r-dimensional Lie group G, points a, identity a_0.

Notes by Robert Jantzen

$\begin{pmatrix}\text{left}\\ \text{right}\end{pmatrix}$ translation by $a_1 \in G$: $\quad a \mapsto L_{a_1}(a) = a_1 a$

$\quad a \mapsto R_{a_1}(a) = a a_1$

inner automorphism: $\quad a \mapsto AD_{a_1}(a) = a_1 a a_1^{-1} = L_{a_1} \circ R_{a_1^{-1}}(a) = R_{a_1^{-1}}(a) \circ L_{a_1}(a)$

The Lie algebras $\hat{g} \cong g, \tilde{g}$ ($\hat{g}$ and g are usually identified and called the Lie algebra of G)

$$\hat{g} = TG_{a_0}, \quad \hat{g}^* = TG^*_{a_0}$$
$$\cap \qquad\qquad \cap$$
$$\{\hat{e}_a\} \text{ basis}, \quad \{\hat{\omega}^a\} \text{ dual basis}$$
$$[\hat{e}_a, \hat{e}_b] \equiv \widehat{[e_a, e_b]} = C^c{}_{ab}\hat{e}_c$$

$\wedge : g \to \hat{g}$

$X \mapsto \hat{X} = X(a_0)$

$\wedge : \tilde{g} \to \hat{g}$

$\tilde{X} \mapsto \hat{X} = \tilde{X}(a_0)$

left invariant vector fields and one-forms

$$g = \{X \in \mathcal{X}(G) \mid L_a X = X ;\ \forall a \in G\}$$
$$\cap$$
$$X(a) = dL_a(a_0)X(a_0) = dL_a(a_0)\hat{X}$$
$$\text{basis } \{e_a\} \qquad [e_a, e_b] = C^c{}_{ab} e_c$$
frame on G, dual frame:
$$\{\omega^a\} \qquad d\omega^a = -\tfrac{1}{2} C^a{}_{bc}\, \omega^b \wedge \omega^c$$
$$\cap$$
$$g^* = \{\omega \in \mathcal{X}^*(G) \mid L_a \omega = \omega, \forall a \in G\}$$

$\sim : g \to \tilde{g}$

$X \mapsto \tilde{X}$

$[g, \tilde{g}] = 0$

right invariant vector fields and one-forms

$$\tilde{g} = \{\tilde{X} \in \mathcal{X}(G) \mid R_a \tilde{X} = \tilde{X};\ \forall a \in G\}$$
$$\cap$$
$$\tilde{X}(a) = dR_a(a_0)\tilde{X}(a_0) = dR_a(a_0)\hat{X}$$
$$\text{basis } \{\tilde{e}_a\} \qquad [\tilde{e}_a, \tilde{e}_b] = -C^c{}_{ab}\tilde{e}_c$$
frame on G, dual frame:
$$\{\tilde{\omega}^a\} \qquad d\tilde{\omega}^a = \tfrac{1}{2} C^a{}_{bc}\, \tilde{\omega}^b \wedge \tilde{\omega}^c$$
$$\cap$$
$$\tilde{g}^* = \{\tilde{\omega} \in \mathcal{X}^*(G) \mid R_a \tilde{\omega} = \tilde{\omega}, \forall a \in G$$

$$X(a) = L_a(a_0)\hat{X} = L_a \circ R_{a^{-1}}(a)\tilde{X}(a) = AD_a(a)\tilde{X}(a) = (AD_a \tilde{X})(a)$$

$$Ad(a)\hat{e}_b \equiv \widehat{(AD_a e_b)} \equiv Ad(a)^c{}_b\, \hat{e}_c \equiv R^c{}_b(a)\,\hat{e}_c$$
$$\shortparallel$$
$$\widehat{(R_{a^{-1}} L_a e_b)} = \widehat{(R_{a^{-1}} e_b)}$$

$$\boxed{\begin{array}{lll} \tilde{e}_a = e_b R^{-1\,b}{}_a & \tilde{\omega}^a = R^a{}_b \omega^b & R^a{}_b = e_b \lrcorner\, \tilde{\omega}^a = \tilde{\omega}^a(e_b) \\ \underline{R}^{-1} d\underline{R} = \underline{k}_a \omega^a & d\underline{R}\,\underline{R}^{-1} = \underline{k}_a \tilde{\omega}^a & \\ \underline{k}_a = C^b{}_{ac}\, \underline{\hat{e}}^c{}_b & \underline{R}\,\underline{k}_a \underline{R}^{-1} = \underline{k}_b R^b{}_a & \underline{R}(\exp X^a \hat{e}_a) = e^{X^a \underline{k}_a} \end{array}}$$

$\underline{\exp : g \cong \hat{g} \to G}$: $\quad X \mapsto \exp X = X_1 a_0 = \tilde{X}_1 a_0$

one dimensional subgroups: $\quad C_X(t) = \exp tX$

fact: $\quad AD_a(\exp X) = \exp Ad(a) X$

$\begin{pmatrix} g \\ \tilde{g} \end{pmatrix}$ generate $\begin{pmatrix}\text{right}\\ \text{left}\end{pmatrix}$ translations: $\boxed{\begin{array}{l} X_t a = R_{\exp tX}(a) \\ \tilde{X}_t a = L_{\exp tX}(a) \end{array}}$ $\begin{array}{l} = (a \exp tX a^{-1}) a = L_{\exp t Ad(a) X}(a) \\ = \ldots = R_{\exp t Ad(a) X}(a) \end{array}$

Thus through each point $a \in G$ ~~and~~ passes a unique curve with tangent $Y \in TG_a$, namely the integral curve of either invariant vector field which coincides with Y at a, which in turn is a coset by a of the corresponding one-dimensional subgroups. These are the geodesics of a certain connection on G.

Natural connections on a Lie Group $\nabla, \tilde\nabla, \bar\nabla = \frac{1}{2}(\nabla + \tilde\nabla)$

∇ and $\tilde\nabla$ are flat transpose connections, ie have ~~tors~~ opposite torsion coinciding with the difference tensor (in coordinates: $\Gamma^a{}_{bc} = \tilde\Gamma^a{}_{cb}$) and therefore the same geodesics and have zero curvature.

$\bar\nabla$ is the unique symmetric connection having same geodesics (described on the previous page)

	$\nabla_{e_a} e_b = C^c{}_{ab} e_c$	$\bar\nabla_{e_a} e_b = \frac{1}{2} C^c{}_{ab} e_c$ $\bar\nabla_{\tilde e_a} \tilde e_b = -\frac{1}{2} C^c{}_{ab} \tilde e_c$	$\tilde\nabla_{\tilde e_a} \tilde e_b = -C^c{}_{ab} \tilde e_c$
~~components of~~	In frame $\{e_a\}$	In frame $\{e_a\}$	In frame $\{\tilde e_a\}$
components of connection:	$\Gamma^a{}_{bc} = C^a{}_{bc}$	$\bar\Gamma^a{}_{bc} = \frac{1}{2} C^a{}_{bc}$	$\tilde\Gamma^a{}_{bc} = -C^a{}_{bc}$
connection one-form:	$\sigma = \omega^a k_a$	$\bar\sigma = \frac{1}{2}\omega^a k_a$	$\tilde\sigma = -\tilde\omega^a k_a$
torsion two-form:	$\Theta = d\omega + \sigma\wedge\omega = \frac{1}{2}\sigma\wedge\omega$	$\bar\Theta = d\omega + \bar\sigma\wedge\omega = 0$	$\tilde\Theta = d\tilde\omega + \tilde\sigma\wedge\tilde\omega = -\frac{1}{2}\tilde\sigma\wedge\tilde\omega$
curvature two-form:	$\Omega = d\sigma + \sigma\wedge\sigma = 0$	$\bar\Omega = d\bar\sigma + \bar\sigma\wedge\bar\sigma = \frac{1}{8} k_e C^e{}_{ab}\,\omega^a\wedge\omega^b$	$\tilde\Omega = d\tilde\sigma + \tilde\sigma\wedge\tilde\sigma = 0$

In frame $\{\tilde e_a\}$

$(\Theta)^\sim = R\,\Theta = -\tilde\Theta$

$(\sigma)^\sim = R(dR^{-1} + \sigma R^{-1}) = 0$

ie $\nabla \tilde e_a = 0$

so ∇-parallel transport is just right translation

$\bar R^c{}_{dab} = -\frac{1}{4} C^c{}_{ed} C^e{}_{ab}$

Ricci tensor:

$\bar R_{db} = \bar R^c{}_{dcb} = -\frac{1}{4}\gamma_{db}$

components of adjoint invariant Killing form:

In frame $\{e_a\}$

$(\tilde\Theta)^\sim = R^{-1}\tilde\Theta = -\Theta$

$(\tilde\sigma)^\sim = R^{-1}(dR + \tilde\sigma R) = 0$

ie $\tilde\nabla e_a = 0$

so $\tilde\nabla$-parallel transport is just left translation

$\gamma_{ab} = C^f{}_{ag} C^g{}_{bf}$; $R^T \gamma R = \gamma$, $\gamma k_a + (\gamma k_a)^T = 0$

Compatible metric g $(\nabla g = \bar\nabla g = \tilde\nabla g = 0)$ $\leftrightarrow$ bi-invariant $\leftrightarrow$ adjoint invariant

$g = g_{ab}\,\omega^a\otimes\omega^b = g_{ab}\,\tilde\omega^a\otimes\tilde\omega^b$ $\qquad R^T g R = g$

$$\left.\begin{aligned} 0 &= \nabla g_{ab} = dg_{ab} - 2C^e{}_{c(a} g_{b)e}\,\omega^c = (\mathcal{L}_{e_c} g)_{ab}\,\omega^c \\ 0 &= \bar\nabla g_{ab} = dg_{ab} - C^e{}_{c(a} g_{b)e}\,\omega^c \\ 0 &= \tilde\nabla g_{ab} = dg_{ab} \end{aligned}\right\} \to \quad \begin{aligned} dg_{ab} = 0 &= C_{(a|c|b)} \\ g k_a + (g k_a)^T &= 0. \end{aligned}$$

Since $\bar\nabla$ is symmetric and $\bar\nabla g = 0$, it is the connection generated by g.

"Yang-Mills groups"

For semi-simple groups the Killing form is nondegenerate (and the adjoint group unimodular: $0 = \mathrm{TR}\,k_a = C^b{}_{ab}$) and hence determines a bi-invariant metric: $\qquad g_{ab} = -1/2\,\gamma_{ab}$.

We include a minus since compact groups have negative definite Killing forms and a 1/2 so $g_{ab} = \delta_{ab}$ when a "canonical basis" is chosen in this case (at least for $r=3$)

For abelian groups $C^a{}_{bc} \equiv 0$ and the adjoint group is trivial (the identity) so g arbitrary is satisfactory. (for example $g_{ab} = \delta_{ab}$).

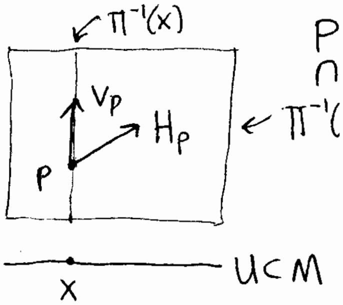
→ No other groups have nondegenerate bi-invariant metrics.

Hence: $\quad R_{ab} = \begin{Bmatrix} \frac{1}{2} g_{ab} \\ 0 \end{Bmatrix}$ semi-simple / abelian $\qquad$ scalar curvature: R $\begin{Bmatrix} r/2 \\ 0 \end{Bmatrix}$ semi-simple / abelian.

(EXCEPT FOR DIRECT PRODUCT GROUPS WITH ANY NUMBER OF FACTOR MANIFOLDS OF THESE TYPES.)

Consider a principal bundle P over base manifold M = spacetime with Yang-Mills group G acting on the fibers on the right (so that in a local trivialization this action becomes right translation). The generators of this action called fundamental vector fields (left invariant vector fields in a local trivialization) span the vertical subspaces. The horizontal subspaces are picked out by a connection on P.

For simplicity we assume from the start a local trivialization over $U \subset M$ which is a coordinate patch for local coordinates $\{x^\mu\}$ on M. These naturally lift up to ~~coordin~~ part of a local coordinate system $\{x^\mu\}$ on $U \times G$: [We indulge in many identifications].

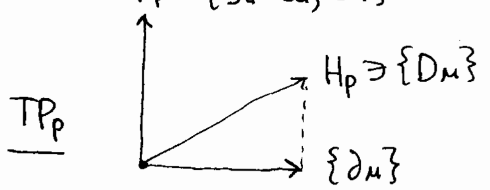


$$P \supset \pi^{-1}(U) \cong U \times G = \{p = (x, a) \mid x \in M, a \in G\}$$

$$\text{"}x^\mu(p) = x^\mu(x)\text{"}$$

$$\partial_\mu = \partial/\partial x^\mu$$

$$V_p \ni \{\xi_a = e_a, \tilde{e}_a\}$$



$$D_\mu = \partial_\mu - A^a{}_\mu(x)\,\tilde{e}_a$$

$$[\xi_a, D_\mu] = \mathcal{L}_{\xi_a} D_\mu = 0$$

D_μ span a right invariant distribution H on P

Define $A^a = A^a{}_\mu dx^\mu \qquad A = A^a \hat{e}_a$

$$F = F^a \hat{e}_a = dA + A \wedge A$$

$$\tilde{F}^a = R^{-1a}{}_b F^b \qquad \tilde{A}^a = R^{-1a}{}_b A^b$$

provided we agree to make the identification
$$2\,\hat{e}_{[a}\hat{e}_{b]} = [\hat{e}_a, \hat{e}_b] = C^c{}_{ab}\,\hat{e}_c$$
or if we have a matrix group, identify $\hat{e}_a$ with its corresponding matrix generator

Introduce the natural frame: $\{D_A\} = \{D_\mu, \xi_a\}$

Its dual frame: $\{W^A\} = \{dx^\mu,\ \bar{A}^a = \omega^a + \tilde{A}^a\}$

The connection form is: $\bar{A} = \bar{A}^a \hat{e}_a$

It annihilates H_p and maps V_p onto the Lie algebra of G: $\bar{A}(X^a \xi_a) = X^a \hat{e}_a$.

The curvature form is: $\bar{F} = d\bar{A} + \bar{A} \wedge \bar{A} = \tilde{F}^a \hat{e}_a$

The structure functions $C^A{}_{BC}$ for this frame:

$$[D_\mu, D_\nu] = -\tilde{F}^c{}_{\mu\nu}\,\xi_c$$
$$[D_\mu, \xi_a] = 0$$
$$[\xi_a, \xi_b] = C^c{}_{ab}\,\xi_c$$

$$\boxed{\begin{aligned} C^c{}_{\mu\nu} &= -\tilde{F}^c{}_{\mu\nu} \\ C^c{}_{ab} &= C^c{}_{ab} \end{aligned}}$$

Metric on P — We can pull up the spacetime metric ${}^{(4)}g$ on M to an inner product on each horizontal subspace H_p and complete it to a metric on P by requiring $V_p \perp H_p$ and the inner product of elements of V_p to be the Killing inner product of the corresponding elements of $\hat{g}$ up to a factor as on the previous page (any inner product on an abelian group):

$$\bar{g} = \underbrace{g_{\mu\nu}(x)\,dx^\mu \otimes dx^\nu}_{\pi^{*\,(4)}g} + \underbrace{g_{ab}\,\bar{A}^a \otimes \bar{A}^b}_{\bar{g}(X,Y) = g(\bar{A}(X), \bar{A}(Y))} \qquad X, Y \in V_p$$

The restriction of the metric to a fiber makes the fiber isometric to the pseudo Riemannian manifold of the previous page.

The volume form on P is just the exterior product of the spacetime volume form with the volume form on the fiber:

$$\bar{\eta} = \sqrt{|\det(g_{AB})|}\,\underbrace{W^0 \wedge \ldots W^3}_{A=\alpha} \wedge \underbrace{W^1 \wedge \ldots \wedge W^r}_{A=a} = |{}^{(4)}g|^{1/2}\,|\det(g_{ab})|^{1/2}\,dx^{0123} \wedge \omega^{1\ldots r}$$
$$= \eta_{ST} \wedge \eta_{fiber}$$

Components of Metric Connection in our frame. Notes by Robert Jantzen (4)

$$\bar\Gamma^C{}_{AB} = \tfrac12 g^{CD}(D_A g_{DB} + D_B g_{DA} - D_D g_{AB}) + \tfrac12 g^{CD}(g_{AE}C^E{}_{DB} + g_{BE}C^E{}_{DA}) + \tfrac12 C^C{}_{AB}$$

nonzero components:

$$\boxed{\bar\Gamma^\gamma{}_{\alpha\beta} = \{{}^{\gamma}_{\alpha\beta}\} \qquad \bar\Gamma^\gamma{}_{a\beta} = \bar\Gamma^\gamma{}_{\beta a} = -\tfrac12 \tilde F_a{}^\gamma{}_\beta \qquad \bar\Gamma^c{}_{ab} = \tfrac12 C^c{}_{ab} \qquad \bar\Gamma^c{}_{\alpha\beta} = -\tfrac12 \tilde F^c{}_{\alpha\beta}}$$

multiply by ± 2 and connection on fiber becomes ∇ or $\tilde\nabla$ which are flat and have zero scalar curvature

Scalar curvature:

$$\bar R = \bar g^{AB}\bar R_{AB} = \dots = \underbrace{R_{S.T.} - \tfrac14 F_a{}^{\alpha\beta}F^a{}_{\alpha\beta}}_{\text{function only of } x.} + R_{FIBER}$$

$\left\{\begin{matrix} \frac{r}{2} \\ 0 \end{matrix}\right\}$ $\begin{matrix}\text{semi-simple}\\ \text{abelian}\end{matrix}$

By introducing torsion on the fiber, the scalar curvature density Lagrangian becomes:

$$\bar L = \bar R\,\bar\eta = \underbrace{[(L_{EINSTEIN} + L_{Y.M})^{(4)}\eta\,]}_{\text{Independent of fiber coordinates.}} \wedge \eta_{FIBER}$$

The Einstein ~~equations~~ equations (~~Ricci~~ tensor $=0$) (for the bundle metric with torsion on fiber) are therefore the combined gravitational and Yang-Mills field equations.

Bonus Geodesics on P (same for metric connection or with added torsion on fiber)

Let $\bar C(t) = (x(t), a(t))$ be a geodesic with tangent:

$$\bar C'(t) = \bar C'^A(t) D_A \quad \left\{\begin{aligned} &\bar C'^\mu(t) = \frac{dx^\mu(t)}{dt} \qquad \cancel{\bar C'^a(t) = \bar A^a(C'(t)) = \omega^a(a'(t)) + \tilde R^a{}_b(a(t))A^b{}_\mu(x(t))\frac{dx^\mu}{dt}} \\ &\bar C'^a(t) = \bar A^a(C'(t)) = \omega^a(a'(t)) + R^a{}_b(a(t))A^b{}_\mu(x(t))\frac{dx^\mu(t)}{dt} \equiv q^a(t)\end{aligned}\right.$$

$$0 = \frac{d\bar C'^\mu}{dt} + \bar\Gamma^\mu{}_{AB}\bar C'^A\bar C'^B = \frac{d^2x^\mu}{dt} + \bar\Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} + \underbrace{2\bar\Gamma^\mu{}_{\alpha a}\frac{dx^\alpha}{dt}}_{-\tilde F_a{}^\mu{}_\alpha \frac{dx^\alpha}{dt}}\bar C'^a$$

$$0 = \frac{d\bar C'^a}{dt} + \underbrace{\bar\Gamma^a{}_{AB}\bar C'^A\bar C'^B}_{\equiv 0}$$

so $\bar C'^a = q^a$ is constant along the geodesic (the vertical component of the tangent vector)

Define the components of the charge vector (really charge/mass vector):

$Q^a(t) = R^a{}_b(t)q^b$ Then: $\boxed{\dfrac{D^2x^\mu(t)}{dt^2} = Q^a(t)F_a{}^\mu{}_\alpha(x(t))\dfrac{dx^\alpha(t)}{dt}}$

"Lorentz force law"

$$Q^a = \tilde\omega^a(a'(t)) + A^a(x'(t))$$

$$\frac{dQ}{dt} = \frac{dR}{dt}q = \frac{dR}{dt}R^{-1}Q = \tilde\omega^a(a'(t))k_a Q = [-A^a(x'(t)) + Q^a(t)]\,k_a Q$$

$$\frac{dQ^a}{dt} + A^c C^a{}_{cb}Q^b = C^a{}_{cb}Q^cQ^b = 0$$

$$\boxed{\frac{\overset{G}{D}Q^a(t)}{dt} = 0}$$

G-gauge covariant derivative.

charge vector parallel transported along trajectory on spacetime.

Presumably these equations describe a test Yang-Mills particle on this spacetime.

CONTINUATION OF PRINCIPAL FIBER BUNDLE NOTES (including explicit computational techniques)

We now generalize by relaxing bi-invariance of the induced metric on the fiber to merely right invariance (maintaining the orthogonality of the horizontal and vertical subspaces) so that the bundle metric $\bar{g}$ is invariant under the right action of G on P, i.e. $\mathcal{L}_{\xi_a}\bar{g} = 0$.

In the locally trivial notation of pages 3 and 4:

$$\bar{g} = g_{\mu\nu}(x)\,dx^\mu \otimes dx^\nu + g_{ab}\,\bar{A}^a \otimes \bar{A}^b \,, \qquad g_{ab} = \tilde{g}_{cd}(x) R^c{}_a R^d{}_b$$

$$\bar{g}\big|_{\pi^{-1}(x)} = \tilde{g}_{cd}(x)\,\tilde{\omega}^c \otimes \tilde{\omega}^d = \text{manifestly right invariant.}$$

To evaluate the connection components in the frame $\{D_A\}$ we have to compute the derivatives $D_c g_{ab}$ and $D_\mu g_{ab}$; first a matrix calculation with d differentiating only along the fiber:

$$\underline{g} = \underline{R}^T \underline{\tilde{g}}\, \underline{R} \qquad \text{transpose} \qquad \text{(underline indicates matrix)}$$

$$d\underline{g} = \underbrace{\underline{R}^T\underline{\tilde{g}}\underline{R}}_{\underline{g}}\,\underbrace{\underline{R}^{-1}d\underline{R}}_{\underline{k}_a\omega^a} + \underbrace{(\underline{R}^{-1}d\underline{R})^T}_{\underline{k}_a^T\omega^a}\,\underbrace{\underline{R}^T\underline{\tilde{g}}\underline{R}}_{\underline{g}=\underline{g}^T} \qquad d\underline{g} = \underline{R}^T\underline{\tilde{g}}\,\underbrace{d\underline{R}\underline{R}^{-1}}_{k_a\tilde{\omega}^a}\underline{R} + R^T\underbrace{(dRR^{-1})^T}_{k_a^T\tilde{\omega}^a}\tilde{g}R$$

$$= (\underline{g}\underline{k}_a + (\underline{g}\underline{k}_a)^T)\,\omega^a \qquad = R^T(\tilde{g}k_a + (\tilde{g}k_a)^T)R\,\tilde{\omega}^a$$

$$D_c\underline{g} = e_c \lrcorner d\underline{g} = \underline{g}\underline{k}_a + (\underline{g}\underline{k}_a)^T \qquad \tilde{e}_c\underline{g} = \tilde{e}_c \lrcorner d\underline{g} = R^T(\tilde{g}k_a + (\tilde{g}k_a)^T)R$$

$$\boxed{D_c g_{ab}} = g_{ad}C^d{}_{cb} + g_{bd}C^d{}_{ca} \qquad \tilde{e}_c g_{ab} = R^d{}_a R^e{}_b\,(\tilde{g}_{ad}C^d{}_{cb} + \tilde{g}_{bd}C^d{}_{ca})$$

$$= 2\,C_{(a|c|b)} \qquad = R^d{}_a R^e{}_b\,(2\tilde{C}_{(d|c|e)})$$

$$\boxed{D_\mu g_{ab}} = R^c{}_a R^d{}_b\,(\partial_\mu \tilde{g}_{ab} - 2A^e{}_\mu\,\tilde{C}_{(a|e|b)})$$

$$= R^c{}_a R^d{}_b \overset{G}{\nabla}_\mu \tilde{g}_{ab}$$

These are both special cases of general formulas.

Suppose $T^{a\ldots}{}_{b\ldots}{}^{\alpha\ldots}{}_{\beta\ldots}(x)$ are the components of a tensor field on the base which transforms under the indicated tensor product of the adjoint representation under gauge transformations. Then define

$$\tilde{T}^{a\ldots}{}_{b\ldots}{}^{\alpha\ldots}{}_{\beta\ldots} = R^{-1\,a}{}_c \cdots R^d{}_b \cdots T^{c\ldots}{}_{d\ldots}{}^{\alpha\ldots}{}_{\beta\ldots}$$

$$D_e\tilde{T}^{a\ldots}{}_{b\ldots}{}^{\alpha\ldots}{}_{\beta\ldots} = -C^a{}_{dc}\tilde{T}^{c\ldots}{}_{d\ldots}{}^{\alpha\ldots}{}_{\beta\ldots} - \cdots + C^c{}_{db}\tilde{T}^{a\ldots}{}_{c\ldots}{}^{\alpha\ldots}{}_{\beta\ldots}$$

$$D_\mu\tilde{T}^{a\ldots}{}_{b\ldots}{}^{\alpha\ldots}{}_{\beta\ldots} = R^{-1\,a}{}_c \cdots R^d{}_b \cdots \overset{G}{\nabla}_\mu T^{c\ldots}{}_{d\ldots}{}^{\alpha\ldots}{}_{\beta\ldots}$$

no Christoffel symbols summed on greek indices

(This is even a special case of formulas for representation-valued fields of any kind.)

For example:

$$D_a\tilde{F}^c{}_{\alpha\beta} = -C^c{}_{ad}\tilde{F}^d{}_{\alpha\beta}\,, \quad D_\mu\tilde{F}^c{}_{\alpha\beta} = R^{-1\,c}{}_d\,(\partial_\mu F^c{}_{\alpha\beta} + A^d_\mu C^c{}_{de}F^e{}_{\alpha\beta})$$

$$D_a\tilde{F}_c{}^\alpha{}_\beta = C^d{}_{ac}\tilde{F}_d{}^\alpha{}_\beta\,, \quad D_\mu\tilde{F}_c{}^\alpha{}_\beta = R^d{}_c(\partial_\mu F_c{}^\alpha{}_\beta - A^d{}_\mu C^e{}_{dc}F_e{}^\alpha{}_\beta)$$

Now we use the standard formula to compute the components of the metric connection in the frame $\{D_A\}$:

$$\bar\Gamma^C{}_{AB} = \tfrac12 g^{CD}\left(D_A g_{DB} + D_B g_{DA} - D_D g_{AB}\right) + \tfrac12 g^{CD}\left(g_{AE}\, C^E{}_{DB} + g_{BE}\, C^E{}_{DA}\right) + \tfrac12 C^C{}_{AB}$$

Since we know $C^E{}_{AB}$ from page 3 & only $g_{\mu\nu}, g_{ab}, g^{\mu\nu}, g^{ab}$ are nonzero & we just evaluated $D_\mu g_{ab}$, $D_\alpha g_{ab}$, with $D_\alpha g_{\mu\nu} =$ ~~the only~~ $\partial_\alpha g_{\mu\nu}$ the only remaining nonzero derivatives, it is just a matter of plugging in. The result (nonzero components):

$$\bar\Gamma^\gamma{}_{\alpha\beta} = \left\{{}^{\gamma}_{\alpha\beta}\right\}$$
$$\bar\Gamma^c{}_{ab} = \tfrac12 C^c{}_{ab} - C_{(a}{}^c{}_{b)}$$

$$\bar\Gamma^\gamma{}_{a\beta} = \bar\Gamma^\gamma{}_{\beta a} = -\tfrac12 \tilde F_a{}^\gamma{}_\beta$$
$$\bar\Gamma^c{}_{\alpha\beta} = -\tfrac12 \tilde F^c{}_{\alpha\beta}$$

$$\bar\Gamma^\gamma{}_{ab} = -\tfrac12 g^{\gamma\delta} D_\delta g_{ab}$$
$$\bar\Gamma^c{}_{a\beta} = \bar\Gamma^c{}_{\beta a} = \tfrac12 g^{cd} D_\beta g_{da}$$

the minus sign arises since we are expressing the components of a right invariant connection on the fiber in a left invariant frame (locally trivial language)

$g_F = \det(g_{ab}) = \det(\tilde g_{ab})$ because $\det R = 1$ since G is unimodular $\rightarrow C^a{}_{ea} = 0$

$$\bar\Gamma^{\gamma a}{}_a = -\tfrac12 g^{\gamma\delta} g^{ab} D_\delta g_{ab} = -g^{\gamma\delta} \partial_\delta \ln g_F{}^{1/2}$$
$$\bar\Gamma^a{}_{\beta a} = \bar\Gamma^a{}_{a\beta} = \tfrac12 g^{ad} D_\beta g_{da} = \partial_\beta \ln g_F{}^{1/2}$$
$$g^{ab} D_\beta g_{ab} = \tilde g^{ab}\left(\partial_\beta \tilde g_{ab} - 2A^e_\beta\, C_{(a|e|b)}\right) = \partial_\beta \ln \tilde g_F - 2 A^e{}_\beta \cancel{C^a{}_{ea}}^{\,0}$$

$$\bar\Gamma^c{}_{cb} = 0$$

To evaluate Ricci tensor components we need in addition $D_E \Gamma^C{}_{AB}$. The only nonzero such derivatives are:

$\bar D_\delta \bar\Gamma^\gamma{}_{\alpha\beta} = \partial_\delta \left\{{}^{\gamma}_{\alpha\beta}\right\}$ etc... fill in yourself.

The components of the Ricci tensor of $\bar g$ in the frame $\{D_A\}$ are:

$$\bar R_{DB} = D_A \bar\Gamma^A{}_{DB} - D_D\bar\Gamma^A{}_{AB} - \bar\Gamma^A{}_{EB}C^E{}_{AD} + \bar\Gamma^A{}_{AE}\bar\Gamma^E{}_{DB} - \bar\Gamma^A{}_{DE}\bar\Gamma^E{}_{AB}$$

$$\bar R_{\delta\beta} = D_A\bar\Gamma^A{}_{\delta\beta} - D_\delta\bar\Gamma^A{}_{A\beta} - \bar\Gamma^A{}_{E\beta}C^E{}_{A\delta} + \bar\Gamma^A{}_{AE}\bar\Gamma^E{}_{\delta\beta} - \bar\Gamma^A{}_{\delta E}\bar\Gamma^E{}_{A\beta}$$

$$= R_{\delta\beta} - D_\delta\bar\Gamma^a{}_{a\beta} + \bar\Gamma^a{}_{a\epsilon}\bar\Gamma^\epsilon{}_{\delta\beta} - \bar\Gamma^a{}_{\delta e}\bar\Gamma^e{}_{a\beta} - \bar\Gamma^\alpha{}_{a\beta}C^a{}_{\alpha\delta}$$

$$= R_{\delta\beta} \underbrace{- \nabla_\delta\bar\Gamma^a{}_{a\beta}}_{-\nabla_\delta\nabla_\beta(\ln\tilde g_F^{1/2})} \underbrace{- \frac14 g^{ad}D_\delta g_{de}\, g^{ef}D_\beta g_{fa}}_{-\frac14\tilde g^{ad}\tilde g^{ef}\overset{G}{\nabla}_\delta\tilde g_{de}\overset{G}{\nabla}_\beta\tilde g_{fa}} - \frac12 F_a{}^\alpha{}_\beta F^a{}_{\alpha\delta}$$

$$g^{\beta\delta}\bar R_{\delta\beta} = R - \frac12 F_a{}^{\alpha\beta}F^a{}_{\alpha\beta} - \overset{G}{\nabla}{}^\alpha\overset{G}{\nabla}_\alpha(\ln\tilde g_F^{1/2}) - \frac14\tilde g^{ad}\tilde g^{ef}\overset{G}{\nabla}{}^\alpha\tilde g_{de}\overset{G}{\nabla}_\alpha\tilde g_{fa}$$

$$\bar R_{db} = D_A\bar\Gamma^A{}_{db} - D_d\bar\Gamma^A{}_{Ab} - \bar\Gamma^A{}_{Eb}C^E{}_{Ad} + \bar\Gamma^A{}_{AE}\bar\Gamma^E{}_{db} - \bar\Gamma^A{}_{dE}\bar\Gamma^E{}_{Ab}$$

$$= (D_a\bar\Gamma^a{}_{db} - D_d\bar\Gamma^a{}_{ab} - \bar\Gamma^a{}_{eb}C^e{}_{ad} + \bar\Gamma^a{}_{ae}\bar\Gamma^e{}_{db} - \bar\Gamma^a{}_{de}\bar\Gamma^e{}_{ab}) \qquad = R_{db} = R^f{}_dR^g{}_b\underbrace{\tilde R_{db}}_{\text{quadratic in } \tilde C\text{'s}}$$

$$\underbrace{-\bar\Gamma^\alpha{}_{d\epsilon}\bar\Gamma^\epsilon{}_{\alpha b}}_{\frac14\tilde F_d{}^\alpha{}_e\tilde F_{b\alpha}{}^e} + D_\alpha\bar\Gamma^\alpha{}_{db} + (\bar\Gamma^a{}_{a\epsilon} + \bar\Gamma^\alpha{}_{\alpha\epsilon})\bar\Gamma^\epsilon{}_{db} \underbrace{- \bar\Gamma^\alpha{}_{de}\bar\Gamma^e{}_{\alpha b} - \bar\Gamma^a{}_{de}\bar\Gamma^e{}_{ab}}_{R^c{}_dR^e{}_b(\frac12\tilde g^{gf}\overset{G}{\nabla}_\alpha\tilde g_{fc}\overset{G}{\nabla}{}^\alpha\tilde g_{eg})}$$

$$\frac14 R^f{}_dR^g{}_b F_f{}^{\alpha\beta}F_{g\alpha\beta}$$

$$R^c{}_dR^e{}_b(-\frac12\overset{G}{\nabla}_\alpha\overset{G}{\nabla}{}^\alpha\tilde g_{ce})$$

$$R^c{}_dR^e{}_b(-\frac12\overset{G}{\nabla}_\alpha(\ln\tilde g_F^{1/2})\overset{G}{\nabla}{}^\alpha\tilde g_{ce})$$

$$-\frac12 R^c{}_dR^e{}_b\left(\frac{1}{\tilde g_F^{1/2}}\overset{G}{\nabla}_\alpha(\tilde g_F^{1/2}\overset{G}{\nabla}{}^\alpha\tilde g_{ce})\right)$$

$$= R^c{}_dR^e{}_b\Big(\tilde R_{ce} + \frac14F_c{}^{\alpha\beta}F_{e\alpha\beta} - \frac12\tilde g_F^{-1/2}\overset{G}{\nabla}_\alpha(\tilde g_F^{1/2}\overset{G}{\nabla}{}^\alpha\tilde g_{ce}) + \underbrace{\frac12\tilde g^{fg}\overset{G}{\nabla}_\alpha\tilde g_{fc}\overset{G}{\nabla}{}^\alpha\tilde g_{eg}}_{\text{missing in Tabensky}}\Big)$$

$$g^{bd}\bar R_{db} = \tilde R_F + \frac14F_a{}^{\alpha\beta}F^a{}_{\alpha\beta} - \frac12\tilde g_F^{-1/2}\overset{G}{\nabla}_\alpha(\tilde g_F^{1/2}\nabla^\alpha\tilde g_{ab})\tilde g^{ab} + \frac12\tilde g^{ab}\tilde g^{cd}\overset{G}{\nabla}_\alpha\tilde g_{ac}\overset{G}{\nabla}{}^\alpha\tilde g_{bd}$$

$$\bar R = R + \tilde R_F - \frac14F_a{}^{\alpha\beta}F^a{}_{\alpha\beta} - \overset{G}{\nabla}_\alpha\overset{G}{\nabla}{}^\alpha(\ln\tilde g_F^{1/2}) - \frac12\tilde g_F^{-1/2}\overset{G}{\nabla}_\alpha(\tilde g_F^{1/2}\nabla^\alpha\tilde g_{ab})\tilde g^{ab} + \frac14\tilde g^{ab}\tilde g^{cd}\overset{G}{\nabla}_\alpha\tilde g_{ac}\overset{G}{\nabla}{}^\alpha\tilde g_{bd}$$

$$\bar R_{d\beta} = D_A\bar\Gamma^A{}_{d\beta} - D_d\bar\Gamma^A{}_{A\beta} - \bar\Gamma^A{}_{E\beta}C^E{}_{Ad} + \bar\Gamma^A{}_{AE}\bar\Gamma^E{}_{d\beta} - \bar\Gamma^A{}_{dE}\bar\Gamma^E{}_{A\beta}$$

$$= -\frac12 D_\alpha\tilde F_d{}^\alpha{}_\beta + D_a\bar\Gamma^a{}_{d\beta} - \bar\Gamma^a{}_{e\beta}C^e{}_{ad} + \bar\Gamma^a{}_{a\epsilon}\bar\Gamma^\epsilon{}_{d\beta} + \bar\Gamma^\alpha{}_{\alpha\epsilon}\bar\Gamma^\epsilon{}_{d\beta} - \bar\Gamma^a{}_{de}\bar\Gamma^e{}_{a\beta} - \bar\Gamma^a{}_{d\epsilon}\bar\Gamma^\epsilon{}_{a\beta} - \bar\Gamma^\alpha{}_{d\epsilon}\bar\Gamma^\epsilon{}_{\alpha\beta}$$

$$-\bar\Gamma^\alpha{}_{de}\bar\Gamma^e{}_{\alpha\beta}$$

$$R^a{}_d\Big(-\frac12\overset{G}{\nabla}_\alpha F_a{}^\alpha{}_\beta - \frac12F_a{}^\alpha{}_\beta\overset{G}{\nabla}_\alpha\ln g_F^{1/2} + \frac12\tilde g^{bc}\overset{G}{\nabla}_\alpha\tilde g_{ab}F_c{}^\alpha{}_\beta \underbrace{- \frac12\tilde\Gamma^b{}_a{}^c\overset{G}{\nabla}_\beta\tilde g_{bc}}_{+\frac12 C^{bc}{}_a\overset{G}{\nabla}_\beta\tilde g_{bc}}\Big)$$

(opp sign relative to Tabensky
given that all F terms switch sign)

We thus have on spacetime, a field theory involving the spacetime metric, a gauge field $A^a{}_\mu dx^\mu$ and a ~~metric valued~~ Lie algebra inner product valued field with components $\tilde{g}_{ab}$. The end result should be identical to what Tabensky does without fiber bundles using an equivalent formalism living entirely on spacetime & not in a bundle over it.

I don't trust my computation completely (as it is 3:15 am) but neither do I trust either Tabensky or the typesetter. You have all the machinery to check every detail for yourself.

We have assumed the group G to be unimodular for this generalization. (existence of a bi-invariant metric is no longer relevant). This is a necessary assumption in order that the ~~equation~~ Lagrange equation's are equivalent to $\mathrm{Ricci}(\bar{g}) = 0$.

For a nonunimodular group the Lagrange equations ~~for $\tilde{g}_{ab}$~~ are not equivalent to $\bar{R}_{AB} = 0$.

Wednesday, Sept. 21, 1978

Dear Arthur,

As I promised, I spent my first "day at the office" figuring out how Tabensky's junk relates to the principal bundle approach, and in so doing cleaned up my computational scheme for the latter approach. The results are sketched in a continuation of the notes Ive already given you on principal fiber bundle computations. I hope you can decipher them. Im waiting to hear from you about the stuff you mentioned on the phone. I hope I can still be of some further use to you. I will think about the variational problem next.

Just a cute side remark. The spitting up of the bundle into a four parameter family of fibers is very similar to the slicing of spacetime into a one-parameter family of hypersurfaces. In adapted coordinates zero shift in the spacetime case corresponds to orthogonality of the horizonal and vertical subspaces (zero "supershift") in the bundle case.

$\mathcal{K}_{\alpha\, ab} = -\frac{1}{2} \overset{G}{\nabla}_\alpha g_{ab}$ corresponds to the extrinsic curvature

$K_{ab} = -\frac{1}{2N} \frac{\partial}{\partial t} g_{ab}$ in the spacetime case

g_{ab} to the induced metric, ${}^F R$ to the hypersurface scalar curvature R^*

and the spacetime Lagrangian:

$$\mathcal{L} = {}^4R\, {}^4g^{1/2} = {}^4R\, N g^{1/2} = -g_{ab}\, \ddot{\pi}^{ab} - N g^{1/2}(\mathrm{Tr}\, K^2 - \mathrm{Tr}^2 K - R^*)$$
$$= -2(g^{1/2})^{\cdot\cdot} + N g^{1/2}\, (\mathrm{Tr}\, K^2 - \mathrm{Tr}^2 K + R^*)$$

to the fiber part of the bundle Lagrangian

$$-2 \overset{G}{\nabla}_\alpha \overset{G}{\nabla}^\alpha ({}^4g\, g_F)^{1/2} + g_F^{1/2}(-\mathrm{Tr}\,\mathcal{K}^2 + \mathrm{Tr}^2\mathcal{K} + {}^F R)$$

etc.

I will be waiting patiently to hear from you again. I miss California weather, Among other things. But Ive got an office in the basement next to the computers so I am assured the best airconditioning available.

bob jantzen

In this continuation we adopt a slight change in notation which is more consistent than above (just what you needed right) by introducing in a given local trivialization two frames: a barred "left invariant frame" $\{\bar{D}_A\} = \{D_\alpha, e_a\}$ whose ~~elements~~ vertical elements are fundamental vector fields and an unbarred "right invariant frame" $\{D_A\} = \{D_\alpha, \tilde{e}_a\}$ whose vertical elements are invariant under the right action of the group G on the fiber $\sim G$. In this local trivialization, $\{e_a\}$ is just a left invariant frame on the group G and $\{\tilde{e}_a\}$ is the corresponding right invariant frame (the right action of G on the bundle becomes right translation on G in the local trivialization and fundamental vector fields become left invariant vector fields on G.)

The right invariant frame has the advantage that components of fields which telescope along the fibers (i.e. have nice transformation properties under the right action of the group G on the bundle), depend only on the base point and are constant along the fibers.

So turn to page 3 of these notes and compare with our new notation

Again let $\{\hat{e}_a\}$ be a basis of TG_e considered as the Lie algebra of G.

$$\{\bar{D}_A\} = \{D_\alpha, e_a = \xi_a\}$$

dual frame:

$$\{\bar{W}^A\} = \{dx^\alpha, \bar{A}^a = R^{-1a}{}_b A^b + \omega^b\}$$

$$D_\alpha = \partial_\mu - A^a{}_\mu \tilde{e}_a = \cancel{\partial_\mu - \bar{A}^a{}_\mu e_a}$$

$$\{D_A\} = \{D_\alpha, \tilde{e}_a\}$$

dual frame:

$$\{W^A\} = \{dx^\alpha, R^a{}_b \bar{A}^b = A^b + \tilde{\omega}^b\}$$

$$A^a = A^a{}_\mu dx^\mu \qquad \bar{A}^a = R^{-1a}{}_b A^b + \omega^b$$

$$F^a = \tfrac{1}{2} F^a{}_{\mu\nu} dx^\mu \otimes dx^\nu \qquad \bar{F}^a = R^{-1a}{}_b F^b$$

$$A = A^a \hat{e}_a \qquad F = F^a \hat{e}_a = dA + [A \wedge A]$$

$$\bar{A}^\bullet = \bar{A}^a \hat{e}_a$$

$$\bar{F} = \bar{F}^a \hat{e}_a = d\bar{A} + [\bar{A} \wedge \bar{A}]$$

connection and curvature forms as TG_e-valued forms

$$g = g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta + \bar{g}_{ab}\, \bar{A}^a \otimes \bar{A}^b$$

$$= g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta + g_{ab}\, (R^a{}_c A^c) \otimes (R^b{}_d A^d)$$

depend only on base point. as do A^a_α and $F^a{}_{\alpha\beta}$

PRINCIPAL FIBER BUNDLE

notes by Robert Jantzen

By choice we restrict to unimodular groups

$C^a{}_{ba} = 0.$

LEFT INVARIANT FRAME

(USING FUNDAMENTAL VECTOR FIELDS $\xi_a = e_a$)

THIS FRAME IS DEFINED UP TO A LIE GROUP AUTOMORPHISM OF EACH FIBER INDEPENDENT OF THE BASE POINT (ie a finite dimensional class) whereas the RIGHT FRAME IS DEFINED UP TO AN INDEPENDENT LIE GROUP AUTOMORPHISM OF EACH FIBER (ie an infinite class) once $C^a{}_{bc}$ is fixed.

$$\boxed{\begin{array}{ll} \{\bar{D}_A\}, \text{ dual frame } & \{\bar{W}^A\} \\ \| & \| \\ \{D_\alpha, e_a\} & \{dx^\alpha, \bar{A}^a\} \end{array}}$$

$$\tilde{e}_a = e_b R^{-1b}{}_a$$

LIE BRACKETS:

$$[D_\alpha, D_\beta] = -\bar{F}^a{}_{\alpha\beta}\, e_a$$
$$[D_\alpha, e_b] = 0$$
$$[e_a, e_b] = C^c{}_{ab}\, e_c$$

STRUCTURE FUNCTIONS:

$$[\bar{D}_A, \bar{D}_B] = \bar{C}^C{}_{AB}\, \bar{D}_C$$

$$\boxed{\begin{array}{l} \bar{C}^c{}_{\alpha\beta} = -\bar{F}^c{}_{\alpha\beta} \\ \bar{C}^c{}_{ab} = C^c{}_{ab} \end{array}}$$

METRIC

$$\bar{g} = g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta + \bar{g}_{ab}\, \bar{A}^a \otimes \bar{A}^b = \bar{g}_{AB}\, \bar{W}^A \otimes \bar{W}^B$$

METRIC CONNECTION COMPONENTS

$$\nabla_{\bar{D}_A} \bar{D}_B = \bar{\Gamma}^C{}_{AB}\, \bar{D}_C$$

$$\boxed{\begin{array}{l} \bar{\Gamma}^\gamma{}_{\alpha\beta} = \left\{ {\gamma \atop \alpha\beta} \right\} \\ \text{- - - -} \\ \bar{\Gamma}^\gamma{}_{a\beta} = \bar{\Gamma}^\gamma{}_{\beta a} = -\tfrac{1}{2} \bar{F}_a{}^\gamma{}_\beta \\ \bar{\Gamma}^c{}_{\alpha\beta} = \qquad -\tfrac{1}{2} \bar{F}^c{}_{\alpha\beta} \\ \text{- - - -} \\ \bar{\Gamma}^\gamma{}_{ab} = \qquad -\tfrac{1}{2} \overset{G}{\nabla}{}^\gamma \bar{g}_{ab} \\ \bar{\Gamma}^c{}_{a\beta} = \bar{\Gamma}^c{}_{\beta a} = \tfrac{1}{2} g^{cd} \overset{G}{\nabla}_\beta \bar{g}_{da} \\ \text{- - - -} \\ \bar{\Gamma}^c{}_{ab} = \tfrac{1}{2} C^c{}_{ab} - \tfrac{1}{2} \bar{C}_{(a}{}^c{}_{b)} \end{array}}$$

$g_F \equiv \det(g_{ab})$

$$\bar{\Gamma}^{\gamma a}{}_a = \Gamma^{\gamma a}{}_a$$
$$\bar{\Gamma}^a{}_{\beta a} = \bar{\Gamma}^a{}_{a\beta} = \Gamma^a{}_{\beta a}$$
$$\bar{\Gamma}^c{}_{bc} = 0 = \bar{\Gamma}^c{}_{cb}$$

UNIMODULARITY USED HERE

RIGHT INVARIANT FRAME

USING "RIGHT INVARIANT" VECTOR FIELDS e_a; $\quad \pounds_{\xi_a} D_A = 0$

THIS HAS THE ADVANTAGE THAT THE COMPONENTS OF INVARIANT FIELDS DEPEND ONLY ON THE BASE POINT (THE SAME IS TRUE OF ~~PSEUDO~~-TENSORIAL FORMS WHICH PROJECT DOWN TO FIELDS ON SPACETIME).

$$\boxed{\begin{array}{ll} \{D_A\}, \text{ dual frame } & \{W^A\} \\ \| & \| \\ \{D_\alpha, \tilde{e}_a\} & \{dx^\alpha, R^a{}_b \bar{A}^b\} \end{array}}$$

LIE BRACKETS:

$$[D_\alpha, D_\beta] = -F^a{}_{\alpha\beta}\, \tilde{e}_a$$
$$[D_\alpha, \tilde{e}_b] = A^a{}_\alpha C^c{}_{ab}\, \tilde{e}_c$$
$$[\tilde{e}_a, \tilde{e}_b] = -C^c{}_{ab}\, \tilde{e}_c$$

STRUCTURE FUNCTIONS:

$$[D_A, D_B] = \mathbb{C}^C{}_{AB}\, D_C$$

$$\boxed{\begin{array}{l} \mathbb{C}^c{}_{\alpha\beta} = -F^c{}_{\alpha\beta} \\ \mathbb{C}^c{}_{\alpha b} = A^a{}_\alpha C^c{}_{ab} \\ \mathbb{C}^c{}_{ab} = -C^c{}_{ab} \end{array}}$$

METRIC

$$\bar{g} = g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta + g_{ab} (R^a{}_c \bar{A}^c) \otimes (R^b{}_d \bar{A}^d) = g_{AB}\, W^A \otimes W^B$$

METRIC CONNECTION COMPONENTS

$$\nabla_{D_A} D_B = \Gamma^C{}_{AB}\, D_C$$

for example: $\Gamma^C{}_{AB} = \tfrac{1}{2} g^{CD} (D_A g_{DB} + D_B g_{DA} - D_D g_{AB}) + \tfrac{1}{2} C^C{}_{AB} + \tfrac{1}{2} g^{CD} (g_{AE} C^E{}_{DB} + g_{BE} C^E{}_{DA})$

$$\boxed{\begin{array}{l} \Gamma^\gamma{}_{\alpha\beta} = \left\{ {\gamma \atop \alpha\beta} \right\} \\ \text{- - - -} \\ \Gamma^\gamma{}_{a\beta} = \Gamma^\gamma{}_{\beta a} = -\tfrac{1}{2} F_a{}^\gamma{}_\beta \\ \Gamma^c{}_{\alpha\beta} = \qquad -\tfrac{1}{2} F^c{}_{\alpha\beta} \\ \text{- - - -} \\ \Gamma^\gamma{}_{ab} = -\tfrac{1}{2} \overset{G}{\nabla}{}^\gamma g_{ab} \\ \Gamma^c{}_{a\beta} = \tfrac{1}{2} g^{cd} \overset{G}{\nabla}_\beta g_{da} \\ \Gamma^c{}_{\alpha b} = \tfrac{1}{2} g^{cd} \overset{G}{\nabla}_\alpha g_{db} + A^d{}_\alpha C^c{}_{db} \\ \text{- - - -} \\ \Gamma^c{}_{ab} = -\tfrac{1}{2} C^c{}_{ab} - C_{(a}{}^c{}_{b)} \end{array}}$$

$$\Gamma^{\gamma a}{}_a = -g^{\gamma\delta} \partial_\delta \ln g_F{}^{1/2}$$
$$\Gamma^a{}_{\beta a} = \Gamma^a{}_{a\beta} = \tfrac{1}{2} g^{ad} \partial_\beta g_{da} = \partial_\beta \ln g_F{}^{1/2}$$
$$\Gamma^c{}_{cb} = 0 = \Gamma^c{}_{bc}$$

see page 13 for more explanation

TABENSKY'S EXTENDED RIEMANNIAN GEOMETRY

He assumes semi-simplicity so that "ad" is an isomorphism (semisimple $\subset$ unimodular) and hence his extended tangent space on spacetime is isomorphic to the tangent space of the corresponding principal fiber bundle over spacetime.

A BASIS FOR HIS EXTENDED TANGENT SPACE (AN EXTENDED FRAME) IS $\{\overset{E}{D}_A\} = \{\overset{G}{\nabla}_\alpha, \, ad(e_a)\}$ ACTING ON LIE ALGEBRA VALUED FUNCTIONS ON SPACE TIME:

$$\overset{G}{\nabla}_\alpha \, (X^d e_d) = (\partial_\alpha X^d + A^b{}_\alpha C^d{}_{bc} X^c)\, e_d$$

$$ad(e_a)\,(X^d e_d) = (C^d{}_{ab} X^b)\, e_d.$$

COMMUTATORS (OR "EXTENDED LIE BRACKETS")

$$[\overset{E}{D}_\alpha, \overset{E}{D}_\beta] = F^a{}_{\alpha\beta}\, ad(e_a)$$

$$[\overset{E}{D}_\alpha, ad(e_b)] = A^a{}_\alpha C^c{}_{ab}\, ad(e_b)$$

$$[ad(e_a), ad(e_b)] = C^c{}_{ab}\, ad(e_c)$$

EXTENDED STRUCTURE FUNCTIONS

$$[\overset{E}{D}_A, \overset{E}{D}_B] = \overset{E}{C}{}^C{}_{AB}\, \overset{E}{D}_C$$

$$\boxed{\begin{aligned} \overset{E}{C}{}^c{}_{\alpha\beta} &= F^c{}_{\alpha\beta} \\ \overset{E}{C}{}^c{}_{\alpha b} &= A^a{}_\alpha C^c{}_{ab} \\ \overset{E}{C}{}^c{}_{ab} &= C^c{}_{ab} \end{aligned}}$$

THESE ARE THE SAME AS THE STRUCTURE FUNCTIONS FOR THE RIGHT FRAME EXCEPT FOR A SIGN:

$(-1)^{\text{number latin indices}}$

THIS COULD BE REMEDIED BY USING INSTEAD THE FRAME $\{D_\alpha, -\tilde{e}_a\}$ IN WHICH CASE THE CORRESPONDENCE WOULD BE EXACT OR WE JUST USE THE SIGN RULE TO GO FROM THE RIGHT FRAME TO THE EXTENDED FRAME.

EXTENDED METRIC:

$$\cancel{\overset{E}{g}_{AB}} \qquad \overset{E}{g}(\overset{E}{D}_A, \overset{E}{D}_B) = g_{AB}$$

METRIC CONNECTION COMPONENTS

$$\overset{E}{\nabla}_{\overset{E}{D}_A} \overset{E}{D}_B = \overset{E}{\Gamma}{}^C{}_{AB}\, \overset{E}{D}_C$$

$$\overset{E}{\Gamma}{}^{\cdot}{}_{\cdot\cdot} = \Gamma^{\cdot}{}_{\cdot\cdot}$$

$$\overset{E}{\Gamma}{}^{\cdot}{}_{\cdot\cdot} = -\Gamma^{\cdot}$$

$$\overset{E}{\Gamma}{}^{\cdot}{}_{\cdot\cdot} = -\Gamma$$

$$\overset{E}{\Gamma}{}^{\cdot}{}_{\cdot\cdot} = \Gamma^{\cdot}{}_{\cdot\cdot}$$

$$\overset{E}{\Gamma} = \Gamma$$

$$\overset{E}{\Gamma} = \Gamma$$

$$\overset{E}{\Gamma}{}^{\cdot}{}_{\cdot\cdot} = -\Gamma^{\cdot}{}_{\cdot\cdot}$$

The right invariant frame has the advantage that the components of everything we're interested in depend only on the base point (whereas the components of these guys in the left frame pick up adjoint matrices $R^a{}_b$ and $R^{-1\,a}{}_b$ on all latin indices). If we evaluate the Ricci tensor in the right invariant frame, then its components in the left invariant frame may be obtained by performing the adjoint transformation on all Latin indices while the components of Tabensky's extended Ricci tensor may be obtained by using the above sign correspondence. [For comparison with Tabensky's formulas one should note that his bar covariant derivative is inconsistently defined in his paper, obscures the dependence on the individual fields and is just generally fucked.)

$$R_{DB} = R^A{}_{BAD} = D_A\Gamma^A{}_{DB} - D_D\Gamma^A{}_{AB} - \Gamma^A{}_{EB}C^E{}_{AD} + \Gamma^A{}_{AE}\Gamma^E{}_{DB} - \Gamma^A{}_{DE}\Gamma^E{}_{AB}$$

Conveniently and suggestively define:

$$K_{\alpha ab} = -\tfrac{1}{2}\overset{G}{\nabla}_\alpha g_{ab} \qquad \mathrm{TR}\,K^2 = K_\alpha{}^a{}_b K^{\alpha b}{}_a = \tfrac{1}{4} g^{ab}g^{cd}\overset{G}{\nabla}_\alpha g_{ab}\overset{G}{\nabla}{}^\alpha g_{cd}$$

$$\mathrm{TR}^2K = K_\alpha{}^a{}_a K^{\alpha b}{}_b = \tfrac{1}{4} g^{ac}g^{db}\overset{G}{\nabla}_\alpha g_{ab}\overset{G}{\nabla}{}^\alpha g_{cd}$$

$$R_{\delta\beta} = {}^4R_{\delta\beta} - \tfrac{1}{2}F_a{}^\alpha{}_\beta F^a{}_{\alpha\delta} - \overset{G}{\nabla}_\delta\overset{G}{\nabla}_\beta \ln g_F{}^{1/2} - K_\delta{}^{ab}K_{\beta ab}$$

$$g^{\delta\beta}R_{\delta\beta} = {}^4R - \tfrac{1}{2}F^2 - \overset{G}{\nabla}{}^\alpha\overset{G}{\nabla}_\alpha \ln g_F{}^{1/2} - \mathrm{TR}K^2$$

$$R_{db} = {}^FR_{db} + \tfrac{1}{4}F_d{}^{\alpha\beta}F_{b\alpha\beta} - \tfrac{1}{2}g_F{}^{-1/2}\overset{G}{\nabla}_\alpha(g_F{}^{1/2}\overset{G}{\nabla}{}^\alpha g_{db}) + 2K^{\alpha c}{}_d K_{\alpha cb}$$

$$g^{bd}R_{db} = {}^FR + \tfrac{1}{4}F^2 - g_F{}^{-1/2}\overset{G}{\nabla}_\alpha\overset{G}{\nabla}{}^\alpha g_F{}^{1/2}$$

$$R = {}^4R - \tfrac{1}{4}F^2 - 2g_F{}^{-1/2}\overset{G}{\nabla}_\alpha\overset{G}{\nabla}{}^\alpha g_F{}^{1/2} + \mathrm{TR}^2K - \mathrm{TR}K^2 + {}^FR$$

$$\mathcal{L} = (g_F\,{}^4g)^{1/2}R = g_F{}^{1/2}(\underbrace{\mathcal{L}_{EYM}}_{{}^4g^{1/2}({}^4R-\frac{1}{4}F^2)} + \mathrm{TR}^2K - \mathrm{TR}K^2 + {}^FR) - \underbrace{2\nabla_\alpha\nabla^\alpha({}^4g\,g_F)^{1/2}}_{\text{spacetime divergence}}$$

$$R_{d\beta} = -\tfrac{1}{2}g_F{}^{-1/2}\overset{G}{\nabla}_\alpha(g_F{}^{1/2}F_d{}^\alpha{}_\beta) + \underbrace{\tfrac{1}{2}C^{ab}{}_d\nabla_\beta g_{ab}}_{= \mathrm{TR}\,\underline{R}_d\underline{K}_\beta = C^a{}_{db}K_\beta{}^b{}_a}$$

<u>equations of motion</u>	<u>for</u>
$R_{\delta\beta}=0$	${}^4g_{\alpha\beta}$
$R_{d\beta}=0$	A^a_α
$R_{db}=0$	g_{ab}

These follow from independent variation of these fields in the scalar curvature density Lagrangian $\mathcal{L}$ on the bundle. (this is why we impose unimodularity)

FIBER CURVATURE

The fiber Ricci tensor components ${}^F R_{db}$ are given by the formula for R_{DB} with all indices Latin and $D_c g_{ab} = D_d \Gamma^c{}_{ab} = 0$:

$${}^F R_{db} = -\Gamma^a{}_{eb} \underbrace{C^e{}_{ad}}_{\Gamma^e{}_{ad}-\Gamma^e{}_{da}} + \underbrace{\Gamma^a{}_{ae}}_{C^a{}_{ea}} \Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{ab}$$

CANCEL

$$= -C^a{}_{ea}\Gamma^e{}_{db} - \Gamma^a{}_{eb}\Gamma^e{}_{ad}$$

PLUG IN $\Gamma^c{}_{ab} = \underbrace{\tfrac12 C^c{}_{ab}}_{\Gamma^c{}_{[ab]}} + \underbrace{C_{(a}{}^c{}_{b)}}_{\Gamma^c{}_{(ab)}}$ (note $C^a{}_{ea}\Gamma^e{}_{db} = C^a{}_{ea} C_{(a}{}^c{}_{b)}$ by the contracted Jacobi identity $C^e{}_{ae} C^a{}_{bc} = 0$)

$$\boxed{{}^F R_{db} = -C^a{}_{ea} C_{(a}{}^e{}_{b)} + \frac14 C_{dfg}C^{fg}{}_b - C_{fgd}C^{(fg)}{}_b}$$

$${}^F R = \underbrace{-C^a{}_{ea}C^{be}{}_b}_{\text{these terms vanish for unimodular group}} + \frac14 C_{dfg}C^{fgd} - C_{fgd}C^{(fg)d}$$

In Tabensky's final line of section 3, in the formula for the fiber curvature in ~~$R_{\alpha\beta}$~~ the vertical-vertical part of RICCI ($R_{\alpha\beta}$ in his notation), he has omitted the term $-C^{\delta\gamma}{}_{\alpha} C_{\gamma\delta\beta}$ inside the parentheses.

All indices are raised and lowered with g^{ab} and g_{ab} of course.

This is formula 2 of §1 of Jensen's paper *The Scalar Curvature of Left-Invariant Riemannian Metrics*, except for the fact that he assumes an orthonormal frame.

$M =$ spacetime manifold

$G =$ semi-simple Lie group with Lie algebra $\mathfrak{g}$ with basis $\{e_a\}$
$[e_a, e_b] = C^c{}_{ab} e_c$

Tabensky introduces "extended functions" as elements of $L(M) =$ Lie algebra-valued functions on M, and "extended vector fields" as derivations on $L(M)$. Since all derivations of a semi-simple Lie algebra are inner (and "ad" is an isomorphism), an extended vector field is just a pair (X, Ω) consisting of a vector field X and $\Omega \in L(M)$ acting on $\theta \in L(M)$ as follows:

$$(X, \Omega)\, \theta = (\overset{G}{\nabla}_X + \mathrm{ad}(\Omega))\, \theta$$

$$\overset{G}{\nabla}_X \theta = (X + \mathrm{ad}(A))\, \theta$$

$$\theta = \theta^a e_a, \qquad \Omega = \Omega^a e_a, \qquad X = X^\alpha \partial_\alpha$$

$$(X, \Omega)(\theta^a e_a) = \underbrace{(\partial_\alpha \theta^a + A^c_\alpha C^a{}_{cb} \theta^b)}_{\overset{G}{\nabla}_\alpha \theta^a} e_a + (\Omega^c C^a{}_{cb} \theta^b) e_a$$

A basis for each extended tangent space is therefore:

$$\{\overset{E}{D}_A\} = \{\overset{G}{\nabla}_\alpha, \mathrm{ad}(e_a)\}.$$

This extended tangent space is isomorphic to the tangent space of a principal fiber bundle over spacetime with group G and his entire extended Riemannian geometry is equivalent to putting a right invariant metric on this bundle but he only works in a single gauge patch so the bundle approach is the only natural way to globalize his approach.

Whenever a symmetry group acts simply transitively on its orbits, one can introduce a frame on the manifold on which it is acting that is invariant under the action of the group; imposing symmetry on a tensor field simply kills the component derivatives along the orbits. One can then easily compare the Lagrange derivative of a Lagrangian before and after the symmetry is imposed on the Lagrangian.

In our case choose the frame $\{\tilde{D}_A\} = \{\partial_\alpha, \tilde{e}_a\}$ with dual frame $\{\tilde{W}^A\} = \{dx^\alpha, \tilde{\omega}^a\}$.

$$\tilde{C}^A{}_{BC} = -\delta^A_a \delta^b_B \delta^c_C C^a{}_{bc} \qquad \partial_A \equiv \tilde{D}_A, \qquad \tilde{\partial}_A \equiv \partial_A - C^B{}_{AB} = \partial_A - \delta^a_A C^b{}_{ab}.$$

$$\bar{g} = g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta + g_{ab}\,(A^a{}_\alpha dx^\alpha + \tilde{\omega}^a) \otimes (A^b{}_\beta dx^\beta + \tilde{\omega}^b)$$
$$= \underbrace{g_{\alpha\beta}}_{\equiv \tilde{g}_{\alpha\beta}} dx^\alpha \otimes dx^\beta + \underbrace{A_{a\alpha}}_{\equiv \tilde{g}_{a\alpha}} (dx^\alpha \otimes \tilde{\omega}^a + \tilde{\omega}^a \otimes dx^\alpha) + \underbrace{g_{ab}}_{\equiv \tilde{g}_{ab}} \tilde{\omega}^a \otimes \tilde{\omega}^b$$

Imposing the bundle symmetry on $\bar{g}$ means $\partial_a \tilde{g}_{AB} = 0 = \partial_D \partial_a \tilde{g}_{AB} = \partial_a \partial_D \tilde{g}_{AB}$.

The scalar curvature Lagrangian form:

$$\mathcal{L} = L\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge \tilde{\omega}^1 \wedge \dots \wedge \tilde{\omega}^r$$

$$L(\tilde{g}_{AB}, \partial_C \tilde{g}_{AB}, \partial_C \partial_D \tilde{g}_{AB}, C^A{}_{BC}) = R\tilde{g}^{1/2}$$

The components of the Lagrange derivative of L with respect to $\bar{g}$ in this frame are:

$$\frac{\delta L}{\delta \tilde{g}_{AB}} = \frac{\partial L}{\partial \tilde{g}_{AB}} - \tilde{\partial}_C \left(\frac{\partial L}{\partial \partial_C \tilde{g}_{AB}} \right) + \tilde{\partial}_D \tilde{\partial}_C \left(\frac{\partial L}{\partial \partial_C \partial_D \tilde{g}_{AB}} \right) = -\tilde{g}^{1/2} G^{AB}.$$

$\tilde{\partial}_A$ rather than ∂_A appears since the divergence of $\tilde{\omega}^1 \wedge \dots \wedge \tilde{\omega}^r$ enters in any "integration by parts". Imposing the bundle symmetry on the Lagrangian means inserting zeros in the arguments $\partial_a \tilde{g}_{AB}$, $\partial_D \partial_a \tilde{g}_{AB}$, $\partial_a \partial_D \tilde{g}_{AB}$; the same for its Lagrange derivative. ~~The Lagrange~~ The Einstein tensor density $(\times(-1))$ of the symmetric metric minus the Lagrange derivative of the symmetrized Lagrangian is easily computed:

$$\underbrace{\mathrm{SYM}\frac{\delta L}{\delta \tilde{g}_{AB}}}_{-\tilde{g}^{1/2} G^{AB}} - \frac{\delta}{\delta \tilde{g}_{AB}} \mathrm{SYM}\, L = \delta^a_A \delta^b_B \Big(-C^d{}_{cd} \frac{\partial L}{\partial \partial_c g_{ab}} + C^f{}_{cf} C^g{}_{dg} \frac{\partial L}{\partial \partial_c \partial_d g_{ab}} + C^f{}_{cf} \partial_\gamma \Big(\frac{\partial L}{\partial \partial_\gamma \partial_c g_{ab}} + \frac{\partial L}{\partial \partial_c \partial_\gamma g_{ab}} \Big) \Big)$$

Thus unless $C^b{}_{ab} = 0$, the Lagrange derivative of the Lagrange with the symmetry imposed does not equal the Einstein tensor density, ie one does not obtain the Einstein equations by varying the Lagrangian evaluated on the symmetric field.

LAGRANGE MULTIPLIER DILEMMA

If we want to extremize $\int R g^{1/2}$ subject to the constraints $\mathcal{L}_{\xi_a} g_{\alpha\beta} = 0$ we just add the constraints to the Lagrangian with Lagrange multipliers and vary both g and the Lagrange multipliers freely:

$$I = \int_C (R g^{1/2} + \lambda^{a\alpha\beta} \mathcal{L}_{\xi_a} g_{\alpha\beta})$$

C compact ~~submanifold~~ region ~~or subspace of~~

$$I' = \int_C -(g^{1/2} G^{\alpha\beta} + \mathcal{L}_{\xi_a} \lambda^{a\alpha\beta})\, g'_{\alpha\beta} + \lambda'^{a\alpha\beta} \mathcal{L}_{\xi_a} g_{\alpha\beta}$$

$$+ \underbrace{\int_C g^{1/2} (-g^{\alpha\beta} g^{\gamma\delta} + g^{\alpha\gamma} g^{\delta\beta})\, g'_{\alpha\beta;\gamma\delta} + \int (\lambda^{a\alpha\beta} \xi^{\gamma}{}_a\, g'_{\alpha\beta})_{;\gamma}}$$

divergence integrals — forget about since integrate to ∂C where $g'_{\alpha\beta}$ and derivatives vanish.

$$\boxed{\begin{array}{c} g^{1/2} G^{\alpha\beta} + \mathcal{L}_{\xi_a} \lambda^{a\alpha\beta} = 0 \\ \mathcal{L}_{\xi_a} g_{\alpha\beta} = 0 \end{array}}$$

now what. usually one can solve for the Lagrange multipliers. Here one seems to be at a dead end.

January 11, 1979

Dear Arthur,

Thanks for getting in touch with me again. Ive dashed off answers to all of your questions but got stuck on one term in the variation of the Lagrangian. Since Ive only got a week left to prepare a new talk Im giving in Waterloo, Canada, Ill have to leave it for now and get back to it later.

- Misner Thorne and Wheeler use a different (and I think unnatural) convention:

$$\Gamma^A{}_{BC} = \omega^A(\nabla_{e_B} e_C) = {}^{MTW}\Gamma^A{}_{CB}$$

Since $\Gamma^A{}_{[BC]} = \frac{1}{2} C^A{}_{BC}$ for a symmetric connection,

$${}^{MTW}\Gamma^A{}_{BC} = \Gamma^A{}_{BC} - C^A{}_{BC}$$ which accounts for the different sign $-\frac{1}{2}C^A{}_{BC}$ instead of $\frac{1}{2}C^A{}_{BC}$ in the connection formula.

- On page 14 it is noted that one may rewrite $R_{\delta\beta}$ (page 12) as:

$$R_{\delta\beta} = {}^4R_{\delta\beta} - \tfrac{1}{2} F_a{}^{\alpha}{}_{\beta} F^a{}_{\alpha\delta} - \underbrace{g_F^{-1/2}\nabla_\delta\nabla_\beta g_F^{1/2}}_{\text{now its } g_F^{1/2} \text{ instead of } \ln g_F^{1/2}} + \underbrace{\mathrm{Tr}K_\delta \mathrm{Tr}K_\beta - \mathrm{Tr}K_\delta K_\beta}_{\text{this combination occurs all over the place}}$$

does that make you happier?

- Gauge derivatives aren't needed for $g_F^{1/2}$ which is a weight one scalar density under gauge transformations only since the groups we are considering are unimodular (ie no difference between gauge scalars and gauge scalar densities).

- I have no idea what $\overset{G}{\nabla}_\delta \overset{G}{\nabla}_\beta g_F^{1/2}$ means geometrically.

Im looking forward to hearing from you again.

Best regards,

bob

USEFUL IDENTITY: $\nabla_\alpha \nabla_\beta \ln g_F{}^{1/2} = -\mathrm{Tr}\, K_\alpha \mathrm{Tr}\, K_\beta + g_F{}^{-1/2} \nabla_\alpha \nabla_\beta\, g_F{}^{1/2}$

pf. Expand ln derivative and use: $\mathrm{Tr}\, K_\alpha = -\nabla_\alpha \ln g_F{}^{1/2}$.

BUNDLE EINSTEIN TENSOR COMPONENTS

■ $$G_{\delta\beta} = R_{\delta\beta} - \tfrac{1}{2} R g_{\delta\beta} = {}^4G_{\delta\beta} - T^{YM}{}_{\delta\beta} + \mathrm{Tr}\, K_\delta \mathrm{Tr}\, K_\beta - \mathrm{Tr}\, K_\delta K_\beta - \tfrac{1}{2} g_{\delta\beta}\left(\mathrm{Tr}^2 K - \mathrm{Tr}\, K^2 + {}^F R\right) - g_F{}^{-1/2}\, {}^4g^{-1/2}\, G^{\alpha\gamma}{}_{\beta\delta} \nabla_\alpha \nabla_\gamma\, g_F{}^{1/2}$$

DEFINITIONS: $$T^{YM}{}_{\delta\beta} = \tfrac{1}{2}\left(F_a{}^\alpha{}_\beta F^a{}_{\alpha\delta} - \tfrac{1}{4} F^2 g_{\beta\delta}\right)$$

$$G^{\alpha\beta\gamma\delta} = {}^4g^{1/2}\left(g^{\alpha(\gamma} g^{\delta)\beta} - g^{\alpha\beta} g^{\gamma\delta}\right)$$

NOTE THAT USING THE USEFUL IDENTITY ONE CAN REWRITE $R_{\delta\beta}$ and R_{db}:

$$R_{\delta\beta} = {}^4R_{\delta\beta} - \tfrac{1}{2} F_a{}^\alpha{}_\beta F^a{}_{\alpha\delta} - g_F{}^{-1/2} \nabla_\delta \nabla_\beta\, g_F{}^{1/2} + \mathrm{Tr}\, K_\delta \mathrm{Tr}\, K_\beta - \mathrm{Tr}\, K_\delta K_\beta$$

$$R_{db} = {}^F R_{db} + \tfrac{1}{4} F_d{}^{\alpha\beta} F_{b\alpha\beta} - \tfrac{1}{2} \nabla_\alpha \nabla^\alpha g_{db} + 2 K^{\alpha c}{}_d K_{\alpha c b} - \mathrm{Tr}\, K_\alpha K^\alpha{}_{db}$$

■ $$G_{db} = R_{db} - \tfrac{1}{2} R g_{db} = {}^F G_{db} + \tfrac{1}{4}\left(F_d{}^{\alpha\beta} F_{b\alpha\beta} + \tfrac{1}{2} g_{db} F^2\right) - \tfrac{1}{2} g_{db}\, {}^4R - \tfrac{1}{2} \overset{G}{\nabla}_\alpha \overset{G}{\nabla}{}^\alpha g_{db} + g_F^{-1/2} g_{db} \nabla_\alpha \nabla^\alpha g_F{}^{1/2} + 2 K_\alpha{}^c{}_d K^\alpha{}_{cb} - \mathrm{Tr}\, K_\alpha K^\alpha{}_{db} - \tfrac{1}{2} g_{db}\left(\mathrm{Tr}^2 K - \mathrm{Tr}\, K^2\right)$$

■ $$G_{d\beta} = R_{d\beta}$$

USEFUL VARIATION FORMULAS (for any metric $g_{\alpha\beta}$)

$$\begin{cases} (g^{1/2}R)' = -g^{1/2}G^{\alpha\beta}g'_{\alpha\beta} + g^{1/2}g^{\alpha\beta}R'_{\alpha\beta} \\ g^{1/2}g^{\alpha\beta}R'_{\alpha\beta} = G^{\alpha\beta\gamma\delta}\nabla_\gamma\nabla_\delta g'_{\alpha\beta} \\ Ng^{1/2}g^{ab}R'_{ab} = (G^{\alpha\beta\gamma\delta}\nabla_\gamma\nabla_\delta N)g'_{\alpha\beta} + \nabla_\alpha X^\alpha \end{cases}$$

VARIATION OF BUNDLE LAGRANGIAN

$$\mathcal{L} = g_F{}^{1/2}\,{}^4g^{1/2}R = g_F{}^{1/2}\underbrace{\mathcal{L}_{EYM}}_{{}^4g^{1/2}({}^4R-\frac{1}{4}F^2)} + {}^4g^{1/2}g_F{}^{1/2}(\mathrm{Tr}^2K - \mathrm{Tr}K^2 + {}^FR) + \mathrm{div}$$

$$\frac{\delta}{\delta g_{\alpha\beta}}(g_F{}^{1/2}\mathcal{L}_{EYM}) = g_F{}^{1/2}\,{}^4g^{1/2}(-{}^4G^{\alpha\beta} + T^{\alpha\beta}) + G^{\alpha\beta\gamma\delta}\nabla_\gamma\nabla_\delta g_F{}^{1/2}$$

$$\frac{\delta}{\delta g_{\alpha\beta}}(({}^4g^{1/2}\cdots g^{\gamma\delta})g_F{}^{1/2}(\mathrm{Tr}K_\gamma\mathrm{Tr}K_\delta - \mathrm{Tr}K_\gamma K_\delta))$$

$$= -{}^4g^{1/2}g_F{}^{1/2}(\mathrm{Tr}K^\alpha\mathrm{Tr}K^\beta - \mathrm{Tr}K^\alpha K^\beta - \tfrac{1}{2}g^{\alpha\beta}(\mathrm{Tr}^2K - \mathrm{Tr}K^2))$$

$$\frac{\delta}{\delta g_{\alpha\beta}}({}^4g^{1/2}g_F{}^{1/2}\,{}^FR) = \tfrac{1}{2}\,{}^4g^{1/2}g_F{}^{1/2}g_{\alpha\beta}\,{}^FR$$

compare with previous page:

$$\boxed{\frac{\delta}{\delta g_{\alpha\beta}}\mathcal{L} = -{}^4g^{1/2}g_F{}^{1/2}G^{\alpha\beta}}$$

$$\frac{\delta}{\delta A^d{}_\beta}(-\tfrac{1}{4}g_F{}^{1/2}g^{1/2}F^2) = g^{1/2}\overset{G}{\nabla}_\alpha(g_F{}^{1/2}F_d{}^{\alpha\beta})$$ just as in usual computation

$$\overset{G}{\nabla}_\alpha g_{ab} = \nabla_\alpha g_{ab} - A^c{}_\alpha(C_{acb} + C_{bca})$$

$$(\overset{G}{\nabla}_\alpha g_{ab})' = -2A^c_\alpha{}'\,C_{(a|c|b)}$$

$$(K_{\alpha ab})' = C_{(a|c|b)}A^c_\beta{}'$$

$$(\mathrm{Tr}K_\alpha)' = \underset{\substack{\| \\ 0 \text{ unimodular groups}}}{(\mathrm{Tr}\,k_c)}A^c{}_\beta{}'$$

$$(\mathrm{Tr}K^2)' = (K_{\alpha ab}K^{\alpha ab})' = 2\,\mathrm{Tr}k_dK^\beta A^d_\beta{}'$$

$$\frac{\delta}{\delta A^d{}_\beta}\mathcal{L} = g^{1/2}\overset{G}{\nabla}_\alpha(g_F{}^{1/2}F_d{}^{\alpha\beta}) - 2g_F^{1/2}g^{1/2}\mathrm{Tr}k_dK^\beta$$

$$= -2g^{1/2}g_F{}^{1/2}R_d{}^\beta = \boxed{-2g^{1/2}g_F{}^{1/2}G_d{}^\beta = \frac{\delta}{\delta A^d{}_\beta}\mathcal{L}}$$

The following computation of $\frac{\delta}{\delta g_{ab}} \mathcal{L} = -\, {}^4 g^{1/2} g_F^{1/2}\, G^{ab}$

is a near miss. The variation of $Tr^2 K$ isn't coming out right and I'm temporarily puzzled. Perhaps you can figure it out. I'll get back to it when I have more time.

$$\left(g_F^{1/2} g^{1/2} \left({}^4R - \tfrac{1}{4} g_{ab} F^a{}_{\wedge}{}^{\alpha\beta} F^b{}_{\alpha\beta} + {}^F R \right)\right)' =$$

$$g_F^{1/2} g^{1/2} \left(\tfrac{1}{2} {}^4R\, g^{ab} - \tfrac{1}{4} \left(F^{a\alpha\beta} F^b{}_{\alpha\beta} + \tfrac{1}{2} g^{ab} F^2 \right)\right) g'_{ab} +$$

$$g_F^{1/2} g^{1/2} \left(- {}^F G^{ab} g'_{ab} \right)$$

This uses the fact that $(g_F^{1/2}\, {}^F R)' = - g_F^{1/2}\, {}^F G^{ab} g'_{ab}$.
The divergence vanishes identically by unimodularity and right invariance.

$$\left(g^{1/2} g_F^{1/2} \left(Tr^2 K - Tr K^2 \right) \right)' =$$

$$g^{1/2} g_F^{1/2} \left(\tfrac{1}{2} g^{ab} \left(Tr^2 K - Tr K^2 \right) g'_{ab} + (Tr^2 K)' - (Tr K^2)' \right)$$

$$(Tr K^2)' = \left(g^{\alpha\beta} g^{ac} g^{bd} K_{\alpha ab} K_{\beta cd} \right)'$$

$$= -2 K^{\alpha ac} K_{\alpha a}{}^{d} g'_{cd} + \underbrace{2 K^{\alpha ab} \left(-\tfrac{1}{2} \nabla_\alpha g'_{cd} \right)}_{\nabla_\alpha K^{\alpha cd}\, g'_{cd}} + \text{div}$$

$$\overset{G}{\nabla}_\alpha K^{\alpha cd} = -\tfrac{1}{2} \overset{G}{\nabla}_\alpha \left(g^{ca} g^{db} \overset{G}{\nabla}{}^\alpha g_{ab} \right)$$

$$= -\tfrac{1}{2} g^{ca} g^{db} \overset{G}{\nabla}_\alpha \overset{G}{\nabla}{}^\alpha g_{ab} \underbrace{- \tfrac{1}{2} \underbrace{\nabla_\alpha (g^{ca} g^{db})}_{-(K_\alpha{}^{ca} g^{db} + g^{ca} K_\alpha{}^{db})} \underbrace{\nabla^\alpha g_{ab}}_{-2K^\alpha{}_{ab}}}_{4K_{\alpha a}{}^{c} K^{\alpha a d}}$$

$$(Tr K^2)' = \left(2K^{\alpha ac} K_{\alpha a}{}^{d} - \tfrac{1}{2} g^{ca} g^{db} \overset{G}{\nabla}_\alpha \overset{G}{\nabla}{}^\alpha g_{ab} \right) g'_{cd}$$

$$(Tr^2 K)' = \left(g^{\alpha\beta} g^{ab} K_{\alpha ab} g^{cd} K_{\beta cd} \right)' = 2\, Tr K^\alpha \left(g^{cd} K_{\alpha cd} \right)'$$

$$= -2\, Tr K^\alpha K_\alpha{}^{cd} g'_{cd} + \underbrace{2\, Tr K^\alpha g^{cd} \left(-\tfrac{1}{2} \overset{G}{\nabla}_\alpha g'_{cd} \right)}_{\underbrace{\overset{G}{\nabla}_\alpha \left(Tr K^\alpha g^{cd} \right) g'_{cd}}_{\left((\nabla_\alpha Tr K^\alpha) g^{cd} + 2\, Tr K^\alpha K_\alpha{}^{cd} \right) g'_{cd}} + \text{div}}$$

CANCEL

$$= (\nabla_\alpha Tr K^\alpha) g^{cd} g'_{cd} = \left(Tr^2 K - g_F^{-1/2} \nabla_\alpha \nabla^\alpha g_F^{1/2} \right) g^{cd} g'_{cd}$$

If the term $(\mathrm{Tr}^2 K) g^{cd}$ had been instead $\mathrm{Tr}\, K_\alpha K^{\alpha\, cd}$,
we would have obtained $\dfrac{\delta \mathcal{L}}{\delta g_{ab}} = -4 g^{1/2} g_F{}^{1/2} G^{ab}$

because all the other terms agree. I don't understand what
I've done wrong yet.