Let G be a Lie group with Lie algebra $\mathfrak{g}$.

Let (ρ, ρ', V) be a representation of $(G, \mathfrak{g})$, ie $\begin{cases} \rho : G \to GL(V) \\ \rho' : \mathfrak{g} \to \mathfrak{gl}(V) \end{cases}$

where ρ' is the induced representation of $\mathfrak{g}$

Let $\{e_a\}$ be a basis of $\mathfrak{g}$ and $[e_a, e_b] = C^c{}_{ab} e_c$, and let $\{e_A\}$ be a basis of V and $\rho'_a{}^A{}_B$ the components of $\rho'(e_a)$ in this basis.

For $(Ad, ad, \mathfrak{g})$: $\rho'_a{}^b{}_c = C^b{}_{ac}$.

Let $T^{p,q}(M,V)$ and $\Lambda^p(M,V)$ be the vector spaces of V-valued $\binom{p}{q}$-tensor fields and p-forms on a manifold M. e_a and e_A are constant elements of $\Lambda^0(M,\mathfrak{g})$ and $\Lambda^0(M,V)$.

For $\mathfrak{g}$-valued forms define:
$$[\overset{\swarrow p}{A} \wedge \overset{\swarrow q}{B}] = A^a \wedge B^b C^c{}_{ab} e_c$$
$$= (-1)^{pq+1} [B \wedge A] \quad . \equiv ad(A) \wedge B$$

For a one-form A: $[A \wedge [A \wedge A]] = ad(A) \wedge ad(A) \wedge A = 0$

since $A^a \wedge A^b \wedge A^c \, C^e{}_{af} C^f{}_{bc} = 0$ by the Jacobi identity.

Note $d[A \wedge B] = [dA \wedge B] + (-1)^p [A \wedge dB]$.

NON RIGOROUS AND LOCAL DEFINITION:

A G-connection $\overset{G}{\nabla}$ on M is an $\mathbb{R}$-bilinear mapping $\nabla : \mathfrak{X}(M) \times \Lambda^0(M,V) \longrightarrow \Lambda^0(M,V)$, $(X, \phi) \mapsto \overset{G}{\nabla}_X \phi$ satisfying:

$$\overset{G}{\nabla}_{fX+gY} \phi = f \overset{G}{\nabla}_X \phi + g \overset{G}{\nabla}_Y \phi \qquad \phi \in \Lambda^0(M,V)$$
$$\overset{G}{\nabla}_X f\phi = f \overset{G}{\nabla}_X \phi + (Xf)\phi \qquad f, g \in F(M)$$
$$X, Y \in \mathfrak{X}(M).$$

for each representation (ρ, ρ', ∇) of $(G, \mathfrak{g})$, ~~and which is independent of the representation in the following sense~~ and is also such that $\overset{G}{\nabla}_X$ acts as an element of $\rho'(\mathfrak{g}) \subset \mathfrak{gl}(V)$ on constant elements of $\Lambda^0(M,V)$, and is independent of the representation in the following sense: if (ρ, ρ', ∇) is any representation for which $\ker \rho' = \{0\}$ then:
$$\overset{G}{\nabla}_X e_A = e_B \, \rho'(A(X))^B{}_A$$
defines $A \in \Lambda^1(M, \mathfrak{g})$; this formula holds for all representations with the same A.

A is called the connection one-form.

Any $A \in \Lambda^1(M,\mathfrak{g})$ determines a G-connection.

This is all I've defined

Note $\overset{G}{\nabla}_X \phi = [X + \rho'(A(X))] \phi$.

This is all I've defined. Throw out the above which got complicated because of the lack of "soldering" for a general gauge group.

Introduce the curvature two-form : $F = dA + \frac{1}{2}[A \wedge A] \in \Lambda^2(M, g)$

Alternatively : $(\overset{G}{\nabla}_X \overset{G}{\nabla}_Y - \overset{G}{\nabla}_Y \overset{G}{\nabla}_X - \overset{G}{\nabla}_{[X,Y]})\phi = \rho'(F(X,Y))\phi$, $\phi \in \Lambda^0(M, g)$.

Introduce the G-covariant exterior derivative $\overset{G}{D} : \Lambda^p(M,V) \to \Lambda^{p+1}(M,V)$:

$$\overset{G}{D}\phi = d\phi + \rho'(A)\wedge\phi \quad , \quad \phi \in \Lambda^p(M,V).$$

Ricci identity : $\overset{G}{D}{}^2\phi = \rho'(F)\wedge\phi$.

Bianchi identity : $\overset{G}{D}F = 0$

$$\begin{cases} dF = [dA \wedge A] = -ad(A)\wedge(F - \frac{1}{2}ad(A)\wedge A) = -ad(A)\wedge F \\ \overset{G}{D}F = dF + ad(A)\wedge F = 0 \end{cases}$$

To extend $\overset{G}{\nabla}$ to $T^{p,q}(M,V)$ we need an affine connection ∇ on M :

$$\overset{G}{\nabla}_X \phi = \nabla_X \phi + \rho'(A(X))\phi \qquad \phi \in T^{p,q}(M,V)$$

$$\overset{G}{D}\phi = D\phi + \rho'(A)\wedge\phi \qquad \text{for a } V\times\tbinom{p}{q}\text{-tensor field valued form.}$$

(Note that $\overset{G}{\nabla}$ obeys the Liebnitz rule with respect to representation tensor products)

Let $\{e_\alpha\}$ be a coordinate frame on M and $\overset{G}{\nabla}_\alpha = \overset{G}{\nabla}_{e_\alpha}$:

$$\overset{G}{\nabla}_\alpha \phi^A = \partial_\alpha \phi^A + A^a{}_\alpha \rho'_a{}^A{}_B \phi^B \qquad \phi \in \Lambda^0(M,V)$$

This ~~formula assumes ∇ is torsion free~~

$$[\overset{G}{D}\phi]_{\alpha_1\cdots\alpha_{p+1}} = (p+1)\,\overset{G}{\nabla}_{[\alpha_1}\phi_{\alpha_2\cdots\alpha_{p+1}]} \qquad \phi \in \Lambda^p(M,V)$$

$$F_{\alpha\beta} = 2\partial_{[\alpha}A_{\beta]} + [A_\alpha, A_\beta]$$

$$F^a{}_{\alpha\beta} = \partial_\alpha A^a_\beta - \partial_\beta A^a_\alpha + C^a{}_{bc}A^b_\alpha A^c_\beta$$

$$2\overset{G}{\nabla}_{[\alpha}\overset{G}{\nabla}_{\beta]}\phi = \rho'(F_{\alpha\beta})\phi \qquad \phi \in \Lambda^0(M,V)$$

For ~~compact~~ unitary groups define $\mathbb{e}_a = i e_a$, $[\mathbb{e}_a, \mathbb{e}_b] = i C^c_{ab}\,\mathbb{e}_c$

and let $A = g\mathbb{A}$ where g is called a coupling constant.

(with direct product groups each connection can have its own coupling constant)

$$A^a e_a = -ig\mathbb{A}^a\mathbb{e}_a = -ig\vec{\mathbb{A}} \qquad \vec{\mathbb{A}} = \mathbb{A}^a\mathbb{e}_a.$$

$$F = -ig\vec{\mathbb{F}} , \qquad \vec{\mathbb{F}} = \mathbb{F}^a\mathbb{e}_a \qquad F^a = g\mathbb{F}^a$$

An affine connection on M is a special case with $G = GL(n,\mathbb{R})$ and V the tensor products of $\mathbb{R}^n$ with itself and ρ the tensor products of the identity representation, together with a soldering provided by local frames $\{e_\alpha\}$ which enable one to identify each TM_x with $\mathbb{R}^n$.

A global gauge transformation is a mapping of the representation algebra $\{T^{p,q}(M,V)\}$ into itself: $\quad \phi \mapsto \rho(u)\phi \equiv u\cdot\phi \; , \; u\in G, \phi\in T^{p,q}(M,V)$

A local gauge transformation is the same except u is now a map $u: M\to G$ and A behaves as follows:

$$A \mapsto u\cdot A = \mathrm{Ad}(u)A - u^*\omega$$

where ω is the canonical $\mathfrak{g}$-valued one-form on G.

This is just the natural "dragging along of the connection" by the local gauge transformation:

$$(u\cdot \overset{G}{\nabla}_X)\,\Phi_A \equiv \rho'(u\cdot A(X))\, e_A$$
$$\equiv \rho(u)\, \overset{G}{\nabla}_X\,(\rho^{-1}(u)\, e_A)$$
$$= \underbrace{\rho(u)\nabla_X \rho^{-1}(u)\, e_A}_{-\rho'((u^*\omega)(X))} + \underbrace{[\mathrm{Ad}_{\rho(u)}\, \rho'(A)]}_{\rho'(\mathrm{Ad}_u A)\, e_A}\, e_A \quad = \rho'(((\mathrm{Ad}_u A - u^*\omega)(X))\, e_A$$

This is true for all representations only if the above trans law is obeyed. In other words A transforms so that covariant differentiation commutes with local gauge transformations.

One may easily verify: $u\cdot F = \mathrm{Ad}_u F$.

DERIVATIVE GAUGE TRANSFORMATION

Let $u_t = \exp Z_t$ be a curve of gauge transformations and $Z' = d/dt|_0 Z_t$.

Then:
$$d/dt|_0 \; u_t\cdot\phi = \rho'(Z')\phi$$
$$d/dt|_0 \; u_t\cdot A = \mathrm{ad}(Z')A - dZ' = -\overset{G}{D}Z'.$$

~~Wu and Yang~~ Unitary or compact groups:
$$\begin{cases} u\cdot\vec{A} = \mathrm{Ad}(u)\vec{A} + \frac{i}{g}(-u^*\omega) \\ \vec{F} = d\vec{A} - ig\frac{1}{2}[\vec{A}\wedge\vec{A}] \end{cases}$$

Wu and Yang. All of this discussion assumes a ~~single~~ global smooth connection one-form on M (ie a trivial bundle) but this is usually sufficient for field theorists. However in the general case, from this nonbundle point of view one must consider a covering of M with a connection one-form defined on each open set in the covering such that they are related by gauge transformations on the intersections ... as Wu and Yang capitalized on.

(v)

Yang Mills groups. To do a Yang Mills theory one needs an adjoint invariant inner product on the Lie algebra. These exist only on semisimple or abelian groups or direct products of such groups.

Yang Mills Lagrangian: $L_{YM} = -\frac{1}{4} \langle F_{\alpha\beta}, F^{\alpha\beta} \rangle = -\frac{1}{4} F^a_{\alpha\beta} F^{\alpha\beta}_a$.

To consider real Higgs fields $\phi \in \Lambda^0(M,V)$ we need a unitary representation (ρ, ρ', V), ie a ρ-invariant inner product on V: $\langle \rho_u \phi, \rho_u \phi \rangle = \langle \phi, \phi \rangle$.

Higgs Lagrangian: $L_H = -\frac{1}{2} \langle \overset{G}{\nabla}_\alpha \phi, \overset{G}{\nabla}^\alpha \phi \rangle - V(\phi) = -\frac{1}{2} \overset{G}{\nabla}_\alpha \phi^A \overset{G}{\nabla}^\alpha \phi_A - V(\phi)$

where ~~$V(\rho\phi)$~~ $V(\rho_u \phi) = V(\phi)$ is a ρ-invariant potential.

Variation formulas (1) $F = dA + \frac{1}{2}[A \wedge A]$ $\quad F' = dA' + [A \wedge A'] = DA'$

or $F'^a{}_{\alpha\beta} = 2 \overset{G}{\nabla}_{[\alpha} A'^a_{\beta]}$

(2) $L'_{YM} = -\frac{1}{2} F^{\alpha\beta}_a F'^a{}_{\alpha\beta} = -F^{\alpha\beta}_a \overset{G}{\nabla}_\alpha A^a_\beta \sim \overset{G}{\nabla}_\alpha F^{\alpha\beta}_a A'^a_\beta$

where $\sim$ means equal up to a divergence.

Now let prime indicate variation in A only:

$L'_\phi = -\langle \overset{G}{\nabla}^\alpha \phi, \rho'(A'_\alpha) \phi \rangle \equiv J^\alpha_{\phi a} A^{a\prime}_\alpha \qquad J^\alpha_\phi = -\langle \nabla^\alpha \phi, \rho'(e_a) \phi \rangle$

Now let prime indicate variation in ϕ only:

$L_\phi' \sim \langle \overset{G}{\nabla}_\alpha \overset{G}{\nabla}^\alpha \phi, \phi' \rangle - V'_A(\phi) \phi'^A$

The field equations are thus: $\begin{cases} F^{\alpha\beta}_a{}_{;\beta} = J^\alpha_{\phi a} & \text{or} \quad \overset{G}{D} {*F} = {*J_\phi} \quad \text{or} \quad \overset{G}{\delta} F = J_\phi \\ \overset{G}{\nabla}_\alpha \overset{G}{\nabla}^\alpha \phi^A - V'^A(\phi) = 0 \end{cases}$

For complex Higgs fields one needs a hermitian ρ-invariant $\langle , \rangle$ on V, and the generalization is immediate.

EINSTEIN, YM, HIGGS. Fischer-Marsden idea for gauge-generators in 3+1 split:

Take the relevant inner product of the momentum with the derivative of an inverse gauge transformation on the field and integrate by parts to remove derivatives from the gauge parameter field:

$N \lrcorner J_E \approx \langle \Pi_E, \mathcal{L}_N g \rangle = \Pi^{ij} 2 N_{i|j} \sim N^i (-2\Pi_i{}^j{}_{|j}) = N \lrcorner 2\delta \Pi_E$

$N \lrcorner J_{YM} \sim \langle \Pi_{YM}, \mathcal{L}_N A \rangle \approx N \lrcorner (\langle \Pi_{YM} \times B \rangle + A^a C_a)$ (computation)

$N \lrcorner J_H \sim \langle \Pi_H, \mathcal{L}_N \phi \rangle \sim N \lrcorner \{ \langle \Pi, d\phi \rangle \}$

$\lambda^a C_a \sim \underbrace{\langle \Pi_{YM}, D\lambda \rangle}_{\Pi_{YM}{}^i{}_a \lambda^a{}_{;i} \sim -\lambda^a \overset{G}{\nabla}_i \Pi_{YM}{}^i{}_a} + \langle \Pi_H, -\rho'(\lambda)\phi \rangle \sim \lambda^a (\delta \Pi_{YM\,a} \underbrace{- \Pi_{HA} \rho'_a{}^A{}_B \phi^B}_{Q_{Ha}})$

So $\begin{cases} J = 2\delta \Pi_E + \Pi_{YM\,a} \times B^a + A^a C_a \\ C_a = \delta \Pi_{YM\,a} + Q_{Ha} \end{cases}$

$\{ \int \lambda_1^a C_a, \int \lambda_2^a C_a \} = -\int [\lambda_1, \lambda_2]^a C_a$

$\{ \int N_1 \lrcorner J, \int N_2 \lrcorner J \} = -\int [N_1, N_2] \lrcorner J$. (right action)

↑ minus since J, C generate inverse gauge transformations

SO(4,ℝ) acting on $\mathbb{R}^4$

(1) $g_{\alpha\beta} = \delta_{\alpha\beta}$, raise lower indices (trivial).

(2) $(I_{\alpha\beta})^{\gamma\delta} = -\delta_{\alpha\beta}{}^{\gamma\delta}$, $I_{\alpha\beta} = I_{\alpha\beta}{}^{\gamma}{}_{\delta}\,\hat{e}^{\delta}{}_{\gamma} \in so(4)$

since $\omega = \omega^{\alpha}{}_{\beta}\hat{e}^{\beta}{}_{\alpha} \in so(4) \rightarrow \quad g_{\alpha(\alpha}\omega^{\gamma}{}_{\beta)} = 0$

(3) $[I^{\alpha\beta}, I^{\gamma\delta}]_{\mu\nu} = I^{\alpha\beta}{}_{\mu\rho}\,I^{\gamma\delta\rho}{}_{\nu} - I^{\gamma\delta}{}_{\mu\rho}\,I^{\alpha\beta\rho}{}_{\nu} = 2\,I^{\alpha\beta}{}_{[\mu|\rho|}\,I^{\gamma\delta\rho}{}_{\nu]}$

$= +2\,\delta^{\alpha\beta}{}_{\rho[\mu}\,\delta_{\nu]}{}^{\rho}{}_{\gamma\delta} = 2(\delta^{\alpha}_{\rho}\delta^{\beta}_{[\mu} - \delta^{\beta}_{\rho}\delta^{\alpha}_{[\mu})(g_{\nu]\gamma}\delta^{\rho}_{\delta} - g_{\nu]\delta}\delta^{\rho}_{\gamma})$

$= -\,\delta^{\alpha}_{\gamma}\,\delta^{\beta}{}_{\delta\,\mu\nu} + \cdots$

$= +4\cdot\delta^{[\alpha}_{[\gamma}(I^{\beta]}{}_{\delta]})_{\mu\nu}$ $\quad [I^{23}, I_{31}] = 4\,\delta^{[2}_{[3}I^{3]}_{1]} = -I^{2}{}_{1} = I^{12}$

so $\boxed{\omega = -\tfrac{1}{2}\omega^{\alpha\beta}I_{\alpha\beta}}$ $\quad \omega_{(\alpha\beta)} = 0.$ $\quad$ (be careful diff ω)

$O = \exp\omega \quad \in SO(4).$

Let $\{e_\alpha\}$ be an ON basis of $\mathbb{R}^4$, dual basis $\{\omega^\alpha\}$. Let $\omega^{\alpha_1\ldots\alpha_p} \equiv \omega^{\alpha_1}\wedge\ldots\wedge\omega^{\alpha_p}$.

$X = \frac{1}{p!}X_{\alpha_1\ldots\alpha_p}\,\omega^{\alpha_1\ldots\alpha_p}$, $\quad {*X}_{\alpha_{p+1}\ldots\alpha_n} = \frac{1}{(n-p)!}\,X^{\alpha_1\ldots\alpha_p}\,\epsilon_{\alpha_1\ldots\alpha_p\alpha_{p+1}\ldots\alpha_n}$

$$\left[(*)^2 = (-1)^{p(4-p)} \quad \begin{array}{ccccc} = 1 & -1 & 1 & -1 & 1 \\ p= 0 & 1 & 2 & 3 & 4 \end{array}\right.$$

$\Lambda_2(\mathbb{R}^4)$ is naturally isomorphic with $so(4)$.

Let $\mathcal{P}_{\pm}X = \frac{1}{2}(X + {*X})$, $X \in \Lambda_2($

$= X^{(\pm)}$

$\boxed{{*X}^{(\pm)} = \pm X^{(\pm)}}$

$\left\{\begin{array}{l} \mathcal{P}_{\pm}^{\alpha\beta}{}_{\gamma\delta} = \frac{1}{2}(\delta^{\alpha\beta}{}_{\gamma\delta} \pm E^{\alpha\beta}{}_{\gamma\delta}) \\ (\mathcal{P}_{\pm}X)_{\alpha\beta} = X_{\gamma\delta}\,P_{\pm}^{\gamma\delta}{}_{\alpha\beta} \end{array}\right.$

~~$\omega^{(\pm)ab} = \frac{1}{2}$ $\omega^{ab(\pm)} = \frac{1}{2}$~~

${*X}_{ab} = X^{c4}\,\epsilon_{abc4} = X^{c4}$

${*X}_{a4} = X^{bc}\,\epsilon_{bca4} = X^{bc}$

$X^{(\pm)}{}_{ab} = \frac{1}{2}(X_{ab} \pm X_{c4})$

Let $L^{\pm}_{a} = \frac{1}{2}(I_{bc} \pm I_{a4})$, $\quad {*L^{\pm}_{a}} = \pm L^{\pm}_{a}$.

S $\quad [L^{\pm}_{a}, L^{\pm}_{b}] = \epsilon_{abc}L^{\pm}_{c} \quad [L^{+}_{a}, L^{-}_{b}] = 0$.

So the Lie algebra is a direct sum of two mutually commuting isomorphic subalgebras

$so(4) \approx \Lambda_2^{+} \oplus \Lambda_2^{-}.$

Let $\omega = \theta^{+}n^{a}_{+}L^{+}_{a} + \theta^{-}n^{a}_{-}L^{-}_{a}$, $\quad \delta_{ab}n^{a}_{\pm}n^{b}_{\pm} = 1$.

$\exp\omega = \exp\theta^{+}n^{a}_{+}L^{+}_{a}\ \exp\theta^{-}n^{a}_{-}L^{-}_{a}$.

$e^{\omega}L^{\pm}_{a}e^{-\omega} = e^{\mathrm{ad}\,\theta^{\pm}n^{b}_{\pm}L^{\pm}_{b}}L^{\pm}_{a} = L^{\pm}_{b}(e^{\theta^{\pm}n^{a}_{\pm}I^{\pm}_{c}})^{b}{}_{a}$

$\left\{\begin{array}{l}\text{so the } \pm \text{ 3-dim groups act independently on } \Lambda_2^{+} \text{ and } \Lambda_2^{-}. \\ \text{as } SO(3,\mathbb{R}).\end{array}\right.$

$\mathbb{R}^4$ with natural basis $\{e_\alpha\}$, dual basis $\{\omega^\alpha\}$, natural inner product : $\langle e_\alpha, e_\beta\rangle = \delta_{\alpha\beta} \equiv g_{\alpha\beta}$.

Let $\{\gamma_\alpha\} \subset gl(4,\mathbb{C})$ be a set of 4 matrices satisfying : $\boxed{\gamma_{(\alpha}\gamma_{\beta)} = -g_{\alpha\beta}1}$

. These generate a 16-dimensional (real) Clifford algebra isomorphic to $gl(4,\mathbb{C})$.

Define $\gamma_{\alpha_1\cdots\alpha_p} = \gamma_{[\alpha_1\cdots\alpha_p]}$

$$\gamma_5 = *\gamma = \tfrac{1}{4!}\epsilon_{\alpha\beta\gamma\delta}\gamma^{\alpha\beta\gamma\delta} = \gamma_1\gamma_2\gamma_3\gamma_4$$

$$\gamma_5^2 = 1 \quad , \{\gamma_5, \gamma_\alpha\} = 0$$

$$\text{pf.}\begin{cases} \gamma_5^2 = (\gamma_1\gamma_2\gamma_3\gamma_4)(\gamma_1\gamma_2\gamma_3\gamma_4) = (\gamma_1\gamma_2)^2(\gamma_3\gamma_4)^2 = (-\gamma_1^2\gamma_2^2)(-\gamma_3^2\gamma_4^2) = 1 \\ (\gamma_1\gamma_2\gamma_3\gamma_4)\gamma_4 = -\gamma_4\gamma_5 \quad \text{etc.} \end{cases}$$

Let $P_\pm = (1\pm\gamma_5)/2$. These project into the ± 1 eigenspaces $S_\pm \subset \mathbb{C}^4$ of γ_5.

Evaluate the duals of $\{\gamma_{\alpha_1\cdots\alpha_p}\}$: $\quad *\gamma_{\alpha_{p+1}\cdots\alpha_4} = \frac{1}{(4-p)!}\gamma^{\alpha_1\cdots\alpha_p}\epsilon_{\alpha_1\cdots\alpha_p\alpha_{p+1}\cdots\alpha_4}$.

$$*\gamma_{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta\gamma\delta} \quad , *1 = \gamma_5$$

$$\begin{cases} *\gamma_{\alpha\beta} = -\gamma_5\gamma_{\alpha\beta} \\ \gamma_{\alpha\beta} = -\gamma_5 {*\gamma_{\alpha\beta}} \end{cases} , \quad \gamma^\pm_{\alpha\beta} = \tfrac{1}{2}(\gamma_{\alpha\beta} \pm *\gamma_{\alpha\beta}) = P_\mp\gamma_{\alpha\beta}.$$

$$\begin{cases} *\gamma_\delta = \tfrac{1}{3!}\epsilon_{\alpha\beta\gamma\delta}\gamma^{\alpha\beta\gamma} = -\gamma_5\gamma_\alpha & \gamma_\alpha = -\gamma_5 {*\gamma_\alpha} \\ *\gamma_{\alpha\beta\gamma} = \gamma^\delta\epsilon_{\delta\alpha\beta\gamma} = \gamma_5\gamma_{\alpha\beta\gamma} & \gamma_{\alpha\beta\gamma} = \gamma_5 {*\gamma_{\alpha\beta\gamma}}. \end{cases}$$

Let $X = \cancel{X_\alpha\omega^\alpha}\; x + X_\alpha\omega^\alpha + \tfrac{1}{2}X_{\alpha\beta}\omega^{\alpha\beta} + \tfrac{1}{3!}X_{\alpha\beta\gamma}\omega^{\alpha\beta\gamma} + \tfrac{1}{4!}X_{\alpha\beta\gamma\delta}\omega^{\alpha\beta\gamma\delta} \in \Lambda(\mathbb{R}^4)$

be an element of the exterior algebra over $\mathbb{R}^4$.

$\Lambda(\mathbb{R}^4)$ is naturally isomorphic with Cliff :

$X \in \Lambda(\mathbb{R}^4) \mapsto SX \in \text{Cliff}$

$$SX = x1 + X_\alpha\gamma^\alpha + \tfrac{1}{2}X_{\alpha\beta}\gamma^{\alpha\beta} + \underbrace{\tfrac{1}{3!}X_{\alpha\beta\gamma}\gamma^{\alpha\beta\gamma}}_{-*X_\alpha\gamma_5\gamma^\alpha} + \underbrace{\tfrac{1}{4!}X_{\alpha\beta\gamma\delta}\gamma^{\alpha\beta\gamma\delta}}_{*X\gamma_5}$$

$$= \underbrace{\tfrac{1}{2}(x\pm *x)}_{y^\pm}P_\pm \;\; \underbrace{\pm(X_\alpha \mp *X_\alpha)}_{y_\alpha^\pm}P_\pm\gamma^\alpha + \underbrace{\tfrac{1}{2}X^{(\mp)}_{\alpha\beta}}_{\frac{1}{2}y^\pm_{\alpha\beta}}P_\pm\gamma^{\alpha\beta}.$$

If $\gamma_5 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2\end{pmatrix}$ then $\{\gamma_5, \gamma_\alpha\} = 0 \to \quad \gamma_\alpha \sim \begin{bmatrix} 0 & * \\ * & 0\end{bmatrix} \quad \gamma_{\alpha\beta} \sim \begin{bmatrix} * & 0 \\ 0 & *\end{bmatrix}$

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix} \quad P_+\gamma^\alpha = \begin{pmatrix} 0 & * \\ 0 & 0\end{pmatrix} \quad P_+\gamma_{\alpha\beta} = \begin{bmatrix} * & 0 \\ 0 & 0\end{bmatrix}$$

$$P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix} \quad P_-\gamma^\alpha = \begin{pmatrix} 0 & 0 \\ * & 0\end{pmatrix} \quad P_-\gamma_{\alpha\beta} = \begin{bmatrix} 0 & 0 \\ 0 & *\end{bmatrix}$$

$$gl(4,\mathbb{C}) \approx \mathbb{C}^4\otimes\mathbb{C}^{4*} = (S_+\oplus S_-)\otimes(S_+^*\oplus S_-^*) = \underset{\begin{bmatrix} * & 0 \\ 0 & 0\end{bmatrix}}{(S_+\otimes S_+)} \oplus \underset{\begin{bmatrix} 0 & * \\ 0 & 0\end{bmatrix}}{(S_+\otimes S_-)} \oplus \underset{\begin{bmatrix} 0 & 0 \\ * & 0\end{bmatrix}}{(S_-\otimes S_+)} \oplus \underset{\begin{bmatrix} 0 & 0 \\ 0 & *\end{bmatrix}}{(S_-\otimes S_-)}$$

So $\Lambda_2^\pm \approx \text{TRless}(S_\mp\otimes S_\mp)$. But the other components are isomorphic to linear combinations of Λ_p and $*\Lambda_p$.

$I_{\alpha\beta} \leftrightarrow \frac{1}{2}\gamma_{[\alpha}\gamma_{\beta]}$ or $\omega = -\frac{1}{2}\omega^{\alpha\beta} I_{\alpha\beta} \to \mathrm{spin}(\omega) = -\frac{1}{4}\omega^{\alpha\beta}\gamma_\alpha\gamma_\beta$
is a Lie algebra isomorphism from $so(4,\mathbb{R})$ into $\mathrm{spin}(4) = \mathrm{span}\{\gamma_{\alpha\beta}\}$.
Let $\mathrm{Spin}(4) = \exp \mathrm{spin}(4)$, and $\boxed{e^{\mathrm{spin}(\omega)}\gamma_\alpha e^{-\mathrm{spin}(\omega)} = \gamma_\beta (e^{\omega})^\beta{}_\alpha}$

Let $e^\pm_a = (\gamma_{bc} \pm \gamma_{a4})/2$. Then $\frac{1}{2}e^\pm_a \leftrightarrow L^\pm_a$.
Let $S(\theta^+ n_+, \theta^- n_-) = \exp \theta^+ n_+^a \frac{1}{2} e^+_a \; \exp \theta^- n_-^a \frac{1}{2} e^-_a \equiv S^+(\theta^+, n_+) S^-(\theta^- n_-)$.

$$\left(P_+ \cos\frac{\theta^+}{2} + n_+^a e^+_a \sin\frac{\theta^+}{2}\right)\left(P_- \cos\frac{\theta^-}{2} + n_-^a e^-_a \sin\frac{\theta^-}{2}\right)$$

$S^+(4\pi, n_+) = P_+$
$S^+(2\pi, n_+) = -P_-$

acts isomorphically to $SU(2)$ on S_+

acts isomorphically to $SU(2)$ on S_-.

$\mathrm{Spin}(4) \approx SU(2) \times SU(2)$.

The above box provides the homomorphism from $\mathrm{Spin}(4)$ onto $SO(4,\mathbb{R})$.
Its kernel is $\{1, -1\}$ so $SO(4\mathbb{R}) \approx \mathrm{Spin}(4)/\mathbb{Z}_2$.

Suppose $A \in \mathrm{Spin}(4)$, then $A\gamma_\alpha A^{-1} = \gamma_\beta L^\beta{}_\alpha$ with $L \in SO(4,\mathbb{R})$
and $A\gamma^\alpha A^{-1} = L^{-1\alpha}{}_\beta \gamma^\beta$. So:

$$A\, S X\, A^{-1} = x1 + X_\beta L^{-1\beta}{}_\alpha \gamma^\alpha + \frac{1}{2} X_{\gamma\delta} L^{-1\gamma}{}_\alpha L^{-1\delta}{}_\beta \gamma^{\alpha\beta} + \dots$$

Conjugation of Cliff by the Spin(4) group produces the action of $SO(4\mathbb{R})$ on the exterior algebra via the identification S.

Explicit Realization Let $\{\sigma_a\}$ be the Pauli matrices and $\rho_1 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0\end{pmatrix}$, $\rho_2 = \begin{pmatrix} 0 & -i1_2 \\ i1_2 & 0\end{pmatrix}$ $\rho_3 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2\end{pmatrix}$.
Then let $(\gamma_\alpha) = (i\rho_2, -i\rho_1\underline{\sigma}) \sim \begin{bmatrix} 0 & (1,\underline{e}) \\ (-1,\underline{e}) & 0 \end{bmatrix}$ where $e_a = -i\sigma_a$.
$\gamma_5 = \rho_3$, $\gamma_{5\alpha} = (\rho_1, \rho_2\underline{\sigma})$
$\gamma_{bc} = -i\sigma_a$, $\gamma_{a4} = -\rho_3(-i\sigma_a)$, $e^\pm_a = P_\mp(-i\sigma_3)$

$S^+(\theta^+, n_+) = \begin{bmatrix} 1 & 0 \\ 0 & \exp -i\theta n^a\sigma_a \end{bmatrix}$ $S^-(\theta, n) = \begin{pmatrix} \exp -i\theta n^a\sigma_a & 0 \\ 0 & 1 \end{pmatrix}$.

This is the usual physicist convention for SU(2) parametrization

We may attach a quaternary spinor structure to a Riemannian 4-manifold $\mathcal{M}$ by introducing a global orthonormal frame $\{e_\alpha\}$, dual frame $\{W^\alpha\}$ and using this frame to identify the Clifford algebra with the exterior algebra of forms on M. To do this simply identify $\{e_\alpha\}$, $\{W^\alpha\}$ with the natural basis ~~and~~ of $\mathbb{R}^4$ and its dual basis and take over all of the above formulas.

Let $S^4 = \mathbb{C}^4$ with natural basis $\{E_L\}$ and dual basis $\{W^L\}$. One may construct the joint tensor algebra (= spinor algebra) over both S^4 and its complex conjugate space $\overline{S^4}$. The elements of this algebra are called spinors; their valence is now characterized by $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ where $\begin{pmatrix} p \\ q \end{pmatrix}$ is the valence of the S^4 (undotted) indices and $\begin{pmatrix} r \\ s \end{pmatrix}$ is the valence of the $\overline{S}^4$ (dotted) indices. For example:

$\Psi = \Psi^{L\dot{M}}{}_P \, E_L \otimes \bar{E}_M \otimes W^P$ is a $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ spinor,

while Cliff $= S^4 \otimes S^{4*}$ consists of the $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ spinors:

$$\gamma_\alpha = \gamma_\alpha{}^L{}_M \, E_L \otimes W^M .$$

Now consider spinor valued fields on M or "spinorfields". The above identification of $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ spinor fields with the exterior algebra of forms on M "solders" the spinor algebra over $\mathbb{C}^4$ to M.

Let $g = g_{\alpha\beta} W^\alpha \otimes W^\beta$ with $g_{\alpha\beta} = \delta_{\alpha\beta}$ be the metric.

Introduce the components $\Gamma^\gamma{}_{\alpha\beta}$ of the metric connection ∇ by: $\nabla_{e_\alpha} e_\beta = \Gamma^\gamma{}_{\alpha\beta} e_\gamma$.

connection one-forms $\omega^\alpha{}_\beta = \Gamma^\alpha{}_{\gamma\beta} W^\gamma$

curvature two forms $\Omega^\alpha{}_\beta = \frac{1}{2} R^\alpha{}_{\beta\gamma\delta} W^{\gamma\delta}$.

$\underline{\Omega} = d\underline{\omega} + \underline{\omega} \wedge \underline{\omega}$.

$\omega_{(\alpha\beta)} = 0$ since $g_{\alpha\beta;\gamma} = 0 + g_{\delta\beta}\Gamma^\delta{}_{\gamma\alpha} + g_{\alpha\delta}\Gamma^\delta{}_{\gamma\beta} = 2\Gamma_{(\alpha|\gamma|\beta)}$.

$\Omega_{(\alpha\beta)} = 0$ since by the Ricci identity $2\, g_{\alpha\beta;[\gamma\delta]} = 2R_{(\alpha\beta)\gamma\delta} = 0$.

So $\underline{\omega}$ and $\underline{\Omega}$ are $so(4,\mathbb{R})$ valued forms. Now we wish to extend ∇ to spinor fields: $\nabla_{e_a} E_L = K^M{}_{aL} E_M$, $\underline{K} = K_a W^a$ is a matrix valued spin connection form. $\underline{\mathcal{R}} = d\underline{K} + \underline{K} \wedge \underline{K}$ is the spin curvature two form (matrix valued).

We connect up the spin connection by ~~demanding~~ using the isomorphism of $so(4,\mathbb{R})$ and spin(4).

$\underline{K} = -\frac{1}{4} \omega^{\alpha\beta} \gamma_{\alpha\beta}$ $\quad \underline{\mathcal{R}} = -\frac{1}{4} \Omega^{\alpha\beta} \gamma_{\alpha\beta}$.

(We are interpreting $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ spinor fields as matrices)

This has the effect of making the spinor tensor field with (constant) components $\gamma_\alpha{}^L{}_M$ covariant constant:

$$\nabla \gamma_\alpha = 0 + \cancel{K_{\alpha P} \gamma_\alpha} \; K \gamma_\alpha - \gamma_\alpha K = -\tfrac{1}{4} \omega^{\gamma\delta} [\gamma_{\gamma\delta}, \gamma_\alpha] = 0.$$

Example. $$\Psi_\alpha{}^{L\dot{M}}{}_{N;\beta} = e_\beta \Psi_\alpha{}^{L\dot{M}}{}_N - \Gamma^\gamma{}_{\beta\alpha} \Psi_\gamma{}^{L\dot{M}}{}_N + K_\beta{}^L{}_P \Psi_\alpha{}^{P\dot{M}}{}_N - \overline{K_\beta{}^M{}_P} \Psi_\alpha{}^{L\dot{P}} - K^P{}_{\beta N} \Psi_\alpha{}^{L\dot{M}}{}_P$$

are the components of the covariant derivative of the $\binom{0}{1}$-tensor/$\binom{1\,1}{1\,0}$ spinor field. It is convenient to ~~symbolize this~~ introduce a shorthand by realizing that the nonderivative parts are just the action of the $gl(4,\mathbb{R})$ ~~representat~~ and $gl(4,\mathbb{C})$ representations on Ψ, evaluated on ω_α; let $\underline{\Psi}$ symbolize the components of Ψ:

$$\cancel{[\nabla_{e_\alpha} \Psi]_\alpha{}^{L\dot{M}}{}_N =} \quad \nabla_{e_\alpha} \underline{\Psi} = e_\alpha \underline{\Psi} + \rho'(\omega_\alpha)\underline{\Psi} + \sigma'(K_\alpha)\underline{\Psi}$$

$\boxed{\text{if } \Psi \text{ is a } \binom{p}{q} \text{ tensor}/\binom{r\,t}{s\,u} \text{ spinor field}}$

the $\binom{p}{q}$ tensor rep. of $gl(4\mathbb{R})$

$\binom{r\,t}{s\,u}$ spinor rep of $gl(4,\mathbb{C})$.

Now introduce tensor/spinor-valued forms Ψ of type (ρ',σ'); ie forms with extra indices of valence $\binom{p}{q} \leftrightarrow \rho'$ and $\binom{r\,t}{s\,u} \leftrightarrow \sigma'$.

$\Omega^\alpha{}_\beta$ is a $\binom{1}{1}$-~~valued~~ tensor valued 2-form.

$g_{\alpha\beta}$ is a $\binom{0}{2}$-tensor valued 2-form.

$R^L{}_M$ is a $\binom{1\,0}{1\,0}$-spinor valued 2-form ...

Define the covariant exterior derivative D by:

$$D\Psi = d\underline{\Psi} + \rho'(\omega)\wedge\underline{\Psi} + \sigma'(K)\wedge\underline{\Psi}$$

~~where underlining indicates components.~~ For example if Ψ is a scalar valued p-form (ie ordinary p-form) $D\Psi = d\Psi$, while if Ψ is a tensor/spinor valued zero form, $D\Psi = \nabla\Psi$.

In components: $[D\Psi]_{\alpha_1\dots\alpha_{p+1}} = (p+1)\, \nabla_{[\alpha_1} \Psi_{\alpha_2\dots\alpha_p]}$ (same as in definition of d)

Ricci identity: $[D^2\Psi] = \rho'(\Omega)\wedge\Psi + \sigma'(R)\wedge\Psi.$

or $$2\nabla_{[\alpha}\nabla_{\beta]}\Psi = \rho'(\underbrace{\Omega_{\alpha\beta}})\Psi + \sigma'(\underbrace{R_{\alpha\beta}})\Psi$$

two form indices.

Finally let the divergence of a p-form ω be:

$$[\delta\omega]_{\alpha_2\cdots\alpha_p} = -\nabla_{\alpha_1}\omega^{\alpha_1}{}_{\alpha_2\cdots\alpha_p}.$$

; (δ is the adjoint of d. with respect to the natural inner product of forms)

Easy Fact. $$\underbrace{\gamma^\alpha\nabla_\alpha}_{\text{Dirac operator}}\ \underbrace{S\mathbb{X}}_{\text{element Cliff valued fields, i.e. } \left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)\text{-spinor field}} = S(d+\delta)\mathbb{X}$$

$$(\gamma^\alpha\nabla_\alpha)^2 S\mathbb{X} = S(d+\delta)(d+\delta)\mathbb{X} = S\underbrace{(d\delta+\delta d)}_{\Delta}\mathbb{X} = S\Delta\mathbb{X}$$

since $d^2=0=\delta^2$. $\qquad \Delta$: DeRham Laplacian.

pf.

$$\gamma^\alpha\nabla_\alpha\, X_{\alpha_1\cdots\alpha_p}\gamma^{\alpha_1\cdots\alpha_p} = \gamma^\alpha\gamma^{\alpha_1\cdots\alpha_p}\nabla_\alpha X_{\alpha_1\cdots\alpha_p}$$

$$= \underbrace{\gamma^{(\alpha}\gamma^{\alpha_1)}}_{-g^{\alpha\alpha_1}}\gamma^{\alpha_2\cdots\alpha_p}\nabla_\alpha X_{\alpha_1\cdots\alpha_p} + \underbrace{\gamma^{[\alpha}\gamma^{\alpha_1]}\gamma^{\alpha_2}\cdots\gamma^{\alpha_p}}_{(p+1)\gamma^{\alpha\alpha_1\cdots\alpha_p}}\ \underbrace{\nabla_\alpha X_{\alpha_1\cdots\alpha_p}}_{\to\ \nabla_{[\alpha}X_{\alpha_1\cdots\alpha_p]}}$$

$$= -\nabla_{\alpha_1}X^{\alpha_1}{}_{\alpha_2\cdots\alpha_p}\gamma^{\alpha_2\cdots\alpha_p} + (p+1)\nabla_{[\alpha}X_{\alpha_1\cdots\alpha_p]}\gamma^{\alpha\alpha_1\cdots\alpha_p}.$$

Let ψ be a $\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)$-spinor:

$$\gamma^\alpha\gamma^\beta\nabla_\alpha\nabla_\beta\psi = \Big(\underbrace{\gamma^{(\alpha}\gamma^{\beta)}}_{-g_{\alpha\beta}}\nabla_\alpha\nabla_\beta + \gamma^\alpha\gamma^\beta\nabla_{[\alpha}\nabla_{\beta]}\Big)\psi$$

$$= -\nabla^\alpha\nabla_\alpha\psi + \tfrac{1}{2}\gamma^{\alpha\beta}(\mathcal{R}_{\alpha\beta}\psi - \psi\mathcal{R}_{\alpha\beta})$$

$$\underbrace{-\tfrac{1}{8}R_{\alpha\beta}{}^{\gamma\delta}\gamma^{\alpha\beta}\gamma_{\gamma\delta}}_{\text{evaluate } = -\frac{1}{8}R_{\alpha\beta}{}^{\gamma\delta}(-\delta^{\alpha\beta}_{\gamma\delta}) = \frac{1}{4}R.} \qquad \tfrac{1}{8}R_{\alpha\beta}{}^{\gamma\delta}\gamma^{\alpha\beta}\psi\gamma_{\gamma\delta}.$$

$$\boxed{(\gamma\cdot\nabla)^2\psi = -\nabla^\alpha\nabla_\alpha\psi + \tfrac{1}{4}R\psi + \tfrac{1}{8}R_{\alpha\beta}{}^{\gamma\delta}\gamma^{\alpha\beta}\psi\gamma_{\gamma\delta}.}$$

Extension to a Gauge connection

Group G, Lie algebra $\mathfrak{g}$, connection A, curvature F.

Let $\begin{cases} \Lambda^p(M,V) & \text{be } V\text{-valued spinor/tensor } \cancel{\text{fields}} \text{ p-forms} \\ T^{p,q}(M,V) & \text{" " s " " fields.} \end{cases}$

Extend D to gauge covariant exterior derivative: $\overset{G}{D}$

$$\overset{G}{D}\Psi = D\Psi + \rho'(A)\wedge\Psi \qquad \text{for } \Psi \text{ of type } \rho' \text{ wrt } G.$$

Extend δ to gauge covariant divergence $\overset{G}{\delta}$:

$$[\overset{G}{\delta}\Psi]_{\alpha_2 \dots \alpha_p} = \overset{G}{\nabla}_{\alpha_1}\Psi^{\alpha_1}{}_{\alpha_2 \dots \alpha_p}.$$

$$\overset{G}{D}{}^2\Psi = D^2\Psi + \rho'(F)\wedge\Psi.$$

$$2\overset{G}{\nabla}_{[\alpha}\overset{G}{\nabla}_{\beta]}\Psi = 2\nabla_{[\alpha}\nabla_{\beta]}\Psi + \rho'(F_{\alpha\beta})\Psi.$$

Now the $\mathfrak{g}$-valued exterior algebra on M is isomorphic with the Clifford algebra over $F(M)$ and

$$\boxed{(\gamma^\alpha\overset{G}{\nabla}_\alpha)S\mathrm{X} = S(\overset{G}{D}+\overset{G}{\delta})\mathrm{X}}$$

$$(\gamma^\alpha\overset{G}{\nabla}_\alpha)^2\Psi = \cancel{(\gamma^\alpha\nabla_\alpha)^2\Psi +}$$
$$(-\overset{G}{\nabla}_\alpha\overset{G}{\nabla}{}^\alpha + \tfrac{1}{4}R)\Psi - \tfrac{1}{8}R^{\alpha\beta}{}_{\gamma\delta}\gamma^{\gamma\delta}\Psi\gamma_{\alpha\beta} + \gamma^{\alpha\beta}\,\mathrm{ad}(F_{\alpha\beta})\Psi.$$

Question. $\overset{G}{D}{}^2 \neq 0 \neq \overset{G}{\delta}{}^2$, so $(\overset{G}{D}+\overset{G}{\delta})^2 = \underbrace{\overset{G}{D}\overset{G}{\delta}+\overset{G}{\delta}\overset{G}{D}}_{\overset{G}{\Delta}} + \overset{G}{D}{}^2+\overset{G}{\delta}{}^2$

Thus $(\gamma^\alpha\overset{G}{\nabla}_\alpha)^2 S\mathrm{X} = S(\overset{G}{\Delta}+\overset{G}{D}{}^2+\overset{G}{\delta}{}^2)\mathrm{X}$. Is this okay?

$$\overset{G}{\Delta}\omega_{\alpha_1\dots\alpha_p} = -\omega_{\alpha_1\dots\alpha_p;\beta}{}^{;\beta} + \sum_r R^\beta{}_{\alpha_r}\omega_{\alpha_1\dots\beta\dots\alpha_p} - \sum_{r<s} R_{\alpha_r}{}^{\beta}{}_{\alpha_s}{}^{\gamma}\omega_{\alpha_1\dots\beta\dots\gamma\dots\alpha_p}$$
$$+ \sum_r \mathrm{ad}(F^\beta{}_{\alpha_r})\omega_{\alpha_1\dots\beta\dots\alpha_p}$$