Spatially Homogeneous Cosmology: 
Background and Dynamics

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EXPLANATION OF HANDWRITTEN MANUSCRIPT:

lower case: abcdefghijklmnopqrstuvwxyz
upper case: ABCDEFGHIJKLMNOPQRSTUVWXYZ
numerals: 0123456789

Roman numerals: I II III IV V VI VII VIII IX
lower case Greek used: α β γ δ ε ρ μ ο ϋ κ θ ς ρ π ρ ι ι ξ ι
upper case Greek used: Λ Ξ Φ Δ

upper case Gothic (German) used: Ъ ﬂ
lower case Gothic (German) used, indicated by longhand written (not printed) letters: a c e f g h i k l o q s u t

upper case "handwritten type" used:
C D E F H J L M O R T X

sans serif lower case: j (not g) indicated by the symbol j.
all matrices are underlined to indicate BOLDFACE TYPE.

SYMBOLS

( ) parentheses
[ ] square brackets
{ } set brackets
| | absolute value brackets
/ slash
+ plus sign
- minus sign
= equal sign
≠ not equal sign
= "is defined to be"
⊂ set inclusion symbol
∈ "is an element of"
⊕ direct sum symbol
⊗ tensor product symbol
× set product symbol
∫ integral symbol
< > "int," "let" brackets

→ mapping symbol between spaces
→ mapping symbol between elements
≤ less than or equal to
≥ less than or equal to
δ functional derivative symbol
∫∫ slashed functional derivative symbol
∂ partial derivative symbol
∫∫∫ slashed partial derivative symbol
∞ infinity
⊥ perpendicular symbol
Tr trace of matrix
ln natural logarithm
R real line
f(g) composition of maps symbol
≅ isomorphism symbol
∧ wedge symbol
( ) long parentheses
± plus or minus symbol
EXPLANATION OF HANDWRITTEN MANUSCRIPT (2)

SYMBOLS (ABOVE AND BELOW)

\[ A^* \] asterisk
\[ A* \] sharp symbol
\[ A \] bar
\[ \hat{e} \] tilde
\[ \check{e} \] barred tilde
\[ \tilde{N} \] vector arrow symbol
\[ \ddot{q} \] dot
\[ \ddot{e} \] "cove" or "hat"
\[ \ddot{w} \] dotted tilde

\[ S^T \] capital T for matrix transpose
\[ S^{-1} \] minus one = inverse symbol
\[ A^* \] prime
\[ R^+ \] plus sign
\[ N_{ab} \] vertical bar
\[ N_{ia} \] semicolon
\[ M_{ij}, M_{ij(0)}, M_{ijv}, M_{ij} \] uppercase letters
\[ N_{ia} \] comma

The entire manuscript is written with the intention of a 3 line system, namely a baseline, a subscript line, and a superscript line. REGARDLESS OF HOW THE HANDWRITTEN SYMBOLS MIGHT APPEAR.

FOR EXAMPLE: \[ C_{bc} \Rightarrow C_{bc}^{ab} \]

With certain exceptions noted below, a superscript and a subscript are never intended to appear in the same vertical line.

\[ C_{bc}^{ab}, R_{bc}^{ab}, \hat{m}_{bc}, \tilde{M}_{bc}, S_{bc}^{ab} \]

*important:

EXCEPTIONS: \[ S_{bc}, C_{bc}^{ab} \]

OTHER IRREGULARITIES: \[ C_{bc}^{ab}, \hat{e} = \text{pure } \delta_{bc} \text{ in an index position} \]

All momentum symbols \[ P_a, P_b, P_c, P_i \] are lowercase \[ \bar{P}_a, \bar{P}_b, \bar{P}_c \] except for \[ P_a, P_b, P_c, P_i \] which are uppercase.
EXPLANATION OF HANDWRITTEN SYMBOLS (3)

The lower case "a" used to denote points in a Lie group appears in the subscript position in the following places:

$AD_a\ La\ Ra\ Fa$

where it should not be confused with a smaller index letter, i.e., it should be normal size.
SPATIALLY HOMOGENEOUS COSMOLOGY: BACKGROUND AND DYNAMICS

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0. Introduction.
1. A First Look at Lie Groups and Lie Algebras.
2. Integral Curves and Diffeomorphism Groups.
3. Dragging; Lie and Lagrange Derivatives.
4. Lie Groups Revisited.
5. The Adjoint Action and Automorphism Groups.
7. Actions Lifted to TM and TM*.
8. Isometry Groups and Homogeneous Geometries.
11. The Metric Manifold $\mathcal{M}^3$.
15. A Spatially Homogeneous Perfect Fluid Source.
16. Exploration of the Dynamical System.

APPENDICES.

A. Notation and Formulas.
B. Geodesics of $\mathcal{M}$.
C. Bianchi Quaternions.
D. A Spatially Homogeneous Electromagnetic Field.
§0. Introduction

Spatially homogeneous cosmology is a topic in which the disciplines of differential geometry, Lie group theory and classical mechanics unite to provide not only an interesting example for the application of ideas of theoretical relativity but a physically useful model for ideas about the nature of our universe. Unfortunately spatially homogeneous dynamics has not been dealt with systematically, its symmetries have been ignored and its classical mechanical structure misunderstood, resulting in the generation of a host of special cases (and occasional errors) lacking any coherence and conveying little general understanding. The present work attempts to remedy this while providing a complete and essentially self-contained introduction to the topic and the relevant mathematics.

Because there do not appear to be available expositions of Lie group theory which make use of modern differential geometry without being too sophisticated for the physicist who only has a grasp of the basics, we first develop this subject in that spirit. It is not done rigorously and global questions are ignored but hopefully a pedestrian understanding is conveyed. The material is then applied to spatially homogeneous cosmology, which when treated carefully, represents an extremely rich and beautiful dynamical system.

To be more specific this system is interpreted as an ordinary classical mechanical system in which a nonconservative force is present, allowing modified Lagrangian and Hamiltonian techniques to be used. By exploring the action of the automorphism and adjoint groups of the Lie algebra of each Bianchi type on the associated configuration and phase spaces, one is led to construct a class of coordinate systems on those spaces adapted to the symmetry properties of the dynamics which considerably simplify the entire discussion of the problem. A great deal of insight is also gained in interpreting the momentum constraints on this system in terms
of the natural action, the shift freedom on the associated spacetimes and the flow of energy-momentum of the source of the gravitational field parallel to the spatially homogeneous slicing of those spacetimes. In short the techniques introduced allow a somewhat unified approach to the dynamics of spatially homogeneous spacetimes adapted to the specific features of each Bianchi type.

The book GRAVITATION by Misner, Thorne and Wheeler¹³ (hereafter referred to as MTW) is used frequently as a source of background material and unless otherwise specified, its notation and conventions are respected. Appendix A summarises the mathematics and notation used in this chapter. The author thanks Bepo Ruffini for encouraging this work and Michael P. Ryan and Abraham Taub for helpful discussion and for laying the foundation upon which the present discussion of spatially homogeneous dynamics is based.
A Lie group \( G \) is an \( r \)-dimensional real analytic manifold (points symbolized by \( \alpha \)) with a closed multiplication of its points: 
\[ g_1 g_2 = \rho(g_1, g_2) \in G \] 
this multiplication must be associative, an identity element \( e \) must exist and every \( g \in G \) must have an inverse \( \psi(g) = g^{-1} \). Both \( \rho \) for fixed \( a \), \( a \in G \), and \( \psi \) are analytic diffeomorphisms of \( G \) into itself. Since the inverse has the property 
\[ (g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}, \] 
the inverse map satisfies: 
\[ \psi(g_1 g_2) = \psi(g_1) \psi(g_2). \] 
Any diffeomorphism of \( G \) with this property will be called a transposition. (Any transposition \( \psi \) satisfies: 
\[ \psi(g_1) = g_1, \psi(g_2) = g_2^{-1}; \] 
a non-involutive transposition like the inverse satisfies: 
\[ \psi^2 = \psi \psi = Id. \] 
Let \( M \) be an \( n \)-dimensional real analytic manifold (points symbolized by \( x \)). We assume that \( G \) acts on \( M \) on the left (unless otherwise stated) as a group of analytic diffeomorphisms:
\[ x \mapsto a \cdot x = f(x, a) \] 
\[ a \cdot (a_1 \cdot x) = (aa_1) \cdot x, \quad a \cdot x = x. \] 
The latter requirements ensure that the action has the group property and that \( a \) acts as the identity transformation \( \text{Id} \) on \( M \); \( f \) is also analytic in both its arguments. We always assume (unless otherwise stated) that \( G \) acts effectively on \( M \), meaning that only \( e \) leaves every point of \( M \) fixed. The set 
\[ G \times M = \{ a \times x | a \in G \} \] 
is called the orbit of \( x \). Points lying on the same orbit can be mapped into each other under the action of the group. The action is called transitive if \( M \) consists of a single orbit, and intransitive otherwise.

For each \( x \in M \) we may define a map \( F_x : G \rightarrow M \) by \( F_x(a) = f(x, a) \). Its differential \( dF_x(a) \) maps \( T_{g_0} G \) into \( T_{f(x, a)} M \); \( F_x = dF_x(a) \) therefore provides us with a map from the tangent space at the identity of the group to the tangent space at a point \( x \in M \). A corresponding map \( E_x \) from \( T_{g_0} G \) into \( T_x M \), the vector space of analytic vector fields on \( M \), is obtained by setting 
\[ E_x(X) \times x = X(x) \] 
where \( X \in T_{g_0} G \).

It is convenient to introduce the notation \( \tilde{f}_a \) for the diffeomorphism 
\[ x \mapsto \tilde{f}_a(x) = f(x, a) \] 
of \( M \) into itself. To say that \( G \) acts on the left or on
the right simply means that $Sa aS a$ equals $Sa a$ or $Sa a$, respectively. As the second line of (1.1) shows, the notation $aX$ is natural for a left action while $Xa$ would be appropriate for a right action. We will use a tilde to signal right actions. For example, a left action is changed into a right action by composing $S$ with a transposition like the inverse map:

$$S a = S a, S a, S a = S a$$

$G$ naturally acts on itself effectively as a group of analytic diffeomorphisms in two ways, on the left by left translation $a \mapsto L a(a) = a a$ and on the right by right translation $a \mapsto R a(a) = a a$. The associativity of the multiplication implies that right and left translation commute:

$$L a R a(a) = a a a a = R a L a(a) a \mapsto L a R a = R a L a$$

$G$ also acts on itself by inner automorphism;

$$A D a(a) = a a a a = L a R a(a) = R a L a(a)$$

$$A D a = A D a a a$$

By choosing $a a a a$ rather than $a a a a$, we obtain a left action. Let $A D G = \{ A D a \mid a G \}$ be called the adjoint group. The adjoint action is not necessarily effective. The set $C G = \{ a \in G \mid A D a = 1 G \}$ is a subgroup of $G$ called its center and consists of exactly those elements which commute with every other element of $G$; since $a a a a = a$ implies $a a a a = a$.

The adjoint action is effective only when $C G$ is trivial, i.e., $C G = \{ a e \}$. Note that the adjoint group interchanges left and right translations:

$$A D a e R a = L a a a = A D a e L a = R a$$

A homeomorphism is a map $h$ from $G$ to another group $G$ which preserves the group multiplication:

$$h(a a) = h(a) h(a)$$

(Note that $h(a e) = a e$ and $h(a a) = h(a) h(a)$.) The kernel of a homeomorphism is defined to be the inverse image of $e G$: $K e h = \{ a e G \mid h(a a) = a e \}$. When $h$ is a diffeomorphism it is called an isomorphism and $G$ and $G$ are called isomorphic, written $G \cong G$. If in addition $G = G$, $h$ is called
an automorphism. (The composition of two transpositions is an automorphism.)

For example, the adjoint diffeomorphisms are automorphisms of $G$ since:

$\text{(i)} \quad \text{Ad}_g(a_1a_2) = \text{Ad}_g(a_1) \text{Ad}_g(a_2)$,

and hence $\text{Ad}(g) \in \text{Ad}(G)$. They are called inner automorphisms since they involve the group multiplication. Similarly, (i)-(iv) shows that $\text{Ad}$, considered as a map from $G$ onto $\text{Ad}(G)$ is a homomorphism. By definition, its kernel is $\{e\}$. $\text{Ad}(G)$ is called the adjoint representation of $G$. (The multiplication making this "general linear group" $\text{GL}(V)$ into a group is composition of its elements.) The identity $a_0$ is a fixed point of the diffeomorphism $\text{Ad}_g$ for all $g \in G$, so its differential $\text{Ad}(a) = (\text{dAd}_a)(a_0)$ is an element of $\text{GL}((T_aG))$. By the chain rule, the differential of the composition of two maps is the composition of the differentials so (i)-(iv) imply the relation $\text{Ad}(a_1a_2) = \text{Ad}(a_2) \text{Ad}(a_1)$. $\text{Ad}$ is therefore a homomorphism from $G$ into $\text{GL}(T_{a_0}G)$ called the adjoint representation of $G$. The image of the map, denoted by $\text{Ad}(G)$, is a subgroup of $\text{GL}(T_{a_0}G)$ which should be called the linear adjoint group but will also be referred to as the adjoint group. This sloppiness is justified by the fact that $\text{Ad}(G) = \text{Ad}(G)$, as will be shown in Section 5.

Let us adopt the notation $HK = \{a_1a_2 | a_1 \in H, a_2 \in K\}$ for the product of arbitrary subsets $H, K$ of $G$ and review some general facts about groups. A subgroup $H$ is simply a subset which is closed under multiplication ($HH = H$) and contains all inverses ($H^- = H$). The sets $aH$ and $Ha$ are called left and right cosets of $H$ in $G$ and represent the orbits of the point $a$ under right and left translation by the subgroup $H$. Denote the set of left cosets by $G/H$. If $h \in G$ is a homomorphism then $h(H)$ is a subgroup of $G$ for any subgroup $H$ of $G$ while $\text{ker}(h)$ is a subgroup of $G$. For example, $\text{Ad}_g$ is an automorphism so $\text{Ad}_g(H)$ for each $g \in G$ is a subgroup of $G$ called a conjugate of $H$. If $\text{Ad}_g(H) = H$ for all $g \in G$, $H$ is called a normal or invariant subgroup. (The kernel of a homomorphism is a normal subgroup.) The left and right cosets of such a subgroup coincide.
since $a Ha^{-1} = H$ implies $a H = Ha$. In this case $G/H$ inherits a group structure of its own from $G$:

$$a H a^{-1} H = a a^{-1} H = a H a^{-1} H$$

Thus $G/H$ is itself a group with identity element $a H = H$. The map $T_H$ projecting a point of $G$ to the coset to which it belongs is therefore a homomorphism (with kernel $H$). A very useful fact is that if $H = \ker f$ for some homomorphism $f$, then $G/H \cong f(G)$. For example, the center of $G$ is the kernel of the homomorphism $AD$, so $G/C(G) \cong AD(G)$.

As another example, consider the infinite-dimensional group $\mathcal{D}(M)$ of diffeomorphisms of $M$ into itself, with composition as its multiplication. The property $f a f^{-1} = f a f^{-1}$ reveals $f$ to be a homomorphism from $G$ into $\mathcal{D}(M)$, i.e. a left action of a group $G$ on a manifold $M$ is just a homomorphism from $G$ into $\mathcal{D}(M)$. Its kernel is just $\ker f = \{ a \in G | f a = I_d \}$. An effective action has trivial kernel and is an isomorphism onto its image $f a \in \mathcal{D}(M)$, while $G/\ker f$ acts effectively on $M$ in a natural way if the action is ineffective (by defining $(a \ker f) \cdot x = a \cdot x$). Similarly, left translation and inner automorphism yield an isomorphism $L$ and homomorphism $AD$ from $G$ into $\mathcal{D}(G)$, while right actions (like right translation) are "homomorphisms up to a transposition", namely maps between groups which become homomorphisms when composed with a transposition such as the inverse (as in (1.2)).

A Lie algebra $\mathfrak{g}$ is an $n$-dimensional vector space over the real numbers $\mathbb{R}$ with an alternating bilinear bracket operation $[,]$ satisfying the Jacobi identity (elements of $\mathfrak{g}$ will be denoted by $X, Y, Z$ etc.):

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$  

The set $gl(V)$ of linear transformations of an $n$-dimensional vector space $V$ into itself is naturally a Lie algebra where the bracket of two transformations $X, Y$ is just their commutator $[X, Y] = XY - YX$. (The Jacobi identity is identically satisfied for brackets defined by commutators, as may be seen by expanding it out into twelve terms which cancel in pairs.) A choice of basis $\{e_a\}$ of $V$ with dual
basis \{w^i\} (i.e. \(w^i(e^j) = \delta^i_j\)) maps \(g(V)\) isomorphically onto the vector space of \(n\)-dimensional real square matrices \(g(n,\mathbb{R})\) and \(GL(V)\) onto the group \(GL(n,\mathbb{R})\) of nonsingular elements of \(g(n,\mathbb{R})\).

\[(a)\quad A \mapsto A = A^a b \delta_a^b, \quad A^a_b = w^a(A(e^b)).\]

For notation see §6 where we study these spaces.

A transposition of a Lie algebra is defined exactly as for a group, namely a map \(\phi \in GL(g)\) which satisfies \(\phi([X,Y]) = [\phi(X), \phi(Y)] = -[X,\phi(Y)]\).

The natural transposition \(\phi'\) corresponding to the inverse transposition of \(G\) is reflection about the origin: \(\phi'(x) = -x\).

With \(X \in g\) we may associate an element \(\text{ad}(X)\) of \(g^g(g)\) by defining \(\text{ad}(X)Y = [X,Y]\). Rewriting the Jacobi identity and noting that it holds for all \(Z \in g\), we obtain: \(\text{ad}(XY) = [\text{ad}(X), \text{ad}(Y)]\).

This says that the map \(\text{ad} : g \rightarrow g^g(g)\) is a Lie algebra homomorphism, namely a linear map \(\sigma\) from \(g\) into another Lie algebra \(\tilde{g}\) satisfying:

\[(b)\quad \sigma([X,Y]) = [\sigma(X), \sigma(Y)].\]

The kernel of \(\sigma\) is defined by \(\ker \sigma = \{X \in g : \sigma(X) = 0\}\), and \(\sigma\) is an isomorphism on \(X\). When \(\sigma\) is also a vector space isomorphism (\(\ker \sigma = \{0\}\)), it is called a Lie algebra automorphism, and we write \(g \cong \tilde{g}\), while if in addition \(g \cong \tilde{g}\), it is called a Lie algebra automorphism. The set of all \(A \in GL(g)\) which are automorphisms is a subgroup denoted by \(\text{Aut}(g)\).

A Lie subalgebra of \(g\) is a subvector space \(\mathcal{A}\) closed under the bracket. In an obvious notation \([\mathcal{A}, \mathcal{A}] \subset \mathcal{A}\), while an invariant Lie subalgebra or ideal is one for which \(\text{ad}(\mathcal{A}) \mathcal{A} = \mathcal{A}\).

The kernel of a homomorphism \(\sigma : g \rightarrow \tilde{g}\) is an ideal in \(g\) while its image \(\sigma(g)\) is a Lie subalgebra of \(\tilde{g}\). The kernel of the map \(\text{ad}\) is the set of all \(X \in g\) which commute with all other elements of \(g\), an ideal called the center of \(g\), and its image \(\text{ad}(g) \subset g^g(g)\) is called the adjoint Lie algebra. When the center of \(g\) is trivial, \(g \cong \text{ad}(g)\). A Lie algebra representation \(\sigma\) is a homomorphism from \(g\) into \(g(V)\) for some \(V\); \(\text{ad}\) is called the adjoint representation of \(g\). As a last example, the set of all brackets \([g,g]\) is an ideal called the derived Lie algebra.
Let \( \{e_a\} \) be a basis of a Lie algebra \( \mathfrak{g} \), with dual basis \( \{\omega^a\} \). The components of the bracket in this basis are defined by:

\[
[e_a, e_b] = C^{c}_{ab} e_c
\]

They are the components of a \((1, 2)\)-tensor over \( \mathfrak{g} \), called the structure constant tensor (sometimes abbreviated by \( \text{SCT} \)).

\[
C^{c}_{ab} = \omega^c (\{e_a, e_b\})
\]

It is antisymmetric in its covariant arguments and satisfies the following relations imposed by the Jacobi identity:

\[
0 = C^{d}_{ab} C^{c}_{cd} + C^{d}_{bc} C^{c}_{ad} + C^{d}_{ac} C^{c}_{bd} = 3 C^{d}_{abc} C^{c}_{d} = \omega^c (\text{ad}(e_c) e_a e_b),
\]

The matrix of \( \text{ad}(e_a) \) in this basis, denoted by \( \mathcal{R}_a \), has components \( \mathcal{R}_a^b = \omega^b (\text{ad}(e_a) e_c) \), while the matrix form of the homomorphism relation \( \text{ad}(e_a) \circ \text{ad}(e_b) = \text{ad}(\{e_a, e_b\}) \) is:

\[
[C^a_{ab}, C^b_{cd}] = C^c_{ad}.
\]

The matrices \( \{\mathcal{R}_a\} \) therefore generate a matrix Lie algebra which is the matrix representation of the adjoint Lie algebra in the basis \( \{e_a\} \).

Define the following quantities:

\[
[2] \quad 2 \mathcal{R}_a = C^a_{bc} = \text{tr} \mathcal{R}_a, \quad [3] \quad \mathcal{X}^a_{bc} = C^a_{bc} C^c_{de} = \text{tr} \mathcal{R}_a \mathcal{R}_b = C^a_{bc} C^c_{de},
\]

These are the components of a covector \( \mathcal{X}^a \) over \( \mathfrak{g} \), and a symmetric second rank covariant tensor \( \mathcal{X}^a \) called the Killing form:

\[
\text{Kil}(X, Y) = \text{tr} \mathcal{X}^a \mathcal{X}^a = \mathcal{X}^a \mathcal{X}^a
\]

These are important in the classification of Lie algebras. By contracting the indices \( (e_a, e_b) \) in (11.12) and using the symmetry of \( \mathcal{X}^a \) (or by taking the trace of (11.13)) we obtain the contracted Jacobi identities:

\[
\mathcal{X}^a C^a_{bc} = 0.
\]

Let \( \Lambda : \mathfrak{g} \rightarrow \mathfrak{g} \) be a Lie algebra isomorphism and \( \Lambda^b_{\ a} = \delta^b_{\ a} (\Lambda(e_a)) \), the components of its matrix with respect to bases \( \{e_a\} \), \( \{e_a\} \), and \( C^a_{bc} \), \( C^c_{bc} \), the respective \( \text{SCT} \) components in these bases.

The component form of (11.10) yields the result:

\[
\mathcal{X}^a_{bc} = A^a_{\ bd} C^d_{fg} A^f_{\ e} A^g_{\ c}.
\]

The \( \text{SCT} \) components of isomorphic Lie algebras are related exactly as are the components of the same \( \text{SCT} \) in two different bases, for which would be the interpretation of (11.1) if \( \mathfrak{g} = \mathfrak{g} \) and \( \Lambda \) were the identity transformation. If \( \mathfrak{g} = \mathfrak{g} \) and we take \( e_a = e_a \) so that \( \Lambda \) is
an automorphism of \( g \), then:

\[
C_{c} = A_{a} d C_{f} s_{q} A_{t}^{-1} s_{p} A_{c}^{-1} A_{d} = R_{a} A_{t}^{-1}.
\]

The second equation is an equivalent matrix expression.

Let \( X(M) \) be the infinite-dimensional real vector space of vector fields on a manifold \( M \). Since the Lie bracket \([X,Y]\) of two vector fields \( X, Y \) is again a vector field (see (A.3)), and is defined by a commutator expression, \( X(M) \) is an infinite-dimensional Lie algebra of vector fields on \( M \). Suppose \( h : M \to N \) is a diffeomorphism and \( h \in X(N) \) for \( X \in X(M) \). We define by \((hX)(h(x)) = dh(x)X(x)\), then \( h \circ X(M) = X(N) \) is a Lie algebra isomorphism:

\[
h[X,Y] = [hX,hY].
\]

This can be generalized, when \( h \) is not a diffeomorphism. Although one usually cannot push \( X \) forward to a vector field on \( N \), if \( X \in X(N) \) is such that \( dh(x)X(x) = X(h(x)) \) for all \( x \in M \), \( X \) is called \( h \)-related to \( X \) and the statement analogous to (1.19) is that if \( \tilde{X}, \tilde{Y} \) are \( h \)-related to \( X, Y \) their brackets are also \( h \)-related.

Finite-dimensional Lie subalgebras of \( X(M) \) for some \( M \) are particularly important to Lie group actions. Three such r-dimensional Lie algebras play a fundamental role in the action of a group \( G \) on itself by left and right translation and on another manifold \( M \). Let \( X \in T_{a}G \), be a tangent vector at the identity of \( G \) and \( C(f) \) any parametrized curve such that \( (a) = a \) and \( C(a) = X \). The prime indicates the tangent vector to the curve (see (A.9)). Now let this curve act on \( G \) by left and right translation and on \( M \) by a left action, generating three families of orbits whose tangents at \( t = 0 \) may be evaluated by the chain rule (A.3):

\[
\begin{align*}
C(a,t) &= L_{c(a,t)}(a) = R_{a}(c(t)) \quad \dot{X}(a) = \dot{C}(a,0) = dR_{a}(a)X \in T_{a}G \\
C(a,t) &= R_{c(a,t)}(a) = L_{a}(c(t)) \quad \ddot{X}(a) = C'(a,0) = dL_{a}(a)X \in T_{a}G \\
\dddot{X}(a,t) &= c(t):X = F_{c}(c(t)) \quad \dddot{X}(a,t) = \dddot{C}(a,0) = dF_{c}(a)X \in T_{M}. 
\end{align*}
\]

Thus, for each tangent vector, at the identity, we obtain two vector fields \( X \) and \( \dot{X} \) on \( G \) satisfying \( \dot{X}(a) = \dot{X}(a) \) and a vector field \( \dddot{X} \) on \( M \).

These have the interpretation that if a curve with tangent \( X \) at the identity acts in various ways, it begins by pushing the points along the
corresponding "generating vector fields" or "generators of the action" at these points. This observation provides us with two linear maps from $T_{e}a_{e} \to \mathfrak{g}(e(G))$ and $a_{e}$ linear map $\mathfrak{g}$ into $\mathfrak{x}(e(M))$, called the generating maps of their respective actions:

$$x \cdot T_{a_{e}} = X, X \in \mathfrak{g}(e(G)), z \cdot \mathfrak{x}(e(M)).$$

Denote the images of these maps by $g_{1}, g_{2}$ and $S(T(\mathfrak{g}))$. The first two are $r$-dimensional real vector spaces having been constructed isomorphic to $T_{e}a_{e}$; we deny proving that $S(T(\mathfrak{g}))$ is also $r$-dimensional over $\mathbb{R}$, a claim which depends on the assumption that $G$ acts effectively on $M$.

We next establish that these vector spaces are isomorphic Lie algebras.

Some preliminary results are needed first. From the definitions and the chain rule:

$$(1.20) \quad \mathfrak{v}_e L_{a} (a) \mathfrak{x} = d_{L_{a}} \mathfrak{x} = \mathfrak{x} \circ L_{a} = (L_{a} \mathfrak{y}) (a) = (L_{a} \mathfrak{x}) (a)$$

The first states that $\mathfrak{v}_e L_{a}$ is $L_{a}$ related to itself for all $a \in G$ and that the vector field $L_{a} \mathfrak{x} = (d_{L_{a}} \mathfrak{x}) = (L_{a} \mathfrak{x})$ coincides with $\mathfrak{x}$. (From $L_{a} \mathfrak{x} = \mathfrak{x} \circ L_{a}$ and $L_{a} \mathfrak{y} = \mathfrak{y} \circ L_{a}$.)

Conversely, if $\mathfrak{x}$ is any left invariant vector field, it satisfies $d_{L_{a}} \mathfrak{x} = \mathfrak{x}$, which is how we defined the elements of $\mathfrak{g}$ in the first place. Similarly $\mathfrak{x} \circ L_{a}$ is a right invariant vector field. In short, $\mathfrak{g}$ and $\mathfrak{g}$ are the vector spaces of left and right invariant vector fields on $G$ (generating the right and left translations respectively).

Since the elements of $\mathfrak{g}$ are $L_{a}$-related to themselves, the Lie bracket of two elements of $\mathfrak{g}$ is also $L_{a}$-related to itself for all $a \in G$ and hence is also an element of $\mathfrak{g}$, Thus $\mathfrak{g}$ and similarly $\mathfrak{g}$ are closed under the Lie bracket and therefore $\mathfrak{g}$-dimensional Lie algebras.

Let $C_{x}(t)$, $C_{y}(t)$ be curves with tangents $X_{t}$, $Y_{t}$ at the identity of $G$ where $C_{x}(0) = 0 = C_{y}(0)$, $X(a)$, $Y(a)$ were defined to be the tangents to the curves $R_{\mathfrak{g}_{a}}(0)$ and $L_{\mathfrak{g}_{a}}(0)$ at $a$; for example, if $S$ is
a function on $G$ then by (A.1):

$$(1.21) \quad X(0)f = (dA(a))_x f \circ R_{x_0}^{-1}(a).$$

Using the commutativity of the left and right translations we calculate the Lie bracket of $X$ and $Y$ acting on a function $f$:

$$X(a)(Yf) = (dA(a))_x f \circ L_{x(a)}^{-1}R_{x(a)}(a) = Y(a)(Xf).$$

Since $f$ is arbitrary, $[X, Y] = 0$ proving that $x$ and $y$ are mutually commuting Lie subalgebras of $K(G)$.

Now manipulate the expression $[X, Y](a)f$ using this fact and the agreement of corresponding left and right invariant vector fields at the identity (and the definition (A.5) of the Lie bracket):

$$[X, Y](a)f = [X, Y](a)f = X(a)(Yf) - Y(a)(Xf) = X(a)(a) - Y(a)(a)$$

Since $[X, Y]$ and $(Y, X)$ are both right invariant vector fields, which agree at the identity, they are equal and therefore:

$$(1.22) \quad [X, Y] = [Y, X] \quad \text{and} \quad [-X, Y] = [-Y, X].$$

The map $X \mapsto -X$ is a Lie algebra isomorphism from $g$ onto $\tilde{g}$.

Next we make the following observations:

$$(1.23) \quad F_{aX}(a) = a\cdot (aX) = (aa)X = F_a \circ R_{a}^{-1}(a),$$

$$dF_{aX}(a)X = dF_{a}(a) \circ dR_{a}(a)X = dF_{a}(a)X.$$

This says $S(X)$ is $F_a$-related to $X$ and therefore $(S(X), S(Y))$ is $F_a$-related to $(X, Y)$.

$$(1.24) \quad [S(X), S(Y)] = F_a(0) = dF_{a}(0)\circ [X, Y] = S(X, Y) \circ F_a(0).$$

We adopt the convention that when the argument of $S$ is an element of $g$ or $\tilde{g}$, it is to be evaluated at the identity, so $S(X) = S(X)$. This allows us to interpret $S$ as a map from either $T_e \tilde{g}$, or $\tilde{g}$.

Evaluation of (1.24) at the identity then leads to the following relation which, together with (1.22), yields another:

$$(1.25) \quad S([X, Y]) = S(0) = S(Y, X) = -S(X, 0).$$

Thus is therefore a homomorphism from $\tilde{g}$ onto its image $S(\tilde{g})$. For an effective action we will later show this to be an isomorphism.
$G$ had acted on $M$ on the right, we would have obtained a homomorphism $	ilde{\mathcal{G}}$ from $\mathcal{G}$ onto its image, satisfying (1.25) with tildes interchanged.

Finally, we may convert $\mathcal{G}_a$ into a Lie algebra by its identification with $\mathcal{G}$. If $X,Y \in \mathcal{G}_a$, let $[X,Y] = X(Y(\mathcal{G}))$ where $X,Y$ in the right bracket are the corresponding left-invariant vector fields. The identification we have made by using the same symbol for $\mathcal{G} \times \mathcal{G}_a$ and its image in $\mathcal{G}$ is therefore complete. $\mathcal{G}_a \cong \mathcal{G}$, is referred to as the Lie algebra of the Lie group $G$. However, occasionally it is convenient to distinguish these two spaces which we do when necessary by an explicit map $\tilde{\mathcal{G}}$ from $\mathcal{G}$ onto $\mathcal{G}_a = \tilde{\mathcal{G}}_a$, namely evaluation at $a$. A tangent vector $X \in \mathcal{G}_a$ then generates left and right invariant vector fields $X$ and $\tilde{X}$.

Let $\{e_a\}$ be a basis of $\mathcal{G}_a$, with dual basis $\{\omega^a\}$. It naturally induces bases $\{e_a\}$, $\{\omega^a\}$ and $\{\omega^a\}$ of $\mathcal{G}$, $\tilde{\mathcal{G}}$, and $\mathcal{G}_a$, where $e_a = \omega^a(e_a)$. If $C^b$ are the components of the SCT of $\mathcal{G}$ in the basis $\{e_a\}$, then by (1.22) and (1.25):

\[ [e_a, e_b] = C^c_{\ a} e_c, \quad [\omega^a, \omega^b] = -C^c_{\ a} \omega^c, \quad [e_a, \omega^b] = -C^c_{\ a} \omega^c. \]

Since the translations are diffeomorphisms, their differentials at the identity are vector space isomorphisms, so $\{e_a\}$ and $\{\omega^a\}$ are each global frames on $G$. However, $\{\omega^a\}$ is required to be linearly independent as a set only over $\mathbb{R}$ and not over $\mathcal{G}_a$ and hence spans a p-dimensional subspace of $\mathcal{G}_a$ at each $x \in M$, i.e. in some cases $p$ may even vary over $M$.

We may introduce the 1-form frames dual to $\{e_a\}$ and $\{\omega^a\}$ by $\omega^a(e_b) = \delta^a_b = \omega^a(e_a)$. By (A.11) these satisfy:

\[ dw^a = -\frac{1}{2} C^c_{\ a} \omega^c \omega^b, \quad d\omega^a = \frac{1}{2} C^b_{\ a} \omega^c \omega^c. \]

By defining $\mathcal{G}$-valued 1-forms $\omega = \omega^a e_a$ and $\tilde{\omega} = \omega^a e_a$ and using the notation $[\alpha \wedge \beta]$ for the combined wedge and Lie bracket of $\mathcal{G}$-valued forms, we may write these relations in a basis independent way:

\[ dw^a = 0 = d\tilde{\omega} - \frac{1}{2} [\alpha \wedge \beta]. \]

Note that evaluating $\omega$ or $\tilde{\omega}$ on an element $X = X^a e_a$, e.g. or $Y = Y^a e_a$ e.g. respectively, is equivalent to evaluating those elements of the identity.
\[ \omega(X) = X^* \xi_a = \hat{X} = \hat{\omega}(\hat{X}), \]

The transpose of the differential (defined in (6.2)) at a left translation \( L_a \), maps the cotangent space at \( L_a(a) \) onto the cotangent space at \( a \).

By the left invariance of \( \{ e_a \} \):

\[ [dL_a(a)^* \omega^*(a)] \{ e_b(a) \} = \omega^*(a) \{ dL_a(a) e_b(a) \} \]
\[ = \omega^*(a) \{ e_k(a) \} = \delta^b_k \omega^*(a) \{ e_k(a) \}. \]

This establishes the result:

\[ \omega^*(a) = \omega^*(a) \{ e_k(a) \} = \omega^*(a). \]

We naturally call the \( 1 \)-forms \( \omega^* \) left invariant. Any left-invariant \( 1 \)-form \( \sigma \) is determined by its value at the identity:

\[ \sigma(a) = dL_a(a)^* \omega^*(a) = \omega^*(a). \]

The \( r \)-dimensional vector space of left-invariant \( 1 \)-forms for which \( \omega^* \) provides a basis is the dual vector space to \( g \) and will be denoted by \( g^* \).

Similarly, \( \omega^* \) is a basis for the dual vector space \( g^* \) of right invariant \( 1 \)-forms. \( \omega \) and \( \omega^* \) are the uniquely defined \( g \)-valued invariant \( 1 \)-forms which map \( g \) and \( g^* \) onto \( g^\ast \) according to our correspondence.

Modern differential geometry provides a very simple framework to develop Lie group theory. However, in any application, and in most classical texts on the subject, one usually deals with local coordinate systems.

It is therefore important to make a connection.

Let \( \{ a^\alpha \} \) and \( \{ X^\mu \} \) be local coordinates on \( G \) and \( M \), and define the functions \( a^\alpha \) and \( X^\mu \). With the shorthand notation \( \partial_a = \partial / \partial a^\alpha \), \( \partial_\alpha = \partial / \partial a^\alpha \), \( \partial_\mu = \partial / \partial x^\mu \), the components of the left and right differentials of \( \varphi \) and the differential of \( F_\zeta \) are given by the expressions:

\[ \partial_\alpha \varphi^\alpha = \partial_\alpha (a^\beta \varphi^\beta(a) \varphi^\alpha(a), \partial_\mu \varphi^\mu = \partial_\mu (x^\alpha \varphi^\alpha(a, x)), \partial_\alpha F_\zeta^\alpha = \partial_\alpha (a^\beta \varphi^\beta(a, x)), \partial_\mu F_\zeta^\mu = \partial_\mu (x^\alpha \varphi^\alpha(a, x)). \]

The coordinate basis \( \xi_a = \partial_a \) of \( T_G a \) induces bases \( \xi^\alpha = \partial_\alpha \) and \( \xi^\mu = \partial_\mu \).

One may construct the matrices of components of the dual frames \( \varphi^\alpha(a) \) and \( \varphi^\mu(a) \), as the inverse matrices to the component matrices of \( \xi_a \) and \( \xi^\mu \).
\[(\omega_\alpha e^\beta) = \delta_\beta^\gamma e^\alpha \omega^\gamma, \quad \omega_\alpha e^\beta = \delta_\beta^\gamma \omega^\gamma = \delta_\alpha^\gamma \omega^\gamma.\]

The left invariance of the frame \(\{e_\alpha\}\), namely \(dL_\alpha(a) e_\alpha(a) = e_\alpha \cdot q(a)\), has the coordinate expression:

\[\left[\omega_\alpha^\beta q^\gamma(a, a) e^\alpha(a) = e^\alpha \cdot q(a, a)\right]\]

\[\left[\omega_\alpha^\beta q^\gamma(a, a) = e^\alpha \cdot q(a, a) \omega^\gamma(a)\right]\]

\[\left[\omega_\alpha^\beta q^\gamma(a, a) = e^\alpha \cdot q(a, a) \omega^\gamma(a)\right].\]

The second line uses (1.32), while the third follows from a parallel discussion for the right invariant frame. Similarly from (1.33), namely \(dF_\alpha(a) e_\alpha(a) = e_\alpha \cdot F(a)\), we obtain:

\[\left[\omega_\alpha^\beta f^\gamma(a, a) = e_\alpha \cdot f(a, a) \omega^\gamma(a)\right].\]

The second use of (1.33) is a system of partial differential equations for the multiplication functions \(g^\alpha(a, a)\) with integration constants \(g^\alpha(a, a)\), while (1.34) is a system for the functions \(f^\alpha(a, a)\) with integration constants \(f^\alpha(a, a)\). In classical presentations the commutators of the coordinate induced bases of \(g\) and \(F(a)\) are obtained from the integrability conditions for these systems.
Let $\mathbf{X}$ be an analytic vector field on an analytic manifold $M$. An integral curve of $\mathbf{X}$ is a curve whose tangent coincides with the value of $\mathbf{X}$ at each point along it. Let $C_\mathbf{X}(x,t)$ be the integral curve which passes through $x$ at $t = 0$:

$$ C_\mathbf{X}(x,t) = \mathbf{X}_t \circ C_\mathbf{X}(x,t) \quad C_\mathbf{X}(x,0) = x. $$

In a coordinate system $\{x^i\}$ the components of $\mathbf{X}$ at $x$ are $X_i = \frac{\partial \mathbf{X}}{\partial x^i}(x)$. Using the notation $X^a = \sum X_i^a \delta_i^a$, equation (2.1) becomes:

$$ \frac{d}{dt} X^a(t) = \frac{\partial \mathbf{X}}{\partial x^i}(x^a) \delta_i^a C_\mathbf{X}(x,t). $$

This has the power series solution:

$$ C_\mathbf{X}(x,t) = (e^{t \mathbf{X}})(x), $$

where the exponential operator is defined by the usual series expansion.

(2.3) is just a Taylor series expansion of $X^a(x,t)$ about $t = 0$, valid for $t$ sufficiently small and obtained using iterations of (2.3) evaluated at $t = 0$ (making use of the old-fashioned chain rule):

$$ \left( \frac{d}{dt} \right)^n X^a(x,t) = (t^n X^a)(x). $$

We may interpret $t \mapsto C_\mathbf{X}(x,t)$ as a diffeomorphism of $M$ into itself for $|t|$ small enough. By varying $t$ from zero to a finite value the points of $M$ flow along the integral curves or streamlines of $\mathbf{X}$. $X_t$ may be interpreted as a curve in $\mathcal{D}(M)$ passing through the identity diffeomorphism since $X_0 = \text{Id}$. This curve is called the flow of $\mathbf{X}$. The coordinate power series representation (2.3) may be written:

$$ X^a = (e^{t \mathbf{X}})(x). $$

From this (ignoring global problems) it is clear that these diffeomorphisms form a 1-parameter group and hence a 1-dimensional subgroup of $\mathcal{D}(M)$:

$$ \left( e^{t \mathbf{X}} \right) \circ \left( e^{s \mathbf{X}} \right) = \left( e^{(t+s) \mathbf{X}} \right), \quad \left( e^{t \mathbf{X}} \right) \circ \left( \text{Id} \right) = \left( e^{t \mathbf{X}} \right). $$

This representation also makes evident the useful property $X_t = (t \mathbf{X})$.

When global statements hold $X$ is said to be complete.

Let $F$ be a function on $M$. Then we may power series expand $F \circ X_t$ about $t = 0$ exactly as we did $X^a \circ X_t$ to obtain locally:

$$ F \circ X_t = (e^{t \mathbf{X}} F)(x). $$

Suppose $\mathbf{Y}$ is another vector field and consider $\mathbf{Y}_t \circ X_t$, introducing the notation $C_{\mathbf{Y}}(x,t) = X^a \circ \mathbf{C}_\mathbf{Y}(x,t) = X^a \circ \mathbf{Y}_t(x)$. By an application
of (2.6) and (2.8):

\[ \text{C}_Y (x, s) = e^{s Y} \text{C}_0 (x) = e^{s Y} (e^{t X} x) = e^{t X} e^{s Y} x. \]

Note the reversal in order of the exponential operators, relative to the

corresponding maps. Suppose we naively multiply the exponential expansions

de \( e^{X} \) and \( e^{Y} \) and insert the product into the power series for

the logarithm. The result valid locally is:

\[ Z(x, y) = \log e^{X} e^{Y} = X + Y + \frac{1}{2} (X, Y) + \frac{1}{6} (X, X, Y) + \ldots. \]

The important feature of this result is that each term in the series is built from

brackets of \( X \) and \( Y \) and is therefore itself a vector field on \( M \) belonging to

the Lie subalgebra of \( \mathfrak{X}(M) \) generated by \( X \) and \( Y \). When the series converges,

it therefore converges to a vector field \( Z(x, y) \). We call \( Z(x, y) \) the Campbell-

Baker-Hausdorff functional or C-B-H functional for short.

Suppose \( X \) and \( Y \) belong to an \( r \)-dimensional Lie subalgebra \( \mathfrak{g} \subset \mathfrak{X}(M) \).

Since \( \mathfrak{g} \) is closed under the Lie bracket, each term in the C-B-H series

belongs to \( \mathfrak{g} \) and when the series converges, \( Z(X, Y) \) will also be in the

Lie algebra. By (2.3) and (2.7), \( \mathfrak{g} \cdot \mathfrak{g} = \mathfrak{g} \cdot \mathfrak{g} \) holds for all \( X, Y \in \mathfrak{g} \),

suitably near the origin. In other words the set of diffeomorphisms \( \{ \exp (X, X, \ldots, X) \} \)

is locally closed under composition and inverses as well since \( X_k \cdot X_l = \ldots = X \).

The products (i.e., compositions) of all such diffeomorphisms with any number

of factors from an \( r \)-parameter diffeomorphism group (i.e., \( r \)-dimensional

subgroup of \( \mathfrak{X}(M) \)) which acts on \( M \) by pushing its points along the

integral curves of the elements of its Lie algebra \( \mathfrak{g} \). Let \( \{ e^\alpha \} \) be a basis

of \( \mathfrak{g} \), \( \{ (w^\alpha) \} \) its dual basis and \( C^\alpha_{\beta \gamma} \) its \( B C T \) components. We may define

canonical coordinates \( \{ u^\alpha \} \) on the group by:

\[ u^\alpha (x) = w^\alpha (x) = X^\alpha. \]

The C-B-H functional then provides a power series representation for

the multiplication function in these coordinates:

\[ C^\alpha (u, u) = u^\alpha (Z[u^\beta u^\gamma, u^\delta u^\epsilon]) = u^\alpha + u^\beta + \frac{1}{2} C^\alpha_{\beta \gamma} u^\beta u^\gamma + \ldots. \]

Note that if two vector fields commute, \( Z(X, Y) = X + Y \) and their

1-parameter groups commute.

If we choose an arbitrary curve \( X(t) \) in the Lie algebra \( \mathfrak{g} \) rather than

a straight line through its origin, its action on \( M \) may also be described

in coordinates by a formal solution analogous to (2.4): \( X(t) \).
called a time-dependent vector field. Let \( C(x,t) \) be the curve whose tangent is \( X(t) \) at \( C(x,t) \) and such that \( C(x,0) = x \), i.e., the integral curve of \( X(t) \) through \( x \):

\[
C'(x,t) = X(t) \circ C(x,t)
\]

\[
\frac{d}{dt} C''(x,t) = X(t) \circ C(x,t) \quad X'' , \quad C''(x,0) = X''
\]

These equations have the usual iteration solution familiar from perturbation theory:

\[
C''(x,t) = \left( T \exp \left( \int_0^t dt X(t) \right) X'' \right)(x)
\]

\( \exp \) rather than \( e \) is used for typographical reasons and \( T \) indicates \( t \)-ordering along the path \( X(t) \). When \([X(t), X(t')] = 0\) it may be dropped. This is exactly the case for the curves \( X(t) = tX \), yielding the previous formula for the orbits of 1-parameter subgroups. The 1-parameter family of diffeomorphisms \( t \cdot I \), which map \( x \) onto \( C(x,t) \) is called the flow of the time-dependent vector field \( X(t) \) and represents a curve in \( \mathcal{X}(M) \).