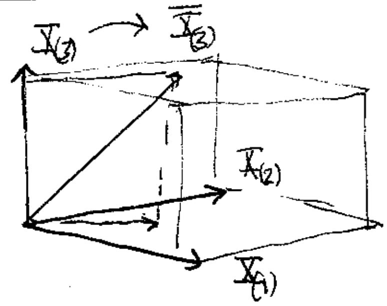
volume and determinants 1

$\mathbb{R}^3$



volume of parallelopiped formed by vectors as edges with origin as a vertex

$$\mathrm{Vol}\,(\underbrace{X_{(1)}, X_{(2)}, X_{(3)}}_{\text{3 perpendicular vectors}}) = \|X_{(1)}\|\,\|X_{(2)}\|\,\|X_{(3)}\|$$

Adding any linear combination of the other vectors to $X_{(3)}$ doesnt change the volume, since the height of the parallelopiped remains the same. One can thus add arbitrary linear combinations of the remaining vectors to a given vector without changing the volume. If any vector is multiplied by a positive number, the height in its direction is scaled by the same factor, and so the volume scales by that factor. Interchanging any two vectors in order doesnt change the volume which doesnt depend on which order the vectors are stated. These properties explain why the ~~determinant~~ absolute value of the determinant measures volume. The sign of the determinant specifies the orientation of the three vectors, i.e., if $X_{(3)}$ is on the same/opposite side of the plane of $X_{(1)}$ and $X_{(2)}$ as determined by the right hand screw rule (same side as $X_{(1)} \times X_{(2)}$) then the determinant is positive/negative and the ordered set of vectors $\{X_{(1)}, X_{(2)}, X_{(3)}\}$ is said to have the same/opposite orientation as the standard basis of $\mathbb{R}^3$.

This same discussion (without the right hand screw rule) applies to any dimension. The sign of the determinant of an ordered set of n-vectors in $\mathbb{R}^n$ specifies the orientation of the linearly independent vectors when nonzero (positive or negative) and the absolute value of the determinant gives the volume of the parallelopiped formed.

Zero determinant means zero volume and hence the vectors are not linearly independent.

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Any $n\times n$ matrix $\underline{A}$ with nonzero determinant can be obtained from the identity matrix by the following 3 operations:

1) Adding a linear combination of the remaining rows/columns to a given row/column — det invariant
2) interchanging any two rows/columns — det changes sign, $|\det|$ invariant
3) multiplying a row/column by a nonzero number α — det scales by α, $|\det|$ scales by $|\alpha|$.

Any $n\times n$ matrix $\underline{A}$ can be thought of as an ordered set of column vectors

$$\underline{A} = \begin{pmatrix} A_{11} \cdots A_{1n} \\ \vdots \\ A_{n1} \cdots A_{nn} \end{pmatrix} = \left(A_{i1}, \ldots A_{in}\right) \equiv \left(X_{(1)} \cdots X_{(n)}\right).$$

$$\underline{A}\, e_i = \underline{A}\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th row} = \left(A_{ji}\right) = i\text{th column vector} = X_{(i)}$$

These column vectors are just the images of the standard basis vectors under the linear transformation associated with the matrix.
We can build up an arbitrary matrix with nonzero determinant in the following way: Start with the identity matrix, whose columns are just the standard basis vectors. Its determinant is one, equal to the volume of a unit cube in $\mathbb{R}^n$. Now scale the columns of the identity matrix by arbitrary nonzero real numbers, mapping the n standard basis vectors to n orthogonal vectors, scaling the determinant by the product of the absolute values of the numbers. Adding arbitrary linear combinations of the remaining columns to each column maps these orthogonal vectors to general nonorthogonal positions leading to a general matrix $\underline{A}$ of the same nonzero determinant whose absolute value clearly represents the volume of the parallelopiped formed by the column vectors.

$$|\det(X_{(1)},\ldots,X_{(n)})| = \text{volume}\,(X_{(1)},\ldots,X_{(n)})$$

$$\text{sgn}\det(X_{(1)},\ldots,X_{(n)}) \begin{array}{ll} >0 & \text{positively oriented lin. ind. set} \\ <0 & \text{negatively oriented lin. ind. set} \\ =0 & \text{linearly dep set.} \end{array}$$

We can extend this from $\mathbb{R}^n$ with its Euclidean geometry to any n-dimensional vector space V with a nondegenerate inner product or "metric". It is enough to choose an orthonormal basis $\{e_a\}$ of the space, i.e. $e_a \cdot e_b = \pm 1\, \delta_{ab}$.

Then expressing vectors in this ~~frame~~ basis, the entire discussion for $\mathbb{R}^n$ in the standard basis carries over and determines volumes by the determinant function of the matrix of orthonormal basis components.

On $\mathbb{R}^n$ the determinant function thought of as an antisymmetric multilinear function on n-vectors is the unique antisymmetric $\binom{0}{n}$-tensor whose value on the standard basis is one. Once we introduce the wedge product, then we can represent this tensor by

$$\det(X_{(1)} \dots X_{(n)}) = \det(X^i{}_{(j)}) = \underbrace{\omega^1 \wedge \dots \wedge \omega^n}_{\equiv\, \omega^{1\dots n}} (X_{(1)}, \dots, X_{(n)})$$

These same formula holds for any positively oriented orthonormal basis of $\mathbb{R}^n$, i.e. the n-covector $\omega^{1\dots n}$ defines an invariant ~~single~~ of antisymmetric $\binom{0}{n}$-tensor on $\mathbb{R}^n$ which evaluates volume, say "vol".
It has components $\text{vol} = \omega^{1\dots n} = \underbrace{\epsilon_{\alpha_1 \dots \alpha_n}}_{\text{alternating symbol.}} \omega^{\alpha_1} \otimes \dots \otimes \omega^{\alpha_n}$.

Alternatively one has the result

$$\underbrace{X_{(1)} \wedge \dots \wedge X_{(n)}}_{\substack{\text{necessarily an}\\ \text{antisymmetric } \binom{0}{n}\text{-tensor}\\ \text{and hence a multiple}\\ \text{of } e_{1\dots n}}} = \underbrace{\det(X^{(i)}{}_{(j)})}_{\substack{\text{the coefficient is just the determinant,}\\ \text{equal to the dual basis n-vector } \omega^{1\dots n}\\ \text{evaluated on the n vectors } X_{(i)}.}} \underbrace{e_1 \wedge \dots \wedge e_n}_{e_{1\dots n}}$$

Example with nonstandard inner product on $\mathbb{R}^3$; standard basis $\{e_a\}$.

Let $g = g_{ab}\,\omega^a \otimes \omega^b$, $g_{ab} = g(e_a, e_b)$ have matrix of components

$$\underline{g} \equiv (g_{ab}) = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

$g_{12} = 2 \neq 0$ means that e_1 and e_2 are not orthogonal with respect to this metric, but $g_{13} = g_{23} = 0$ means e_1 and e_2 are orthogonal to e_3. We can get an orthonormal basis as follows:

Normalize e_3: $\quad \bar{e}_3 = \dfrac{e_3}{\underbrace{\sqrt{3}}_{\sqrt{g_{33}}}} = (0, 0, \tfrac{1}{\sqrt{3}})$

Normalize e_1: $\quad \bar{e}_1 = \dfrac{e_1}{\underbrace{\sqrt{4}}_{\sqrt{g_{11}}}} = (\tfrac{1}{2}, 0, 0)$.

Now subtract from e_2 its component along e_1:

$$\bar{\bar{e}}_2 = e_2 - \frac{g(e_2, e_1)\, e_1}{g(e_1, e_1)} = e_2 - \frac{g_{21}}{g_{11}} e_1 = (0,1,0) - \left(\tfrac{2}{4}\right)(\text{[redacted]}, 0, 0)$$

$$= \left(-\tfrac{1}{2}, \text{[redacted]}, 0\right) = \tfrac{1}{2}(-1, 2, 0)$$

Then $\quad g(\bar{\bar{e}}_2, e_1) = g\left(e_2 - \cancel{}\,\frac{g_{21}}{g_{11}} e_1 ,\ e_1\right) = g_{21} - \frac{g_{21}}{g_{11}} g_{11} = 0$

shows it to be orthogonal to e_1, normalize it:

$$\bar{e}_2 = \frac{\bar{\bar{e}}_2}{\sqrt{g(\bar{\bar{e}}_2, \bar{\bar{e}}_2)}} = \frac{(-1,2,0)}{\sqrt{(-1\ 2\ 0)\begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}}} = \left(-\tfrac{1}{2\sqrt{3}}, \tfrac{1}{\sqrt{3}}, 0\right)$$

$$\Big\} = (-1\ 2\ 0)\begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} = 12, \quad \sqrt{12} = 2\sqrt{3}$$

Then $\{\bar{e}_a\}$ is ON wrt g, ie $g(\bar{e}_a, \bar{e}_b) = \delta_{ab}$.

The matrix transforming from the standard basis to the new one is

$$\bar{e}_a = A^b{}_a e_b \qquad \underline{A} = (\bar{e}_1\ \bar{e}_2\ \bar{e}_3) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \Longrightarrow A^{-1} = \begin{pmatrix} 2 & 1 & 0 \\ \text{[redacted]} & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{pmatrix}$$

The dual basis is: $\quad \bar{\omega}^a = A^{-1\,a}{}_b\, \omega^b$

$$\bar{\omega}^1 = 2\omega^1 + \omega^2$$

$$\bar{\omega}^2 = \text{[redacted]}\ \sqrt{3}\,\omega^2 \qquad \text{oops}$$

$$\bar{\omega}^3 = \text{[redacted]}\ \sqrt{3}\,\omega^3$$

$$\mathrm{Vol}(g) = \bar{\omega}^{123} = \bar{\omega}^1 \wedge \bar{\omega}^2 \wedge \bar{\omega}^3 = \underbrace{(2\omega^1) \wedge (\omega^1 + \sqrt{3}\omega^2)}_{2\sqrt{3}\,\omega^1 \wedge \omega^2} \wedge (\sqrt{3}\omega^3) = 6\,\omega^{123}$$

so the volume of the parallelopiped formed by the standard basis vectors measured by the inner product g is

$$\mathrm{Vol}(g)(e_1, e_2, e_3) = 6\,\omega^{123}(e_1, e_2, e_3) = 6\,\delta^{123}_{123} = 6$$

on the other hand

$$\bar{\bar{e}}_{123} = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 = \underbrace{\left(\tfrac{1}{2}e_1\right) \wedge \left(\tfrac{1}{\sqrt{3}}\left[-\tfrac{1}{2}e_1 + e_2\right]\right)}_{\frac{1}{2\sqrt{3}}\, e_1 \wedge e_2} \wedge \left(\tfrac{1}{\sqrt{3}}\, e_3\right) = \frac{1}{6}\, e_{123}$$

$= \det(\bar{e}_1, \bar{e}_2, \bar{e}_3) =$ volume of parallelopiped formed by $\{\bar{e}_a\}$ measured with respect to the Euclidean metric.

Since $\{\bar{e}_a\}$ is ON wrt g, one can express g as

$$g = \bar{g}_{ab}\, \bar{\omega}^a \otimes \bar{\omega}^b \qquad \bar{g}_{ab} = g(\bar{e}_a, \bar{e}_b) = \delta_{ab}$$

Raising and lowering indices

with any vector $X = X^a e_a = \bar{X}^a \bar{e}_a$, we can associate a covector $X^\flat$ by partially evaluating the tensor g on X:

$$X^\flat = g(X, \;) = C_X\, g \qquad \text{evaluation of first argument on } X$$

$$\begin{aligned} X^\flat &= X_a \omega^a \quad \text{where} \quad X_a \equiv g_{ab} X^b. \\ &= \bar{X}_a \bar{\omega}^a \qquad\qquad\quad \bar{X}_a \equiv \bar{g}_{ab} \bar{X}^b = \delta_{ab} \bar{X}^b. \end{aligned}$$

For example the covectors corresponding to the standard basis vectors are

$$e_1{}^\flat = 4\omega^1 + 2\omega^2 \qquad \text{But} \quad \bar{e}_1{}^\flat = \bar{\omega}^1$$
$$e_2{}^\flat = 2\omega^1 + 4\omega^2 \qquad\qquad \bar{e}_2{}^\flat = \bar{\omega}^2$$
$$e_3{}^\flat = 3\omega^3 \qquad\qquad\qquad\quad \bar{e}_3{}^\flat = \bar{\omega}^3.$$

? For an orthonormal basis, "lowering the index" just produces the dual basis.

$$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \sim (2, 1, 0) \to 2\omega = \bar{\omega}^1$$

$$\begin{pmatrix} -\frac{1}{2\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{3} \\ 0 \end{pmatrix} \sim (0, \sqrt{3}, 0) \to \bar{\omega}^2$$

check

$$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{3} \end{pmatrix} \sim (0, 0, \sqrt{3}) = \bar{\omega}^3$$

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The inverse matrix defines an inner product on the dual space, ie, an inner product for covectors:

$$(g^{ab}) \equiv \underline{g}^{-1} = \begin{pmatrix} \frac{2}{\sqrt3} & -\frac{1}{\sqrt3} & 0 \\ -\frac{1}{\sqrt3} & \frac{2}{\sqrt3} & 0 \\ 0 & 0 & \frac{1}{\sqrt3} \end{pmatrix}$$

$$g^{-1} = g^{ab} e_a \otimes e_b = \bar g^{ab} \bar e_a \otimes \bar e_b$$

$$g^{ab} = g^{-1}(\omega^a, \omega^b)$$

$$\bar g^{ab} = g^{-1}(\bar\omega^a, \bar\omega^b) = \delta^{ab}$$

this makes the dual basis to the ON basis $\{\bar e_a\}$ orthonormal wrt g^{-1}.

We can use this to "raise indices" on covectors $\sigma = \sigma_a \omega^a = \bar\sigma_a \bar\omega^a$:

$$\sigma^{\#} = g^{-1}(\sigma, \ ) = \text{evaluation of } g^{-1} \text{ on first argument}$$

$$\sigma^{\#} = \sigma^a e_a \qquad \text{where} \quad \sigma^a = g^{ab}\sigma_b$$

$$= \bar\sigma^a \bar e_a \qquad \bar\sigma^a = \bar g^{ab}\bar\sigma_b = \delta^{ab}\bar\sigma_b$$

For example:

$$\omega^{1\#} = \frac{2e_1 - e_2}{\sqrt3} \qquad \bar\omega^{1\#} = \bar e_1$$

$$\omega^{2\#} = \frac{-e_1 + 2e_2}{\sqrt3} \qquad \bar\omega^{2\#} = \bar e_2$$

$$\omega^{3\#} = \frac{1}{\sqrt3} e_3 \qquad \bar\omega^{3\#} = \bar e_3.$$

? Raising the index on the dual basis to an ON basis gives back the basis.

Now notice that

$$g(\sigma^{\#}, X) = \underbrace{g_{ab} g^{bc}}_{\delta^c_b} \sigma_c)(X^b = \sigma_b X^b = \sigma(X)$$

so $\quad g(\omega^{a\#}, e_b) = \omega^a(e_b) = \delta^a{}_b.$

Sometimes $\{\omega^{a\#}\}$ is called the basis reciprocal to e_a since it picks out the components with respect to $\{e_a\}$ by inner products ~~of vectors~~ rather than natural evaluation of covectors on vectors.

Volume and determinant 7

The Euclidean metric tensor

$$E = \delta_{ab}\, \omega^a \otimes \omega^b$$

is a tensor whose indices can be raised by g^{-1}:

$$E^{ab} = g^{ac} g^{bd} \delta_{cd} = g^{ac} \delta_{cd} g^{db} = \left((\underline{g}^{-1})^2 \right)^{ab}$$

$$(\underline{g}^{-1})^2 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$\text{"} E^{\#} \text{"} = E^{ab} e_a \otimes e_b = e_1 \otimes e_1 - (e_1 \otimes e_2 + e_2 \otimes e_1) + \tfrac{2}{3}\, e_2 \otimes e_2 + \tfrac{1}{3}\, e_3 \otimes e_3$$

practice with wedges

$$\vec{X} \wedge \vec{Y} \wedge \vec{Z} = -Y \wedge X \wedge Z = +Y \wedge Z \wedge X$$

$$X \wedge \vec{S} = S \wedge X \qquad e_1 \wedge e_{23} = e_{123} \qquad X \wedge Y \wedge Z \wedge X = 0$$

$$\| $$

$$X \wedge (S^{23} \underbrace{e_{23}}_{e_2 \wedge e_3} + S^{31} e_{31} + S^{12} e_{12}) \qquad S =$$

$$(X^1 e_1 + \dots) \wedge (\quad) \qquad X^1 S^{23} \; e_{123} \qquad = \underbrace{(X^1 S^{123} + X^2 S^{31} + X^3 S^{12})}\, e_{123}$$

$$X^2 e_2 \qquad X^2 S^{31} \; e_2 \wedge e_3 \wedge e_1 = e_{123}$$

$$X^3 e_3 \wedge \qquad X^3 S^{12} \; e_3 \wedge e_1 \wedge e_2 = e_{123}$$

$$\overset{p}{S} \wedge \overset{q}{T} = (-1)^{pq} T \wedge S \qquad e_{123} = \frac{1}{3!} \delta^{\alpha_1 \alpha_2 \alpha_3}_{123} e_{\alpha_1 \alpha_2 \alpha_3}$$

$$S \underbrace{e \wedge \dots \wedge e}_{p} \; T \underbrace{e \wedge \dots \wedge e}_{q} \qquad e_{123} = \delta^{\alpha_1 \alpha_2 \alpha_3}_{123} e_{\alpha_1} \otimes e_{\alpha_2} \otimes e_{\alpha_3} = e_1 \wedge e_2 \wedge e_3$$

$$(S^{\alpha_1 \dots \alpha_p} e_{\alpha_1 \dots \alpha_p}) \wedge (T^{\beta_1 \dots \beta_q} e_{\beta_1 \dots \beta_q})$$

$$\frac{1}{p!q!} S^{\alpha_1 \dots \alpha_p} T^{\beta_1 \dots \beta_p} \qquad e_{\alpha_1 \dots \alpha_p \beta_1 \dots \beta_q}$$

$$e_{\alpha_1} \wedge \dots \wedge e_{\alpha_p} \wedge e_{\beta_1} \wedge \dots \wedge e_{\beta_q} \qquad e \dots e_{\alpha_p}$$

$$((-1)^q)^p$$

$$= \det$$

$$\boxed{X_{(1)} \wedge \dots \wedge X_{(n)} = \det(X_{(1)}, \dots X_{(n)})\, e_{1 \dots n}} \qquad \text{"}\det\text{"} > \delta^{\alpha_1 \dots \alpha_n}_{1 \dots n} = \epsilon^{\alpha_1 \dots}$$

$$\det = \epsilon_{\alpha_1 \dots \alpha_n} \omega^{\alpha_1} \otimes \dots \otimes \omega^{\alpha_n}$$