Index conventions 1) a repeated pair of indices in an expression, 1 up & 1 down, imply a sum over the allowed values of the index.

2) a nonrepeated index in an expression is understood to take on all possible values, representing a ~~set~~ of expressions → or if it appears in an equation, a set of equations

$$\mathbb{R}^n = \{ \underbrace{(X^1, \ldots, X^n)} \mid X^\alpha \in \mathbb{R},\ \alpha = 1, \ldots, n \}$$

$$X = X^\alpha e_\alpha \qquad e_\alpha = \text{vector with } \alpha\text{th entry} = 1, \text{ all others zero.}$$

For matrix multiplication, we represent vectors as column vectors $\underline{X} = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$.

A linear function on $\mathbb{R}^n$ satisfies: $f(X) = f(X^\alpha e_\alpha) = X^\alpha f(e_\alpha)$.

A linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ may be:

$$\mathbb{B} : \mathbb{R}^n \to \mathbb{R}^n \qquad \text{Define} \quad \mathbb{B}(e_\alpha) = e_\beta B^\beta{}_\alpha$$

$$B(X) = B(X^\alpha e_\alpha) = X^\alpha B(e_\alpha) = X^\alpha B^\beta{}_\alpha e_\beta$$
$$= (B^\beta{}_\alpha X^\alpha) e_\beta$$

or in components $\quad Y^\alpha = B^\alpha{}_\beta X^\beta$

in matrix notation $\quad \underline{Y} = \underline{B}\,\underline{X} \quad$ if first (upper) index labels the rows and the second (lower) index labels the columns.

The matrix itself $\underline{B} = (B^\alpha{}_\beta)$ has as its columns, the column vectors which represent the images of the basis vectors:

$$\underline{B} = (\underline{B(e_1)}, \ldots, \underline{B(e_n)})$$

Each row of $\underline{B}$ on the other hand represents a covector (real valued linear function on vectors) whose row vector matrix multiplying the column vector of a point in $\mathbb{R}^n$ yields the corresponding component of the image vector with respect to the standard basis.

We can also use a linear transformation to change the basis of $\mathbb{R}^n$, provided that it is nonsingular (nonzero determinant), just the condition that the n image vectors of the standard basis be nondegenerate.

$$\mathbb{B} : \mathbb{R}^n \to \mathbb{R}^n \qquad \mathbb{B}(e_\alpha) = e_\beta \mathbb{B}^\beta{}_\alpha \equiv e_{\alpha'}$$

The matrix $\underline{\mathbb{B}} = (\mathbb{B}^\alpha{}_\beta) = (\mathbb{B}(e_1), \ldots, \mathbb{B}(e_n))$ has as its columns the column vectors representing the image vectors of the standard basis. These new vectors form a basis if they are linearly ~~not~~ independent, and can be used to express vectors instead of the standard basis.

A given vector has different components with respect to the two bases.

$$\underline{X} = \underline{X}^\alpha e_\alpha = \underline{X}^{\alpha'} e_{\alpha'} \qquad = \underline{X}^{\alpha'} B^\beta{}_{\alpha} e_\beta$$

so $$X^\beta = B^\beta{}_\alpha X^{\alpha'}$$

$$\underline{X} = \underline{B}\,\underline{X}'$$

~~Thus~~ or $$\underline{X}' = \underline{B}^{-1}\underline{X}$$

Let $\underline{A} \equiv B^{-1}$: $$\boxed{\underline{X}' = \underline{A}\,\underline{X}}$$

Contrast this with the original linear transformation of $\mathbb{R}^n$

$$\boxed{\underline{Y} = \underline{B}\,\underline{X} = \underline{A}^{-1}\underline{X}}.$$

In the second case we are changing the physical vector to a new vector (an active transformation) expressing its components before and after in the standard basis. In the first case the components of the same physical vector are being expressed in two different bases. (a passive transformation). They are related by the fact that the active transformation of the basis vectors leads to a passive transformation of the components of a given fixed vector.

$$\det(\underline{X}_{(1)}, \ldots, X_{(n)}) = \begin{cases} \text{real valued multilinear function of } n \text{ vectors} \\ 1 \text{ on standard basis} \qquad \det(\underline{e}_1, \ldots, e_n) = 1 \\ \text{gives zero if } \{X_{(1)}, \ldots, X_{(n)}\} \text{ linearly dependent} \\ \quad \text{i.e. one can write } \sum_{\alpha=1}^{n} C^{\alpha} X_{(\alpha)} = 0, \text{ not all } C^{\alpha} \text{ zero.} \\ \cancel{\text{linear}} \end{cases}$$

Since it must give zero any time an argument is repeated:

$$\det(\ldots, X, \ldots, X, \ldots) = 0$$

It must be antisymmetric $\qquad \det(\ldots, X, \ldots, Y, \ldots) = -\det(\ldots, Y, \ldots, X, \ldots)$

ie changes sign under a transposition of arguments

It then follows by definition that any permutation of the arguments changes the value by the sign of the permutation.

Let $\quad \epsilon_{\pi(1)\cdots\pi(n)} = \epsilon^{\pi(1)\cdots\pi(n)} = \mathrm{sgn}\begin{pmatrix} 1 & \cdots & n \\ \pi(1) & \cdots & \pi(n) \end{pmatrix}$.

Then $\qquad \det(e_{\pi(1)}, \cdots, e_{\pi(n)}) = \mathrm{sgn}\,\pi$

$$\det(X_{(1)}, \ldots, X_{(n)}) = \det(X^{\alpha_1}{}_{(1)} e_{\alpha_1}, \ldots, X^{\alpha_n}{}_{(n)} e_{\alpha_n})$$

$$= \det(e_{\alpha_1}, \ldots, e_{\alpha_n})\, X^{\alpha_1}{}_{(1)} \cdots X^{\alpha_n}{}_{(n)}$$

$$= \epsilon_{\alpha_1 \cdots \alpha_n} X^{\alpha_1}{}_{(1)} \cdots X^{\alpha_n}{}_{(n)}$$

Let $A^{\alpha}{}_{\beta} = X^{\alpha}{}_{(\beta)}$ be the matrix $\underline{A} = (\underline{X}_{(1)}, \ldots, \underline{X}_{(n)})$ of column vectors representing the n vectors $\{X_{(\alpha)}\}$.

Then we can define the <u>matrix determinant</u> by $\det \underline{A} = \underbrace{\epsilon_{\alpha_1 \cdots \alpha_n}} A^{\alpha_1}{}_1 \cdots A^{\alpha_n}{}_n$.

expanding the sum over all ~~possible~~ indices leads to 1 term for every permutation of $(1, \ldots, n)$, so we could also sum over all permutations of $(1, \ldots, n)$

$$\det \underline{A} = \sum_{\pi} \mathrm{sgn}\,\pi\, A^{\pi(1)}{}_1 \cdots A^{\pi(n)}{}_n.$$

This is the abstract definition of the determinant of a matrix. The geometric interpretation is that it produces the signed volume of the n-parallelopiped formed with the origin as a vertex and the ~~vecto~~ column vectors of the matrix as edges, where the sign tells whether the ordered set of column vectors has the same orientation (later) as the natural basis.

Algebraically it tests the set of image vectors of the standard basis under the linear transformation for linear dependence.

Looking at the original determinant as an alternating (antisymmetric) covariant ~~n~~- tensor (therefore accepting n vector arguments), we see that it tests for linear independence of a set of n vectors in $\mathbb{R}^n$.

Introduce the dual basis $\{\omega^\alpha\}$ to the standard basis $\{e_\alpha\}$ of $\mathbb{R}^n$, namely: $\omega^\alpha((x_1, \dots, x_n)) = x^\alpha$.

$$\det(\underline{X}_{(1)}, \dots, \underline{X}_{(n)}) = \det(e_{\alpha_1}, \dots, e_{\alpha_n}) \dots X^{\alpha_1} \dots X^{\alpha_n}$$

$$= \det(e_{\alpha_1} \dots e_{\alpha_n})\, \omega^{\alpha_1}(\underline{X}_{(1)}) \cdots \omega^{\alpha_n}(\underline{X}_{(n)})$$

$$= \underbrace{\det(e_{\alpha_1}, \dots, e_{\alpha_n})}_{\epsilon_{\alpha_1, \dots, \alpha_n}}\, \omega^{\alpha_1} \otimes \dots \otimes \omega^{\alpha_n} (\underline{X}_{(1)}, \dots, \underline{X}_{(n)})$$

$$= \epsilon_{\alpha_1 \dots \alpha_n}\, \omega^{\alpha_1} \otimes \dots \otimes \omega^{\alpha_n} (\underline{X}_{(1)}, \dots, \underline{X}_{(n)})$$

So $\det = \underbrace{\epsilon_{\alpha_1 \dots \alpha_n}}_{\text{components of } \binom{0}{n} \text{ tensor "det" (wrt standard basis)}} \omega^{\alpha_1} \otimes \dots \otimes \omega^{\alpha_n}$ as a tensor.

How many independent (nonzero) components does a totally antisymmetric $\binom{0}{n}$-tensor over $\mathbb{R}^n$ have? Just 1: $\epsilon_{1 \dots n} = 1$; every other component is related to this by a sign.

In fact if $T = T_{\alpha_1 \dots \alpha_n} \omega^{\alpha_1} \otimes \dots \otimes \omega^{\alpha_n}$ is totally antisymmetric, then all the indices must be distinct to get a nonzero result and $T_{\alpha_1 \dots \alpha_n} = \epsilon_{\alpha_1 \dots \alpha_n} T_{1 \dots n}$, ie every nonzero component differs at most in sign from $T_{1\dots n}$.

The space of antisymmetric $\binom{0}{n}$ tensors over $\mathbb{R}^n$ has dimension $\underline{1}$ as a vector space. The determinant function acts as a basis for this 1-dimensional vector space. Any antisymmetric $\binom{0}{n}$ tensor is just some constant times the determinant tensor.

Linear independence of a set of fewer vectors in $\mathbb{R}^n$ leads to more standard antisymmetric tensors. These come from antisymmetrizing the tensor product of the vectors.

Going back to the determinant, for example,

$$\det(X_{(1)}, \dots, X_{(n)}) = \epsilon_{\alpha_1 \dots \alpha_n} \underbrace{X^{\alpha_1}_{(1)} \cdots X^{\alpha_n}_{(n)}}$$

$$= X^{[\alpha_1}_{(1)} \cdots X^{\alpha_n]}_{(n)}$$

$$= X^{[1}_{(1)} \cdots X^{n]}_{(n)}$$

only the antisymmetric part of the n-fold tensor product of these ~~n~~ vectors enters into the sum.

The determinant evaluated on these vectors just produces the component $T^{1\dots n}$ of the $\binom{n}{0}$-tensor $X^{[\alpha_1}_{(1)} \cdots X^{\alpha_n]}_{(n)}$.

The antisymmetrized tensor product is the thing which tests linear independence since if you add a linear combination of a set of vectors to the set & evaluate the antisymmetrized tensor product you get zero.

$$X^{[\alpha_1}_{(1)} X^{\alpha_2}_{(2)} (aX^{\alpha_3]}_{(1)} + bX^{\alpha_3]}_{(2)}) = a \underbrace{(X^{[\alpha_1}_{(1)} X^{\alpha_2}_{(2)} X^{\alpha_3]}_{(1)})} + b(X^{[\alpha_1}_{(1)} X^{\alpha_2}_{(2)} X^{\alpha_3]}_{(2)})$$

$$= -X^{[\alpha_1}_{(1)} X^{\alpha_2}_{(2)} X^{\alpha_3]}_{(1)} \quad \therefore = 0$$

same here

If you interchange the two $X_{(1)}$ factors you change the sign but interchanging them leaves the tensor product unchanged so it must be zero.

All of this is just to motivate introducing an antisymmetrized tensor product operation $\wedge$ which is just $\otimes$ plus antisymmetrization on all indices or equivalently arguments.

Define $$\mathbb{X}_{(1)}\wedge\cdots\wedge\mathbb{X}_{(k)} = k!\ \mathrm{ALT}\ \mathbb{X}_{(1)}\otimes\cdots\otimes\mathbb{X}_{(n)}$$

or $$\mathbb{X}_{(1)}\wedge\cdots\wedge\mathbb{X}_{(k)}{}^{\alpha_1\cdots\alpha_k} = \underbrace{k!}_{\text{for convention.}}\ \mathbb{X}_{(1)}^{[\alpha_1}\cdots\mathbb{X}_{(k)}^{\alpha_k]} = \delta^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_k}\mathbb{X}_{(1)}^{\beta_1}\cdots\mathbb{X}_{(k)}^{\beta_k}$$

Now read detour on Kronecker deltas.

Wedge is ~~an~~ extended by assuming it to be an associative operation, ie

$$(\underbrace{\mathbb{X}_{(1)}\wedge\cdots\wedge\mathbb{X}_{(p)}}_{\text{defined}}) \underset{\substack{\uparrow \\ \text{still undefined.}}}{\wedge} (\underbrace{\mathbb{X}_{(p+1)}\wedge\cdots\wedge\mathbb{X}_{(p+q)}}_{\text{defined}}) = \underbrace{\mathbb{X}_{(1)}\wedge\cdots\wedge\mathbb{X}_{(p+q)}}_{\text{defined}}.$$

Generalized Kronecker deltas

$$\delta^{a_1 a_2}_{b_1 b_2} = \begin{vmatrix} \delta^{a_1}_{b_1} & \delta^{a_1}_{b_2} \\ \delta^{a_2}_{b_1} & \delta^{a_2}_{b_2} \end{vmatrix} = \delta^{a_1}_{b_1}\delta^{a_2}_{b_2} - \delta^{a_1}_{b_2}\delta^{a_2}_{b_1}$$

$$\delta^{a_1 a_2 a_3}_{b_1 b_2 b_3} = \begin{vmatrix} \delta^{a_1}_{b_1} & \delta^{a_1}_{b_2} & \delta^{a_1}_{b_3} \\ \delta^{a_2}_{b_1} & \delta^{a_2}_{b_2} & \delta^{a_2}_{b_3} \\ \delta^{a_3}_{b_1} & \delta^{a_3}_{b_2} & \delta^{a_3}_{b_3} \end{vmatrix} = \delta^{a_1}_{b_1}\delta^{a_2}_{b_2}\delta^{a_3}_{b_3} - \delta^{a_1}_{b_1}\delta^{a_2}_{b_3}\delta^{a_3}_{b_2} + \delta^{a_1}_{b_2}\delta^{a_2}_{b_3}\delta^{a_3}_{b_1} - \delta^{a_1}_{b_2}\delta^{a_2}_{b_1}\delta^{a_3}_{b_3} + \delta^{a_1}_{b_3}\delta^{a_2}_{b_1}\delta^{a_3}_{b_2} - \delta^{a_1}_{b_3}\delta^{a_2}_{b_2}\delta^{a_3}_{b_1}$$

$$\delta^{a_1\cdots a_r}_{b_1\cdots b_r} = \begin{vmatrix} \delta^{a_1}_{b_1} & \cdots & \delta^{a_1}_{b_r} \\ \vdots & & \\ \delta^{a_r}_{b_1} & \cdots & \delta^{a_r}_{b_r} \end{vmatrix} = \epsilon^{\sigma(1)\cdots\sigma(r)}\,\delta^{a_1}_{b_{\sigma(1)}}\cdots\delta^{a_r}_{b_{\sigma(r)}} = r!\,\delta^{a_1}_{[b_1}\cdots\delta^{a_r}_{b_r]} = \begin{cases} 0 \text{ unless } b_i = \pi(a_i) \\ \operatorname{sgn}\pi \text{ if } b_i = \pi(a_i) = \operatorname{sgn}\begin{pmatrix} a_1 \cdots a_r \\ b_1 \cdots b_r \end{pmatrix} \end{cases}$$

$1 \le r \le n = \dim$ space.

$$\epsilon^{a_1\cdots a_n} = \delta^{a_1\cdots a_n}_{1\cdots n} = \begin{cases} 0 & (a_1\ldots a_r) \neq \pi(1\ldots n) \\ \operatorname{sgn}\begin{pmatrix} a_1 \cdots a_r \\ 1 \cdots n \end{pmatrix} \end{cases} \qquad n = \dim \text{ space}.$$

$$\epsilon_{a_1\cdots a_n} = \delta^{1\cdots n}_{a_1\cdots a_n} = \begin{cases} \operatorname{sgn}\begin{pmatrix} 1 \cdots n \\ a_1 \cdots a_r \end{pmatrix} \\ 0 & (a_1\ldots a_r) \neq \pi(1\ldots n) \end{cases}$$

$$\epsilon^{a_1\cdots a_r a_{r+1}\cdots a_n}\,\epsilon_{b_1\cdots b_r a_{r+1}\cdots a_n} \propto \delta^{a_1\cdots a_r}_{b_1\cdots b_r}$$

computation

$$= (n-r)!\,\delta^{a_1\cdots a_r}_{b_1\cdots b_r}$$

~~Another computation shows~~ For example if $r=0$:

$$\delta^{a_1\cdots a_n}_{b_1\cdots b_n} = \epsilon^{a_1\cdots a_n}\epsilon_{b_1\cdots b_n},$$

Another computation shows:

$$\delta^{a_1\cdots a_r}_{b_1\cdots b_r} = \frac{(n-k)!}{(n-r)!}\,\delta^{a_1\cdots a_r a_{r+1}\cdots a_k}_{b_1\cdots b_r a_{r+1}\cdots a_k}$$

Generalized Kronecker Deltas & antisymmetrization

$$T_{[a_1\cdots a_k]} \equiv \frac{1}{k!}\,\delta^{b_1\cdots b_k}_{a_1\cdots a_k}\,T_{b_1\cdots b_k} \quad = \text{antisymmetric part of } T_{b_1\cdots b_k}.$$

If $T_{a_1\cdots a_k}$ is already antisymmetric then this sum, for a fixed k-tuplet $(a_1,\dots,a_k)$, consists of $k!$ nonzero terms (all the permutations of this ordering), each of which is the sign of the permutation times the permuted component divided by $k!$. But the permuted compnt differs from the component $T_{a_1\cdots a_k}$ only by the sign of the permutation, hence the sign squares to 1, and all $k!$ terms are equal to $\frac{1}{k!}$ ~~yielding~~ times $T_{a_1\cdots a_k}$, yielding $T_{a_1\cdots a_k}$ as the result. (We assume $(a_1,\dots,a_k)$ are distinct, otherwise both sides are zero).

If we have a $\binom{0}{k}$-tensor:

$$T = T_{\alpha_1\cdots\alpha_k}\,\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_k}$$

$$T_{\alpha_1\cdots\alpha_k} = T(e_{\alpha_1},\dots,e_{\alpha_k})$$

then

$$\mathrm{ALT}\,T\,(X_{(1)},\dots,X_{(k)}) \equiv \frac{1}{k!}\sum_{\pi} \operatorname{sgn}\pi\; T(X_{(\pi(1))},\cdots,X_{(\pi(k))})$$

$$= \frac{1}{k!}\,\delta^{\alpha_1\cdots\alpha_k}_{1\cdots k}\,T(X_{(\alpha_1)},\cdots,X_{(\alpha_k)})$$

$$\mathrm{ALT}\,T\,(e_{\alpha_1},\dots,e_{\alpha_k}) = \frac{1}{k!}\,\delta^{\alpha_1\cdots\alpha_k}_{1\cdots k}\,\underbrace{T(e_{\alpha_1},\dots,e_{\alpha_k})}_{T_{\alpha_1\cdots\alpha_k}}$$

$$= T_{[\alpha_1\cdots\alpha_k]}$$

i.e.

$$\mathrm{ALT}\,T = \underbrace{T_{[\alpha_1\cdots\alpha_k]}}_{=\frac{1}{k!}\delta^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k}T_{[\beta_1\cdots\beta_k]}}\,\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_k} = \frac{1}{k!}\,T_{[\beta_1\cdots\beta_k]}\,\underbrace{\delta^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k}\,\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_k}}_{\equiv\,\omega^{\alpha_1}\wedge\cdots\wedge\omega^{\alpha_k}\,\equiv\,\omega^{\alpha_1\cdots\alpha_k}}$$

The $\binom{0}{k}$ tensors $\omega^{\alpha_1\cdots\alpha_k} = \omega^{\alpha_1}\wedge\cdots\wedge\omega^{\alpha_k}$ are a "basis" for antisymmetric $\binom{0}{k}$ tensors.

But for each k-tuplet of distinct index values $(\alpha_1,\ldots,\alpha_k)$ there are $k!$ different permutations which lead to only a sign change in the tensor. Pick the ordered permutation from this set of $k!$ permutations, i.e. first index < second index < ... < last index.

These ordered indexed tensors are the basis for antisymmetric $\binom{0}{k}$-tensors.

What is the dimension of this vector space.

The number of combinations of n things taken k at a time: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

In particular for $k=n$, the dimension is 1, and $\omega^1\wedge\cdots\wedge\omega^n = \omega^{1\cdots n}$ is a basis of the space.

Its value on n vector arguments is just the determinant:

$$\omega^{1\cdots n}(X_{(1)},\ldots,X_{(n)}) = \det(X_{(1)},\ldots,X_{(n)}).$$

since

$$= \delta^{1\cdots n}_{\alpha_1\cdots\alpha_n}\,\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_n}(X_{(1)},\ldots,X_{(n)})$$

$$\epsilon_{\alpha_1\cdots\alpha_n} X^{\alpha_1}_{(1)}\cdots X^{\alpha_n}_{(n)}$$

In other words "det" as a tensor equals $\omega^{1\cdots n}$.

But the space of antisymmetric $\binom{n}{0}$-tensors is also 1-dimensional and

$$X_{(1)}\wedge\cdots\wedge X_{(n)} = \delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n} X^{\beta_1}_{(1)}\cdots X^{\beta_n}_{(n)}\, e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_n}$$

$$= X^{\beta_1}_{(1)}\cdots X^{\beta_n}_{(n)}\ \underbrace{\delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n} e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_n}}_{\equiv e_{\beta_1}\wedge\cdots\wedge e_{\beta_n} \equiv e_{\beta_1\cdots\beta_n}}$$

$$= \underbrace{X^{[\beta_1}_{(1)}\cdots X^{\beta_n]}_{(n)}}_{\frac{1}{n!}\det(X_{(1)},\ldots X_{(n)})\,\epsilon^{\beta_1\cdots\beta_n}} e_{\beta_1\cdots\beta_n} = \det(X_{(1)},\ldots,X_{(n)})\, e_{1\cdots n}$$

since $\epsilon^{\beta_1\cdots\beta_n} e_{\beta_1\cdots\beta_n} = n!\, e_{1\cdots n}$

In words; the wedge product of n vectors is the determinant of the matrix of their components wrt a set of basis vectors times the ordered wedge product of all the basis vectors.

If one has two bases, the corresponding n-vectors are related by the determinant of the matrix of components of one with respect to the other.

belongs to the 1-dim space for which $e_{1\dots n}$ is a basis, & the coefficient in this basis is just the determinant of the set of vectors.

The wedge product has the property that the wedge product of a set of linearly independent vectors is nonzero, since if the set is not linearly independent, antisymmetrization kills the resulting tensor product.

A basis for antisymmetric $\binom{k}{0}$-tensors is

$$\cancel{e_{\alpha_1\dots\alpha_k} = e^{\alpha_1}\wedge\dots\wedge e^{\alpha_k} = \delta^{\alpha_1\dots\alpha_k}_{\beta_1\dots\beta_k}\, e}$$

$$e_{\alpha_1\cdots\alpha_k} = e_{\alpha_1}\wedge\dots\wedge e_{\alpha_k} = \underbrace{\delta^{\beta_1\dots\beta_k}_{\alpha_1\dots\alpha_k}}\, e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_k}$$

generalized Kronecker delta represents components of tensor $e_{\alpha_1\dots\alpha_k}$

or of $\omega^{\alpha_1\dots\alpha_k}$ from previous discussion.

Again the independent basis elements consist of the ordered indexed tensors, and there are $k!$ reorderings for every such k-tuplet.

For an antisymmetric tensor

$$\begin{aligned}
T &= T_{\alpha_1\dots\alpha_k}\,\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_k} \\
&= T_{[\alpha_1\dots\alpha_k]}\,\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_k} \\
&= \frac{1}{k!}\,\delta^{\beta_1\dots\beta_k}_{\alpha_1\dots\alpha_k}\,T_{\beta_1\dots\beta_k}\,\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_k} = \frac{1}{k!}\,T_{\beta_1\dots\beta_k}\,\omega^{\beta_1\dots\beta_k} \\
&\equiv T_{|\beta_1\dots\beta_k|}\,\omega^{\beta_1\dots\beta_k}
\end{aligned}$$

indicates sum only over ordered k-tuplets of index values.

Wedge product of wedge products:

Given
$$S = S_{\alpha_1\cdots\alpha_p}\,\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_p} = \frac{1}{p!}\,S_{\alpha_1\cdots\alpha_p}\,\omega^{\alpha_1}\wedge\cdots\wedge\omega^{\alpha_p}$$
$$T = T_{\alpha_1\cdots\alpha_q}\,\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_q} = \frac{1}{q!}\,T_{\alpha_1\cdots\alpha_q}\,\omega^{\alpha_1}\wedge\cdots\wedge\omega^{\alpha_q}$$
antisymmetric $\binom{0}{k}$ tensors, one can wedge them togeher

$$S\wedge T = \left(\frac{1}{p!}S_{\alpha_1\cdots\alpha_p}\,\omega^{\alpha_1}\wedge\cdots\wedge\omega^{\alpha_p}\right)\wedge\left(\frac{1}{q!}T_{\alpha_{p+1}\cdots\alpha_{p+q}}\,\omega^{\alpha_{p+1}}\wedge\cdots\wedge\omega^{\alpha_{p+q}}\right)$$

↑ need different index symbols

$$= \frac{1}{p!q!}\,S_{\alpha_1\cdots\alpha_p}T_{\alpha_{p+1}\cdots\alpha_{p+q}}\,\underbrace{\left(\omega^{\alpha_1}\wedge\cdots\wedge\omega^{\alpha_p}\wedge\omega^{\alpha_{p+1}}\wedge\cdots\wedge\omega^{\alpha_{p+q}}\right)}_{\omega^{\alpha_1\cdots\alpha_{p+q}} = \frac{1}{(p+q)!}\delta^{\alpha_1\cdots\alpha_{p+q}}_{\beta_1\cdots\beta_{p+q}}\,\omega^{\beta_1\cdots\beta_{p+q}}}$$

$$= \cancel{\frac{1}{p!q!}\,S_{[\alpha_1\cdots\alpha_p}}$$

$$= \cancel{\frac{1}{p!q!}\,S_{[\alpha_1\cdots\alpha_p}T_{\alpha_{p+1}\cdots\alpha_{p+q}]}\,\omega^{\alpha_1\cdots\alpha_{p+q}}}$$

$$= \underbrace{\frac{1}{p!q!}\,S_{[\beta_1\cdots\beta_p}T_{\beta_{p+1}\cdots\beta_{p+q}]}}_{\frac{1}{(p+q)!}(S\wedge T)_{\beta_1\cdots\beta_{p+q}}}\,\omega^{\beta_1\cdots\beta_{p+q}}.$$

so
$$\boxed{(S\wedge T)_{\beta_1\cdots\beta_{p+q}} = \frac{(p+q)!}{p!q!}\,S_{[\beta_1\cdots\beta_p}T_{\beta_{p+1}\cdots\beta_{p+q}]}}$$

or
$$S\wedge T = \frac{(p+q)!}{p!q!}\,\mathrm{ALT}(S\otimes T)$$

↑ this accounts for the weird factorial factor.

For example: $n=3$. The following bases for antisymmetric tensors arise.

$\binom{0}{1}$-tensors $\quad \omega^1, \omega^2, \omega^3$

$\binom{0}{2}$-tensors $\quad \omega^{12}, \omega^{13}, \omega^{23}$

$\binom{0}{3}$-tensors $\quad \omega^{123}$

same #

This is true in general by symmetry of binomial coefficients.

Define $\Lambda^k(V) = \underbrace{V \wedge \dots \wedge V}_{k \text{ times}}$ = space of all possible antisymmetric "k-~~tensors~~" (contravariant)

and $\Lambda^k(V^*) = \underbrace{V^* \wedge \dots \wedge V^*}_{k \text{ times}}$ = same for covariant tensors

where $V = \mathbb{R}^n$ in these notes.

$$\dim \Lambda^k(V) = \dim \Lambda^{n-k}(V)$$

$$\binom{n}{k} \qquad \binom{n}{n-k}$$

$$\frac{n!}{k!\,n-k!} = \frac{n!}{(n-k)!\,k!}$$

so antisymmetric $(n-1)$-tensors have the same # of independent components as 1-tensors, ie vectors & covectors.