NEW MATH ELECTIVE

FALL 1989 MAT 5600:

DIFFERENTIAL GEOMETRY

Prerequisites: MAT 3305 (Advanced Calc II, for math majors, others don't worry)
MAT 3400 (Linear Algebra)

Lecturer: Bob Jantzen

Tensor algebra, curvilinear coordinates, manifolds, Riemannian and pseudo-Riemannian geometry, some Lie group theory.

Elementary linear algebra is reformulated and extended to describe the algebra of tensors over a real vector space (just linear functions of multiple vector arguments). Elementary multivariable calculus on Euclidean space is reformulated to identify the tangent space in a way which may be extended to a manifold (like a surface or a "curved space"). Putting the two together carries over the tensor algebra to the tangent space at each point of a manifold to become the algebra of tensor fields over the manifold. Exterior and covariant derivatives are introduced with a little Riemannian geometry to generalize vector calculus and Gauss's and Stokes' Theorems and to better understand div, grad, curl, ∇, ∇^2, etc in orthogonal coordinate systems. As a bonus one can generalize the straight lines of Euclidean space and the great circles on the sphere to "geodesics" of any curved manifold (locally the connecting paths of shortest distance), which is how general relativity predicts the orbits of planets, for example. For this one must discuss a non-positive-definite extension of the familiar dot product, leading to the pseudo-Riemannian geometry of relativity. If time permits, a little bit of Lie group theory will be discussed, generalizing what we know about the rotations and translations of ordinary space to more general groups of transformations.

Get a feeling for the tool which has pushed forward much of modern mathematics and physics, the language of special and general relativity, electromagnetism, unified theories, mechanics, dynamical systems, etc.

DON'T LET THE VOCABULARY SCARE YOU. FOR MORE INFORMATION CALL BOB AT 645–7335.

At beginning of course defined

direct sum $\oplus$

and compare with cartesian product $\times$.

The Limitations of Standard Elementary Linear Algebra.

The ~~algebra~~ geometry of $\mathbb{R}^n$ and linear maps ~~between different~~ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is treated in a way inseparable from the Euclidean geometry of these spaces. A change of basis from the standard orthonormal basis to a general basis is touched upon but never used to evaluate quantities like the dot product. Doing standard Euclidean geometry in a nonorthonormal basis ~~be~~ is not covered, for example, and so one doesn't have the machinery to do the linear algebra associated with the differential structure on a bare manifold where no inner product is available. One must therefore redo standard elementary linear algebra in a way which distinguishes the natural machinery of linearity from the Euclidean geometry. This allows the generalization to tensor analysis which too often is itself confused with coordinate transformation machinery since they are often developed as an inseparable unit.

② first backup, $\mathbb{R}^n$ ~~fra~~ wedge product

① Too ambitious start

③ volume & determinants

④ duality

⑤ Backup. Elementary linear algebra notation generalized

V real vectorspace, n-dimensional

1) real linear combinations defined and belong to V, $+/\cdot$ operators satisfy usual rules.

2) max # lin. ind. vectors in any set.

set of max # lin. ind vectors called a $\boxed{\text{basis}}$:

$$\{e_1, \dots, e_n\} = \{e_\alpha\} \qquad \alpha = 1, \dots, n$$

$X = X^\alpha e_\alpha$ any other vector lin dep & can therefore be rep as a linear combination, coefficients called $\boxed{\text{components}}$ wrt basis.

Implied sum on any repeated pair of indices, 1 up 1 down

$f : V \to W$: linear map from V to W or a W-valued linear function on V

$f(X) = X^\alpha \underbrace{f(e_\alpha)}$ linearity condition: $f(\text{linear cmb}) = \text{linear cmb}(f)$

elements of W, expand in basis of W say: $\{E_i\}$, $i = 1, \dots, m = \dim W$

$$f(e_\alpha) \equiv f^\beta{}_\alpha E_\beta$$

$$f(X) = X^\alpha F^\beta{}_\alpha E_\beta = (f^i{}_\alpha X^\alpha) E_i = Y = Y^i E_i$$

components wrt $\{E_i\}, \{e_\alpha\}$ of linear map:

$$Y^i = f^i{}_\alpha X^\alpha \quad \left(\xrightarrow[\text{notation}]{\text{matrix}}\right) \quad \underline{Y} = \underline{f}\,\underline{X}$$

row index, column index

matrix

dim m $m \times n$ matrix   dim n

~~column vectors~~ vectors in V, W represented by column vectors of components wrt chosen bases in each space.

$\mathbb{R}$ is itself a real 1-dim ~~vector~~ space, standard basis: 1

$f : V \to \mathbb{R}$ real valued linear function on V ..

$$f(X) = X^\alpha f(e_\alpha) = f_\alpha X^\alpha = \underline{f}\,\underline{X} \in \mathbb{R}$$

row vector (1 row matrix)   column vector

These are called "covectors" or sometimes "1-forms."

Let the "dual vectorspace" V^* be the space of real valued linear functions on V itself an n-dim vector space since

$$\cancel{f(aX + bX)} \quad (af + bg)(X) = af(X) + bg(X) \quad \text{defines linear cmb's} \in V^*$$

and we can produce a basis of V^* given a basis $\{e_\alpha\}$ of V

Define $\omega^\alpha(e_\beta) = \delta^\alpha_\beta = \begin{cases} 1 \text{ if } \alpha=\beta \\ 0 \text{ if } \alpha\neq\beta \end{cases}$ and extend to be linear functions on V:

$$\omega^\alpha(X) = \omega^\alpha(X^\beta e_\beta) = X^\beta \omega^\alpha(e_\beta) = X^\beta \delta^\alpha_\beta = \underbrace{\delta^\alpha{}_\beta X^\beta} = X^\alpha$$

ω^α picks out αth comp of X wrt the basis e_α

$$f(X) = f(X^\alpha e_\alpha) = X^\alpha f(e_\alpha) = \underbrace{f_\alpha}_{\substack{\text{components} \\ \text{of row vector}}} X^\alpha =$$

$$= f_\alpha \omega^\alpha(X) = (f_\alpha \omega^\alpha)(X)$$

They are linearly indepent since

$$g_\beta \omega^\beta = 0 \rightarrow \underbrace{g_\beta \omega^\beta(e_\alpha)}_{g_\beta} = 0$$

each coefficient must vanish.

so $(f - f_\alpha \omega^\alpha)(X) = 0$ for every $X \in V$

but if a linear function is identically zero, it is the zero function:

$$\left[0 = f(X) = X^\alpha \underbrace{f(e_\alpha)}_{0} = 0 \right.$$ if zero on basis, zero for every vector]

so $f = f_\alpha \omega^\alpha$ so every covector can be written as a linear combination of the basis covectors $\{\omega^\alpha\}$, they are linearly ind., therefore a basis (since adding any other covector to the set makes it linearly dep).

$\{\omega^\alpha\}$ is called the basis dual to $\{e_\alpha\}$ or just the dual basis.

V^* is itself an n-dim vector space so it has a dual space $(V^*)^* \equiv V^{**}$ of real valued linear functions on covectors.

$$F(f_\alpha \omega^\alpha) = f_\alpha \underbrace{F(\omega^\alpha)}_{\equiv F^\alpha} = \underbrace{f_\alpha}_{f(e_\alpha)} F^\alpha = F^\alpha f(e_\alpha) = f(\underbrace{F^\alpha e_\alpha}_{\mathbb{F} \in V})$$

with each linear function on V^*, we can associate a vector $\mathbb{F}$ in V so that the value of F on the covector f equals the value of the covector f on the vector $\mathbb{F}$

so there is a natural identification of V^{**} with V itself and we can terminate the process of taking the dual space. We only need V and V^*, identifying V with V^{**} in this way.

So we've got vectors and covectors and we can evaluate covectors on vectors

$$f(X) = f_\alpha X^\alpha$$

or vectors on covectors $X(f) \equiv f(X) = f_\alpha X^\alpha$ by this defining interchange of vector & covector.

(linear in each argument, holding all others fixed)

A $\boxed{\text{tensor over } V}$ is just a multilinear function with a certain number of vector and covector arguments in a definite order (which characterizes the type of tensor). (The total number k of arguments is called the rank of the "tensor")

1 argument: vector (v), covector (c) — ~~tensor~~ tensor is a covector = covariant, vector = contravariant } 1-tensor — 2^1 possibilities

2 arguments: ~~(vec, vec~~

v → v, c; c → v, c

(v,v) — covariant

(v,c), (c,v) } mixed

(c,c) — contravariant

} 2-tensor — 2^2 possibilities.

(describe mixed as ~~etc.~~ covariant in first argu contravariant in second, etc.)

etc.

The space of tensors of a fixed type themselves form a vector space since it make sense to take linear combinations of them since they have the same argument structure.

$$a\,T(X,f) + b\,S(X,f) \equiv (aT+bS)(X,f)$$

1st arg vector, 2nd arg covector.

$\equiv$ covariant in first argument $\equiv$ contravariant in second

We would therefore like to find a basis of each such vector space

In the same way that a basis $\{e_\alpha\}$ ~~of a~~ V induces a basis $\{\omega^\alpha\}$ of V^* the two bases together induce a basis for each such space of tensors.

<u>How?</u> First we need the tensor product of two tensors.

The product $T(X,f,g)\;S(Y,h)$ is a multilinear function with 5 arguments

$$\equiv \underbrace{(T\otimes S)}(X,f,g,Y,h)$$

This is the name of that tensor, called the tensor product of T and S.

Obviously associative, since mult of real numbers is associative.

$$T(X,f,g)\,S(Y,h)\,U(Z,\bar{W}) = (T\otimes S\otimes U)(X,f,g,Y,h,Z,\bar{W})$$

Now consider the space of 3-tensors with 1-vector argument and 2 covector arguments in that order:

$$T(X,f,g) = T(X^\alpha e_\alpha, f_\beta \omega^\beta, g_\gamma \omega^\gamma) = X^\alpha f_\beta g_\gamma \underbrace{T(e_\alpha,\omega^\beta,\omega^\gamma)}_{\equiv T_\alpha{}^{\beta\gamma}}$$

called the components of T wrt the basis $\{e_\alpha\}$

$$= \omega^\alpha(X)\, f(e_\beta)\, g(e_\gamma)\; T_\alpha{}^{\beta\gamma}$$

$$= \underbrace{\omega^\alpha(X)\, e_\beta(f)\, e_\gamma(g)}\; T_\alpha{}^{\beta\gamma} \qquad = T_\alpha{}^{\beta\gamma}\, \omega^\alpha\otimes e_\beta\otimes e_\gamma\,(X,f,g)$$

$$\equiv \underbrace{\omega^\alpha\otimes e_\beta\otimes e_\gamma}_{n\times n\times n \text{ different tensor products.}}(X,f,g)$$

so $T = T_\alpha{}^{\beta\gamma}\, \omega^\alpha\otimes e_\beta\otimes e_\gamma$

so an arbitrary tensor can be expanded in terms of these basis tensors.

linearly independent since if

$$T_\alpha{}^{\beta\gamma}\, \omega^\alpha\otimes e_\beta\otimes e_\gamma = 0$$

$$0 = (T_\alpha{}^{\beta\gamma}\, \omega^\alpha\otimes e_\beta\otimes e_\gamma)(e_\gamma, \omega^\mu, \omega^\nu)$$

$$= T_\alpha{}^{\beta\gamma}\, \omega^\alpha(e_\gamma)\, \omega^\mu(e_\beta)\, \omega^\nu(e_\gamma)$$

$$= T_\alpha{}^{\beta\gamma}\, \delta^\alpha{}_\gamma\, \delta^\mu{}_\beta\, \delta^\nu{}_\gamma \qquad = T_\gamma{}^{\mu\nu}$$

so ~~all~~ all coefficients zero, therefore a basis.

Thus these spaces of real valued multilinear functions with a certain number of vector and covector arguments (in which they are covariant & contravariant respectively) are naturally called tensor product spaces. The space of this example we can denote by $V^*\otimes V\otimes V$ since any element can be written as a linear combination of tensor products of a covector and two vectors in that order.

The shorthand $\otimes^k V^* = \underbrace{V^*\otimes\cdots\otimes V^*}_{k \text{ factors}}$ denotes the space of totally covariant k-tensors, covariant in each argument, accepting k vector arguments (contravariant arguments) etc.

$$\uparrow$$

So:

$$V\otimes V,\ V\otimes V^*,\ V^*\otimes V,\ V^*\otimes V \qquad \text{2-tensors}$$

$$\uparrow$$

$$V \leftrightarrow V^* \qquad \text{1 tensors}$$

above each vector space V, we can build a hierarchy of tensor product spaces, and each basis $\{e_\alpha\}$ of V induces a dual basis & bases of all tensor product spaces. This is called the tensor algebra over V (the tensor product allows tensors to be multiplied, etc...).

<u>CONTRACTION</u> The tensor product increases the number of arguments. Tensor contraction reduces them in pairs, one vector and one covector.

EX. $T = T_\alpha{}^{\beta\gamma}\, \omega^\alpha \otimes e_\beta \otimes e_\gamma$

$$C_{1,2}\, T \doteq T_\alpha{}^{\beta\gamma}\, \underbrace{\omega^\alpha(e_\beta)}_{\delta^\alpha_\beta}\, e_\gamma = T_\beta{}^{\beta\gamma}\, e_\gamma$$

" contraction of T on first & second arguments "

This requires a basis to define, so we should check that it doesn't depend on the basis. LATER.

In components we just sum over the chosen index pair, 1 up, 1 down.

<u>EX</u> $X \otimes T = (X^\delta e_\delta) \otimes (T_\alpha{}^{\beta\gamma}\omega^\alpha \otimes e_\beta \otimes e_\gamma) = X^\delta T_\alpha{}^{\beta\gamma}\, e_\delta \otimes \omega^\alpha \otimes e_\beta \otimes e_\gamma$

$$C_{1,2}\, X \otimes T = X^\alpha T_\alpha{}^{\beta\gamma}\, e_\beta \otimes e_\gamma = C_X T$$

(contraction by X in first vector argument)

This results in the partial evaluation of the tensor T on the vector X to leave a ~~vector~~ tensor with 1 less vector argument.

We can partially evaluate any tensor on any number of arguments to obtain tensors of lesser rank.

SYMMETRY

Suppose we have a k-covariant tensor $T = T_{\alpha_1 \dots \alpha_k}\, \omega^{\alpha_1} \otimes \dots \otimes \omega^{\alpha_k}$;

$$T(X_{(1)}, \dots, X_{(k)}) = T_{\alpha_1 \dots \alpha_k} X_{(1)}^{\alpha_1} \cdots X_{(k)}^{\alpha_k}$$

$$\text{If } T(\dots, X_{(i)}, \dots, X_{(j)}, \dots) = \begin{cases} T(\dots, X_{(j)}, \dots, X_{(i)}, \dots) & \sim \text{ symmetric} \\ -T(\dots, X_{(j)}, \dots, X_{(i)}, \dots) & \sim \text{ antisymmetric (alternating)} \end{cases}$$

is true ~~for~~ upon interchange of every pair of arguments,
T is said to be respectively symmetric or antisymmetric (alternating).

In this case the value of 1 permutation of the arguments determines all permutations, with the sign of the permutation entering in the antisymmetric case.

One can always "project out" the totally symmetric or antisymmetric part of any tensor:

$$(\mathrm{SYM}\ T)(X_{(1)}, \dots, X_{(k)}) \equiv \frac{1}{k!} \sum_{\sigma \in S_k} T(X_{(\sigma(1))}, \dots, X_{(\sigma(k))})$$

$$(\mathrm{ALT}\ T)(X_{(1)}, \dots, X_{(k)}) \equiv \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}\sigma \cdot T(X_{(\sigma(1))}, \dots, X_{(\sigma(k))})$$

where S_k is the set of permutations of the numbers 1 to k.

If a tensor is already symmetric / antisymmetric, these operations respectively will give back the same tensor. Thus ~~SYM^2 and ALT^2~~

$\mathrm{SYM}^2 = \mathrm{SYM}$ and $\mathrm{ALT}^2 = \mathrm{ALT}$ as operations

They are clearly linear operations. $\mathrm{SYM}(aT + bS) = a(\mathrm{SYM}\ T) + b(\mathrm{SYM}\ S)$, etc.
so they are "linear projections" onto the subspaces of totally symmetric / antisymmetric k-covariant tensors.

~~Define $\Theta^{\beta_1 \dots \beta_k}{}_{\alpha_1 \dots \alpha_k}$~~

SYMMETRY AND MIXED TENSORS OR PARTIAL SYMMETRY

For an arbitrary type tensor, one can select a subset of arguments of a given type, either covariant or contravariant, and discuss symmetry for that subset alone.

MORE TERMINOLOGY

In addition ~~the~~ the rank k of a tensor (total # of arguments) one may further classify a tensor by the individual numbers of covariant and contravariant arguments

A $\binom{p}{q}$-tensor (rank $k=p+q$) has p ~~contravariant~~ contravariant arguments (& is said to be contravariant in those arguments) and q ~~contravariant argume~~ covariant arguments (& is said to be covariant in those arguments.).

The only characteristic left is the order of the arguments. ~~When this is not importan~~ Since vector & ~~covector~~ covector arguments cannot be interchanged, it is only the order of the arguments of a given type that count. Occasionally we can get away with putting all the covector arguments first and the ~~contravariant~~ contravariant arguments second:

$$T = T^{\alpha_1 \ldots \alpha_p}{}_{\beta_1 \ldots \beta_q} \, e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_p} \otimes \omega^{\beta_1} \otimes \cdots \otimes \omega^{\beta_q}$$

A $\binom{p}{q}$-tensor has p contravariant (up) indices and q covariant (down) indices which are associated respectively with evaluation on dual basis vectors and basis vectors

$$T^{\alpha_1 \ldots \alpha_p}{}_{\beta_1 \ldots \beta_q} = T(\omega^{\alpha_1}, \ldots, \omega^{\alpha_p}, e_{\beta_1}, \ldots, e_{\beta_q}).$$

EXAMPLES OF VECTOR SPACES

1) $\mathbb{R}^n = \underbrace{\mathbb{R}\times\cdots\times\mathbb{R}}_{n} = \{(x_1,\dots,x_n) \mid x_i \in \mathbb{R}\}$ n-dimensional vector space.

standard basis: $e_\alpha = (0,..0,\underset{\alpha\text{th place}}{1},0,..,0)$.

2) $\underbrace{\text{Span}}_{\substack{\text{set of all real}\\ \text{linear combinations}\\ \text{of set vectors}}} \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\} = \{a^1\frac{\partial}{\partial x} + a^2\frac{\partial}{\partial y} + a^3\frac{\partial}{\partial z} \mid a^i \in \mathbb{R}\}$

· 3-dimensional vector space

basis: $e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y}$, $e_3 = \frac{\partial}{\partial z}$.

3) Span $\{\cos\omega t, \sin\omega t\}$, $\omega > 0$ fixed constant, t arbitrary real variable

this is a 2-dim vector subspace of the vector space of all ~~periodic~~ functions of t ~~t with period ω, obviously~~ obviously an infinite-dimensional space.

basis: $e_1 = \cos\omega t$, $e_2 = \sin\omega t$.

4) Solution space of ODE: $\left(\frac{d}{dt}\right)^2 x + \omega^2 x = 0$.

same as 3).

5) Any problems in which linearity plays a role have vector spaces associated with them. Once we choose a basis, we can work with the components and it doesnt matter what the interpretation of those basis vectors is, the vector algebra is always the same stuff done to components.

EXAMPLES IN R^3

Let $\underline{X} = X^\alpha e_\alpha \in R^3$ (standard basis) have a column vector $\underline{X} = \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix}$ of components.

The dot product $\quad \underline{X}\cdot\underline{Y} = \underline{X}^T\underline{Y} = \delta_{\alpha\beta}X^\alpha Y^\beta \equiv g(X,Y) \equiv g_{\alpha\beta}X^\alpha Y^\beta$.

defines a symmetric covariant 2-tensor, whose components equal the covariant kronecker delta symbol: $\delta_{\alpha\beta} = \begin{cases} 1 & \alpha=\beta \\ 0 & \alpha=\beta \end{cases}$

The scalar triple product

$$\underline{X}\cdot(Y\times Z) = \begin{vmatrix} X^1 & X^2 & X^3 \\ Y^1 & Y^2 & Y^3 \\ Z^1 & Z^2 & Z^3 \end{vmatrix} \equiv D(X,Y,Z) = D_{\alpha\beta\gamma}X^\alpha Y^\beta Z^\gamma$$

defines an antisymmetric covariant 3-tensor

partially
Evaluating this tensor on the last two arguments produces a covector

$$D(\ ,Y,Z) = D_{\alpha\beta\gamma}Y^\beta Z^\gamma \omega^\alpha$$

The value of this covector on X is the dot product with the vector $Y\times Z$

$$D(X,Y,Z) = X\cdot(Y\times Z).$$

In fact any vector can be associated with a covector by partial evaluation of the tensor g

$$g(\ ,\underline{Y}) = \text{"}\underline{Y}\cdot\ \text{"} = \underline{Y}^\flat, \quad g(X,Y) = \underline{X}\cdot\underline{Y} = \underbrace{\underline{Y}^\flat}_{\text{covector}}(\underline{X})$$

$$\text{"} = \text{"}\ g_{\alpha\beta}Y^\beta\omega^\alpha$$

$$(\underline{Y}^\flat)_\alpha = g_{\alpha\beta}Y^\beta = \delta_{\alpha\beta}Y^\beta \text{ "} = Y^\alpha \text{"}$$

Here we have to be careful. In the standard basis $\underline{Y}$ and $\underline{Y}^\flat$ have the same components but this will not be true in every basis as we will see later.

Equations with "free indices" must always have the same index positioning on both sides of the equation, ie same ~~number~~ free covariant or contravariant matching indices.

PROBLEMS

① Consider the following covariant 4-tensor:

$$T(X,Y,Z,W) = (X\cdot Z)(Y\cdot W) - (X\cdot W)(Y\cdot Z)$$

a) what symmetries does this tensor have (Hint: consider interchanging pairs of arguments.

b) $T = T_{\alpha\beta\gamma\delta}\,\omega^\alpha\otimes\omega^\beta\otimes\omega^\gamma\otimes\omega^\delta$. is the expansion of this tensor in the standard basis. What is the value of the component T_{1122}? ~~T_{2211}~~ T_{1212}? T_{1111}? T_{2112}?

c) $S(XYZW) = (X\times Y)\cdot(Z\times W)$ defines another such tensor with the same symmetries. Look up any book on vector algebra to see that they are the same tensor.

② On $\mathbb{R}^2$ with standard basis $e_1=(1,0)$, $e_2=(0,1)$. and dual basis $\{\omega^1,\omega^2\}$. Consider a new basis $e_{1'}=(1,1)$, $e_{2'}=(-1,+1)$

a) Use the duality relations $\omega^{\alpha'}(e_{\beta'})=\delta^\alpha{}_\beta$ to construct the new dual basis as linear combinations of the ~~stand~~ standard dual basis.

b) What is $\omega^{2'}((1,2))$? (value of $\omega^{2'}$ on the vector $(1,2)$).

c) What are the components $g_{\alpha'\beta'}$ of the tensor g in the new basis (defined using the 2-dim dot product as above).

d) What is the significance of the condition $g_{1'2'}=0$?

Linear transformations of V into V

Suppose: $\mathbb{B}: V \to V$ is linear as discussed on page 1.

Then we can express the equation $Y = B(X)$ wrt the basis $\{e_\alpha\}$ by taking components:

$$Y^\beta e_\beta = B(X^\alpha e_\alpha) = \underbrace{B(e_\alpha)}X^\alpha = (B^\beta{}_\alpha X^\alpha)\, e_\beta \quad \to \quad Y^\beta = B^\beta{}_\alpha X^\alpha$$

$$\equiv e_\beta B^\beta{}_\alpha \qquad\qquad \underline{Y} = \underline{B}\ \underline{X}$$

where $B^\beta{}_\alpha = \omega^\beta(B(e_\alpha))$

matrix representation wrt basis $\{e_\alpha\}$.

Now Define the $\binom{1}{1}$-tensor: $\mathbb{B} = B^\beta{}_\alpha\, e_\beta \otimes \omega^\alpha$

Then the same linear transformation may be accomplished by contraction of X with this tensor (evaluation of first vector argument on X):

$$B(X) = C_X \mathbb{B} = B^\beta{}_\alpha\, e_\beta\, \omega^\alpha(X) = B^\beta{}_\alpha X^\alpha e_\beta$$

Thus $\binom{1}{1}$-tensors can always be reinterpreted as linear maps of the vector space into itself.

Linear transformations from V into V*

Suppose $F: V \to V^*$ is linear, i.e. $F(X)$ is a covector:

$$f = F(X) = F(X^\alpha e_\alpha) = X^\alpha \underbrace{F(e_\alpha)} = (F_{\beta\alpha} X^\alpha)\, \omega^\beta \quad \to \quad f_\beta = F_{\beta\alpha} X^\alpha$$

$$\equiv F_{\beta\alpha}\, \omega^\beta$$

so $F_{\beta\alpha} = F(e_\alpha)(e_\beta)$ $\qquad \underline{f} = (\underline{F}\ \underline{X})^T$

row vector

Similarly define the $\binom{0}{2}$-tensor: $\mathbb{F} = F_{\alpha\beta}\, \omega^\alpha \otimes \omega^\beta$

$$F(X) = C^{(2)}_X \mathbb{F} = F_{\alpha\beta}\, \omega^\alpha \omega^\beta(X) = F_{\alpha\beta} X^\beta \omega^\alpha \quad \checkmark$$

contract on second vector argument.

So $\binom{0}{2}$-tensors can always be interpreted as linear maps of the vector space to its dual

You can repeat the discussion for linear transformations from V^* to V and find that $\binom{2}{0}$-tensors may be identified with them.

So the 2-tensors of all 3 possible types $\left[\binom{1}{1} \binom{0}{2} \binom{2}{0} \right]$ are important in discussing linear transformations among the vector space and its dual.

The $\binom{1}{1}$ tensors are also identifiable with linear transformations from V^* to V^*

$$f_\alpha = g_\beta B^\beta{}_\alpha \quad \text{in components} \rightarrow \quad \mathbb{B} = B^\beta{}_\alpha \, e_\beta \otimes \omega^\alpha$$

" $f = \mathbb{B}(g, \ )$ " partial evaluation of $\mathbb{B}$ on its only covector argument produces a covector.

" Raising & lowering indices "

Any invertible transformation $G : V \to V^*$ with inverse $G^{-1} : V^* \to V$

$$\left(\begin{array}{ll} \text{in components} \quad X^\alpha \to G_{\alpha\beta} X^\beta & f_\alpha \to g_\beta G^{\beta\alpha} \\ & G_{\alpha\beta} G^{\beta\gamma} = \delta^\gamma{}_\alpha = G^{\gamma\beta} G_{\beta\alpha} \end{array} \right)$$

establishes a 1-1 correspondence between V and V^*. With each $X \in V$ we associate a unique covector $G(X) \equiv X^\flat$ and vice versa $G^{-1}(X^\flat) = X$.

Because the index changes location under the map in component form, this is referred to as "lowering" (with G) the indices and "raising" the indices (with G^{-1}).

This operation can be extended to any tensors. Each index can be raised and lowered or vice versa, thus establishing a family of tensors of different valence but the same total number of arguments. All indices can be lowered to produce a k-covariant tensor or all indices can be raised to produce a k-contravariant tensor.

<u>EX</u> $\binom{1}{2}$ tensor $T = T^\alpha{}_{\beta\gamma} \, e_\alpha \otimes \omega^\beta \otimes \omega^\gamma$

lower first: $T_{\alpha\beta\gamma} = G_{\alpha\delta} T^\delta{}_{\beta\gamma} \rightarrow T_{\alpha\beta\gamma} \, \omega^\alpha \otimes \omega^\beta \otimes \omega^\gamma = T^\flat$

Raise last two: $T^{\alpha\beta\gamma} = T^\alpha{}_{\mu\nu} G^{\mu\beta} G^{\nu\gamma} \rightarrow T^{\alpha\beta\gamma} \, e_\alpha \otimes e_\beta \otimes e_\gamma = T^\sharp$

etc.

<u>EXAMPLE</u> On $\mathbb{R}^n$, in the standard basis $\{e_\alpha\}$, a vector $X = X^\alpha e_\alpha$ represented by the column vector $\underline{X} = \begin{pmatrix} X^1 \\ \vdots \\ X^n \end{pmatrix}$ can be associated with a covector $X^\flat = X_\alpha \omega^\alpha$, represented by the transposed row vector

$$\underline{X}^T = (X^1, \dots, X^n) \qquad \text{with} \qquad X_\alpha = \delta_{\alpha\beta} X^\beta, \text{ ie } X_1 \equiv X^1, \text{etc.},$$

defining a covector which has the same ordered set of components as the original vector. The covector $X^\flat$ evaluated on a vector Y is equivalent to taking the dot product of Y with X:

$$X^\flat(Y) = X_\alpha Y^\alpha = \delta_{\alpha\beta} X^\beta Y^\alpha = X \cdot Y$$

We can also think of $X^\flat$ as arising from partial evaluation of the Euclidean metric tensor associated with the dot product.

$$g(X,Y) = \delta_{\alpha\beta} X^\alpha Y^\beta$$

$$g(X, \bullet) = \delta_{\alpha\beta} X^\alpha \omega^\beta = X^\flat.$$