LOCAL DIFF. GEOMETRY OF TENSOR BUNDLES

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M a manifold, $TM = \{(x,X) \mid x \in M, X \in TM_x\}$ = tangent bundle of M.

$\pi : TM \to M$ the natural projection $\pi(x,X) = x$.

TM is a vector bundle over M since the fiber $\pi^{-1}(x)$ over each point x is a vector space.

Given a local coordinate chart $\{x^i\}$ on $U \subset M$, a coordinate chart $\{q^i, \dot q^i\}$ is naturally induced on $\pi^{-1}(U)$ by "taking components of tangent vectors in the coordinate ~~frame~~ $\{\partial/\partial x^i\}$."

$$\left[\begin{array}{l} q^i(x,X) = x^i(x) \quad \text{or} \quad q^i = x^i \circ \pi \\ \dot q^i(x,X) = dx^i(X) = X^i(x) \quad \text{if} \quad X(x) = X^i(x) \frac{\partial}{\partial x^i}\Big|_x . \end{array}\right.$$

At a point $(x,X) \in TM$, $\{\frac{\partial}{\partial q^i}, \frac{\partial}{\partial \dot q^i}\}$ are a basis of the tangent space $T(TM)_{(x,X)}$.

$\{\frac{\partial}{\partial \dot q^i}\}$ span the subspace tangent to the fiber, the "vertical subspace" $V_{(x,X)}$.

$\{\frac{\partial}{\partial q^i}\}$ span a complementary subspace isomorphic to the tangent space at the base TM_x (the isomorphism: $\frac{\partial}{\partial q^i} \leftrightarrow \pi_*(\frac{\partial}{\partial q^i}) = \frac{\partial}{\partial x^i}$), but this subspace changes with the choice of a new coordinate system on the base manifold:

$x^{i'} = x^{i'}(x)$ induces the coordinate transformation:

$$q^{i'} = q^{i'}(q) \qquad \frac{\partial}{\partial q^i} = \left(\frac{\partial q^{j'}}{\partial q^i}\right)\frac{\partial}{\partial q^{j'}} + \underbrace{\frac{\partial \dot q^{j'}}{\partial q^i}\frac{\partial}{\partial \dot q^{j'}}}_{\text{vertical piece}}$$

$$\dot q^{i'} = \frac{\partial q^{i'}}{\partial q^j}(q)\, \dot q^j$$

$\{\frac{\partial}{\partial q^i}\}$ and $\{\frac{\partial}{\partial q^{i'}}\}$ therefore span different subspaces.

If we want a horizontal subspace, we'll have to impose one.

A connection ∇ on M induces a distribution of horizontal subspaces $H_{(x,X)}$ on TM via lifting curves $C(t)$ on M to curves $\bar C(t)$ on TM.

Let $C(t)$ be a curve on M with $C(0) = x$. ~~and $C'(0)$~~

Given $X \in TM_x$, define $\bar C(t) \equiv X_t \in TM_{C(t)}$ as the parallel transport of X along $C(t)$. $\bar C(t)$ is a curve on TM with $\bar C(0) = (x,X)$. (The horizontal lift of $C(t)$)

In coordinates: $C^i(t) = x^i \circ C(t)$, $C'^i(t)$ = tangent to curve,

so $q^i(t) = C^i(t)$, $\dot q^i(t) = X^i(t)$.

Parallel transport of X_t says: $\frac{dX^i}{dt} + \Gamma^i{}_{jk} X^k C'^j = 0$, so

$$\frac{dq^i(t)}{dt} = C'^i(t) \qquad \frac{d\dot q^i(t)}{dt} = -\Gamma^i{}_{jk}\, \dot q^k(t)\, C'^j(t)$$

$$\text{ie} \quad \bar C'(t) = C'^i(t)\left(\frac{\partial}{\partial q^i} - \Gamma^j{}_{ik}\, \dot q^k \frac{\partial}{\partial \dot q^j}\right)\Big|_{C(t)} .$$

Thus $D_i = \frac{\partial}{\partial q^i} - \Gamma^j{}_{ik}\dot q^k \frac{\partial}{\partial \dot q^j}$ spans the subspace of tangents to the horizontal lifts of curves to the tangent bundle. ~~A tangent vector~~ vector field $X = X^i \frac{\partial}{\partial x^i}$ on M lifts up to a horizontal vector field on TM: $\bar X = X^i D_i$.

~~$\{D_i, \frac{\partial}{\partial \dot q^i}\}$ is a natural frame to use, given the original coordinate system. Its dual frame is $\{\omega^i, dq^i\}$ where $\omega^i = d\dot q^i + \Gamma^i{}_{jk}\dot q^k dq^j$~~, since

~~$\omega^i(D_j) = 0 \quad \omega^i(\frac{\partial}{\partial \dot q^j}) = \delta^i{}_j \quad dq^i(\frac{\partial}{\partial \dot q^j}) = 0 \quad dq^i(D_j) = \delta^i{}_j$~~

$\{D_i, \frac{\partial}{\partial \dot q^i}\}$ is a natural frame to use having started with the coordinate system $\{x^i\}$.
Its dual frame is $\{dq^i, \omega^i\}$ where $\omega^i = d\dot q^i + \Gamma^i{}_{jk}\dot q^k dq^j$ since

$$dq^i(D_j) = \delta^i{}_j \qquad dq^i(\tfrac{\partial}{\partial \dot q^j}) = 0$$
$$\omega^i(\tfrac{\partial}{\partial \dot q^j}) = \delta^i{}_j \qquad \omega^i(D_j) = 0.$$

The structure functions for the frame are defined by computing the Lie brackets of the frame vector fields:

$$[D_i, D_j] = -R^k{}_{\ell ij}\dot q^\ell \frac{\partial}{\partial \dot q^k}$$
$$[D_i, \tfrac{\partial}{\partial \dot q^k}] = -\Gamma^k{}_{ij}\frac{\partial}{\partial \dot q^k}$$

Curvature of the connection

By $\Gamma^k{}_{ij}$, $R^k{}_{\ell ij}$ etc on the tangent bundle, I really mean $\Gamma^k{}_{ij}\circ\pi$, $R^k{}_{\ell ij}\circ\pi$
ie "$\Gamma^k{}_{ij}\circ\pi(q\dot q) = \Gamma^k{}_{ij}(q)$"

One can play the same game on the cotangent bundle: $TM^* = \{(x,\sigma) \mid \sigma \in TM^*_x\}$.
$\{x^i\}$ on M induce $\{q^i, p_i\}$ on TM^* by

$$\left[\begin{array}{ll} q^i(x,\sigma) = x^i(x) \text{ or } q^i = x^i\circ\pi & \text{where now } \pi: TM^*\to M \\ & \pi(x,\sigma) = x. \\ p_i(x,\sigma) = \sigma(\frac{\partial}{\partial x^i}) = \sigma_i \text{ if } \sigma = \sigma_i dx^i. & \end{array}\right.$$

In the same way one obtains a frame $\{D_i, \frac{\partial}{\partial p_i}\}$ with $D_i = \partial_i + p_j\Gamma^j{}_{ik}\frac{\partial}{\partial p_k}$
with $\{D_i\}$ spanning the horizontal subspace, $\{\frac{\partial}{\partial p_i}\}$ spanning the vertical subspace.
Dual frame $\{dq^i, \omega_i = dp_i - p_j\Gamma^j{}_{ki}dq^k\}$.

$$[D_i, D_j] = p_k R^k{}_{\ell ij}\frac{\partial}{\partial p_\ell} \qquad [D_i, \tfrac{\partial}{\partial p_j}] = \Gamma^j{}_{ik}\frac{\partial}{\partial p_k}.$$

The same holds for any of the tensor bundles. All of these ~~~~ vector bundles are special, though, in that the fibers are intimately connected with the tangent structure of the base manifold. In general, V vector space with basis $\{e_A\}$ then locally a vector bundle B with fibers isomorphic to V will be $U\times V$ where $U\subset B$., Coordinates $\{q^i, q^A\}$ with $q^i = x^i\circ\pi$ for coordinates $\{x^i\}$ on U.

$$\begin{bmatrix} D_i = \frac{\partial}{\partial q^i} - \Gamma^B{}_{iA}q^A\frac{\partial}{\partial q^B} \\ \omega^A = dq^A + \Gamma^A{}_{iB}q^B dq^i \end{bmatrix}$$

frame $\{D_i, \frac{\partial}{\partial q^A}\}$, dual frame $\{dq^i, \omega^A\}$

$$[D_i, D_j] = -R^B{}_{Aij}q^A\frac{\partial}{\partial q^B}$$
$$[D_i, \tfrac{\partial}{\partial q^A}] = -\Gamma^B{}_{iA}\frac{\partial}{\partial q^B}$$

curvature ("F")
connection ("A")

For example, if we have an $SU(N)$ gauge theory and $[T_a, T_b] = C^c_{ab}T_c$ ~~~~ with $\{T_a\}$ a basis of the matrix Lie algebra $su(N)$, and $A = A^a_i dx^i T_a$ is the pullback on M of the connection on the principle bundle by a section, then
(1) in the adjoint representation, the vector space V is the Lie algebra with basis $\{e_A\} = \{T_a\}$, and $\Gamma^a_{ib} = A^c_i C^a_{cb}$
(2) in the identity rep., $V = \mathbb{C}^N$ and $\Gamma^A{}_{iB} = A^c_i T_c{}^A{}_B$

Since $\{D_i, \frac{\partial}{\partial q^A}\}$ is a frame with dual frame $\{dq^i, \omega^A\}$, where $\{D_i\}$ span the horizontal subspaces $\mathcal{H}$ and $\{\frac{\partial}{\partial q^A}\}$ span the vertical subspaces V, the horizonal and vertical parts of any vector field are given by:

$$vX = \omega^A(X)\frac{\partial}{\partial q^A} = X \lrcorner (\omega^A \otimes \frac{\partial}{\partial q^A}) \equiv X \lrcorner P_V$$

$$hX = dq^i(X)\, D_i = X \lrcorner (dq^i \otimes D_i) \equiv X \lrcorner P_H$$

$P_V = \omega^A \otimes \frac{\partial}{\partial q^A}$ and $P_H = dq^i \otimes D_i$ are invariantly defined projection tensors on the bundle B. (or TM or TM^* in the appropriate notation).

Suppose X, Y are vector fields on M; $\bar{X} = X^i D_i$ and $\bar{Y} = Y^i D_i$ are their lifts to B:

$$[\bar{X}, \bar{Y}] = -R^B{}_{Aij} X^i Y^j q^A \frac{\partial}{\partial q^B} + [X,Y]^i D_i$$

so $$h[\bar{X},\bar{Y}] = -R^B{}_A(X,Y)\, \underbrace{q^A \frac{\partial}{\partial q^B}}_{E^A{}_B}, \quad v[\bar{X},\bar{Y}] = \overline{[X,Y]}.$$

these vector fields generate the action of the group of linear transformations of the vector space V as it acts on the bundle.

We might call $R^B{}_A E^A{}_B = R$ (a (1-1) tensor valued 2-form) the curvature of the connection on the vector bundle.

But Kob. & Nomizu dont go into details about vector bundles so I dont know what the actual terminology is.