Left-invariant vector fields
and one-forms

\[ \mathfrak{g}_L = \{ X \in \mathfrak{g} \mid \forall \phi \in \mathfrak{g}, [X, \phi] = \mathfrak{ad}(X) \phi \} \]
\[ \mathfrak{g}_R = \{ X \in \mathfrak{g} \mid \forall \phi \in \mathfrak{g}, [\phi, X] = \mathfrak{ad}(\phi) X \} \]

\[ \mathfrak{g}_L = \{ X \in \mathfrak{g} \mid \forall \phi \in \mathfrak{g}, [X, \phi] = \mathfrak{ad}(X) \phi \} \]
\[ \mathfrak{g}_R = \{ X \in \mathfrak{g} \mid \forall \phi \in \mathfrak{g}, [\phi, X] = \mathfrak{ad}(\phi) X \} \]

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\[ \mathfrak{g}_R = \{ X \in \mathfrak{g} \mid \forall \phi \in \mathfrak{g}, [\phi, X] = \mathfrak{ad}(\phi) X \} \]
Natural connections on a Lie Group $\hat{\nabla}$, $\hat{\nabla}$, $\hat{\nabla} = (\hat{\nabla} + \hat{\nabla})$

$\nabla$ and $\hat{\nabla}$ are flat transverse connections, i.e., have the same opposite torsion coinciding with the difference tensor (in coordinates $\hat{\nabla}^{ab} = \hat{\nabla}^{ab}$), and therefore the same geodesics and have zero curvature.

$\nabla$ is the unique symmetric connection having some geodesics (described on the previous page).

$$\nabla_{\alpha} \beta = C_{\alpha \beta} \gamma \beta$$

$$\nabla_{\alpha} \beta = - C_{\alpha \beta} \gamma \beta$$

Components of $\nabla$

$$\nabla_{\alpha} \beta = \partial_{\alpha} \beta + \Gamma^{\gamma}_{\alpha \beta} \gamma$$

$$\nabla_{\alpha} \beta = - \partial_{\alpha} \beta - \hat{\Gamma}^{\gamma}_{\alpha \beta} \gamma$$

Components of $\hat{\nabla}$

$$\hat{\nabla}_{\alpha} \beta = \partial_{\alpha} \beta + \hat{\Gamma}^{\gamma}_{\alpha \beta} \gamma$$

$$\hat{\nabla}_{\alpha} \beta = - \partial_{\alpha} \beta - \hat{\Gamma}^{\gamma}_{\alpha \beta} \gamma$$

The torsion components of $\nabla$

$$\Omega^{\alpha \beta}_{\gamma} = \frac{1}{2} \hat{\Gamma}^{\alpha \beta}_{\gamma} - \frac{1}{2} \hat{\Gamma}^{\beta \alpha}_{\gamma}$$

The torsion components of $\hat{\nabla}$

$$\Omega^{\alpha \beta}_{\gamma} = \frac{1}{2} \hat{\Gamma}^{\alpha \beta}_{\gamma} - \frac{1}{2} \hat{\Gamma}^{\beta \alpha}_{\gamma}$$

It follows that $\nabla$ is parallel transport is just right translation.

$$\hat{\nabla}^{\alpha \beta}_{\gamma} = C_{\alpha \beta} \gamma$$

$$\hat{\nabla}^{\alpha \beta}_{\gamma} = - C_{\alpha \beta} \gamma$$

Compatible metric $g$: $(\nabla g = \hat{\nabla} g = 0 = 0) \Leftrightarrow$ bi-invariant $\leftrightarrow$ adjoint invariant

$$g = g_{\alpha} w^{\alpha \beta} w^\beta = g_{\alpha} \hat{w}^{\alpha \beta} \hat{w}^\beta$$

$$\nabla g = \partial g_{\alpha} w^{\alpha \beta} w^\beta = (\partial g_{\alpha} w^{\alpha \beta} + \hat{\Gamma}^{\gamma}_{\alpha \beta} \gamma \omega^\gamma)$$

$$\hat{\nabla} g = \partial g_{\alpha} w^{\alpha \beta} w^\beta = (\partial g_{\alpha} w^{\alpha \beta} - \hat{\Gamma}^{\gamma}_{\alpha \beta} \gamma \omega^\gamma)$$

$$\nabla g = g_{\alpha} \partial \omega^\beta + g_{\alpha} \partial \hat{\omega}^\beta = 0$$

$$\hat{\nabla} g = g_{\alpha} \partial \omega^\beta - g_{\alpha} \partial \hat{\omega}^\beta = 0$$

Since $g$ is symmetric and $g = 0$, it is the connection generated by $g$.

For semi-simple groups, the Killing form is nondegenerate (and the adjoint group unimodular: $0 = TR \delta \alpha = C \delta \beta$) and hence determines a bi-invariant metric:

$$g_{\alpha} = \frac{1}{2} \hat{g}_{\alpha \beta}$$

We include a minus since compact groups have negative definite Killing forms and $g = R^{\alpha \beta} g_{\alpha} g_{\beta}$.

For abelian groups, $C = 0$ and the adjoint group is trivial (i.e., $g = 0$). When a canonical basis is chosen in this case, we can consider $g_{\alpha}$. For example, $g_{\alpha} = \delta_{\alpha \beta}$.

No other groups have nondegenerate bi-invariant metrics.

Hence: $R = \Gamma^{\alpha \beta}_{\gamma}$. For a semi-simple scalar curvature:

$$\Gamma^{\alpha \beta}_{\gamma} = \frac{1}{2} \delta_{\alpha \beta} g_{\delta \gamma}$$

$$\Gamma^{\alpha \beta}_{\gamma} = 0$$

$$\Gamma^{\alpha \beta}_{\gamma} = \frac{1}{2} \delta_{\alpha \beta} g_{\delta \gamma}$$

(Except for direct product groups with any number of factors manifolds of these types.)
Consider a principal bundle $P$ over base manifold $M = \text{space-time with Yang-Mills group } G \text{ acting on the fibers on the right}$, so that in a local trivialization, this action becomes right translation. The generators of this action called fundamental vector fields (left-invariant vector fields in a local trivialization) span the vertical subspaces. The horizontal subspaces are picked out by a connection $\omega$.

For simplicity, we assume from the start a local trivialization over $UCM$, which is a coordinate patch for local coordinates $\{x^a\}$ on $M$. These naturally lift up to dual vectors of a local coordinate system $\{X^a\}$ on $UXG$: $\{dx^a, e^a, \tilde{e}^a\}$.

The connection form is: $\Gamma = A^a dx^a$. It annihilates $H_P$ and maps $V_P$ onto the Lie algebra of $G$: $\bar{A}(X^a \Delta a) = X^a \bar{a}$. The curvature form is: $F = dA + A \wedge A = F^a \bar{e}^a$.

The structure functions $C^c_{ab}$ for this frame:

\[
\begin{align*}
\{D_a, D_b\} &= -F^c \Gamma_{ab}^c \\
\{D_a, S_b\} &= 0 \\
\{S_a, S_b\} &= \Omega_{ab} \end{align*}
\]

The restriction of the metric to a fiber makes the fiber isometric to the pseudo-Riemannian manifold of the previous page.

The volume form on $P$ is just the exterior product of the spacetime volume form with the volume form on the fiber:

\[
\Omega = \sqrt{|det(g_{\Delta a})|} \Omega = \Omega_{a\bar{a}} \wedge \Omega = 2! \sqrt{2} \left| det(g_{ab}) \right| \frac{1}{2} \sqrt{g^{\bar{a}b}} \omega^{\alpha_1 \alpha_2 \ldots} 
\]
Components of Metric Connection in our frame:

\[ \Gamma^X_{ab} = \frac{1}{2} g^{cd} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}) + \frac{1}{2} g^{cd} (g_{ae} \Gamma^e_{db} + g_{be} \Gamma^e_{da}) + \frac{1}{2} C^e_{ab} \]

nonzero components:

\[ \Gamma^X_{ab} = \frac{1}{2} g^{cd} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}) \]

\[ \Gamma^X_{ab} = \frac{1}{2} g^{cd} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}) \]

\[ \Gamma^C_{ab} = -\frac{1}{2} \Gamma^F_{ab} \]

Scalar curvature:

\[ R = g^{ab} R_{ab} = \ldots = R_{SS} - \frac{1}{4} F_a E_b F^b E_a + R_{\text{fiber}} \]

function only of \( x \), \( \{ \} \), semi-simple \( \sim \) Lie algebra

By introducing torsion on the fiber, the scalar curvature density Lagrangian becomes:

\[ L = \tilde{R} \tilde{\nabla} = \left[ \left( -L_{\text{Einstein}} + L_{\text{gauge}} \right) h + \tilde{\nabla} \right] \tilde{\nabla} \text{ fiber} \]

independent of fiber coordinates.

The Einstein equations (Real tensor = 0) are therefore the combined gravitational and Yang-Mills field equations.

**Bonus**

Geodesics on \( P \) (same for metric connection or with added torsion on fiber)

Let \( C(t) = (x(t), \varphi(t)) \) be a geodesic with tangent:

\[ C'(t) = C''(t) \Gamma^a_{bc} A^a(t) \]

\[ C''(t) = A^a(t) \Gamma^a_{bc}(t) \]

\[ C''(t) = A^a(t) \Gamma^a_{bc}(t) = \frac{d}{dt} \left( \frac{d}{dt} \right) \varphi(t) \]

\[ C''(t) = A^a(t) \Gamma^a_{bc}(t) = \frac{d}{dt} \left( \frac{d}{dt} \right) \varphi(t) \]

\[ C''(t) = A^a(t) \Gamma^a_{bc}(t) = \frac{d}{dt} \left( \frac{d}{dt} \right) \varphi(t) \]

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\[ C''(t) = A^a(t) \Gamma^a_{bc}(t) = \frac{d}{dt} \left( \frac{d}{dt} \right) \varphi(t) \]

Define the components of the charge vector (really charge/mass vector):

\[ Q^a(t) = R^a(t) \varphi^b \]

Then:

\[ \frac{d^2 x^a(t)}{dt^2} = Q^a(t) \varphi^b \frac{d x^b}{dt} \]

"Lorentz force law"

charge vector parallel transported along trajectory on spacetime.

Presumably these equations describe a test Yang-Mills particle on this spacetime.
we now generalize by relaxing bi-invariance of the induced metric on the fiber to merely right invariance (maintaining the orthogonality of the horizontal and vertical subspaces) so that the bundle metric $\bar{g}$ is invariant under the right action of $G$ on $P$, i.e. $\bar{g}_{x_{a}} \bar{g} = 0$.

In the locally trivial notation of pages 3 and 4:

$$\bar{g} = g_{uv}(x) dx^{a} dx^{b} + g_{ab} \bar{A}^{a} \bar{A}^{b}, \quad g_{ab} = g_{cd}(x) R_{a}^{c} R_{d}^{b}$$

$\bar{g}_{x_{a}} \bar{g} = g_{cd}(x) \partial_{a} \bar{A}^{c} \partial_{a} \bar{A}^{d}$ = manifestly right-invariant.

To evaluate the connection components in the frame $\{ D_{a} \}$ we have to compute the derivatives $D_{a} g_{ab}$ and $D_{a} g_{cd}$ first by a matrix calculation, with $d$ differentiating only along the fiber:

$$D_{a} g_{ab} = D_{a} \left( g_{uv}(x) dx^{a} dx^{b} + g_{ab} \bar{A}^{a} \bar{A}^{b} \right)$$

$$= g_{ab} \partial_{a} \bar{A}^{b} + \partial_{a} g_{uv} \left( dx^{a} dx^{b} \right)$$

$$= g_{ab} \partial_{a} \bar{A}^{b} + g_{uv} \partial_{a} \bar{A}^{u} \partial_{a} \bar{A}^{v}$$

$$= 2 \partial_{a} g_{ab}$$

These are both special cases of a general formulas.

Suppose $T^{a_{1} \ldots a_{n}}$ are the components of a tensor field on the base which transforms under the indicated tensor product of the adjoint representation under gauge transformations. Then define $\hat{T}^{a_{1} \ldots a_{n}}$:

$$\hat{T}^{a_{1} \ldots a_{n}} = R^{a_{1} b} \ldots R^{a_{n} b} T^{b_{1} \ldots b_{n}}$$

$$D_{a} \hat{T}^{a_{1} \ldots a_{n}} = \partial_{a} \hat{T}^{a_{1} \ldots a_{n}}$$

This is even a special case of the formulas for representation valued fields.

For example:

$$D_{a} F_{ab} = C_{ac} F_{c b}$$

$$D_{a} F_{c a} = R^{b} \partial_{c} \phi^{a} \left( d \delta F_{a b} + A_{a}^{d} C_{d b} \right)$$

$$D_{a} F_{c}^{a} = C_{d c} F_{d}^{a}$$

$$D_{a} F_{c}^{d} = R^{a} \partial_{c} \phi^{d} - A^{a} C_{d c} F_{d}^{c}$$
Now we use the standard formula to compute the components of the metric connection in the frame \( \{ A^i \} \):

\[
\Gamma^a_{ab} = \frac{1}{2} g^{ac} \left( \partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab} \right) + \frac{1}{2} g^{ca} \left( \partial_c g_{ab} + \partial_a g_{cb} - \partial_b g_{ac} \right) + \frac{1}{2} g^{cb} \partial_b a
\]

Since we know \( g^{ab} \) from page 3, only \( g_{ab}, \partial_a g_{bc}, \partial_b g_{ac}, \partial_c g_{ab} \) are nonzero & we just evaluated \( \partial_a g_{bc}, \partial_b g_{ac}, \partial_c g_{ab} \) with \( \partial_a g_{bc} = \delta_a^{[b} \delta_c^{c]} \), the only remaining nonzero derivatives, it is just a matter of plugging in. The result (nonzero components):

\[
\Gamma^a_{ab} = \{ \delta_a \}
\]

\[
\Gamma^c_{ab} = \frac{1}{2} C^{cab} \delta_c^b
\]

\[
\Gamma^a_{bc} = -\frac{1}{2} g^{cd} \partial_c g_{ab}
\]

\[
\Gamma^c_{ab} = \Gamma^c_{ba} = \frac{1}{2} g^{cd} \partial_d g_{ab}
\]

\[
\Gamma^b_{ac} = 0
\]

To evaluate Ricci tensor components we need in addition \( \partial_a \Gamma^c_{ab} \). The only nonzero such derivatives are:

\[
\partial_a \Gamma^a_{ab} = \delta_b \{ \delta_a \} \quad \text{etc...}
\]

Fill in yourself.
The components of the Ricci tensor of $\mathcal{G}$ in the frame $\{D_i\}$ are:

\[ R_{ab} = D_a D_b - D_b D_a - D_a D_b \mathcal{G} + D_b D_a \mathcal{G} - D_a D_b \mathcal{G} - D_b D_a \mathcal{G} \]

\[ R_{ab} = D_a F_{bc} - D_b F_{ac} + D_c F_{ab} + D_a d_b - D_b d_a - D_c d_a + D_a d_b - D_b d_a - D_c d_a \]

\[ g^{cd} R_{cd} = R - \frac{1}{2} F_{ab} F^{ab} - \frac{1}{4} \mathcal{G}_{ab} (\mathcal{G}^{cd}) - \frac{1}{4} \mathcal{G}^{ab} \mathcal{G}^{cd} \mathcal{G}_{cd} \]

\[ R_{ab} = D_a F_{bc} - D_b F_{ac} + D_c F_{ab} + D_a d_b - D_b d_a - D_c d_a + D_a d_b - D_b d_a - D_c d_a \]

\[ \mathcal{G}_{ab} R_{cd} = R_{ab} + \frac{1}{4} F_{bc} F_{cd} - \frac{1}{2} \mathcal{G}_{bc} \mathcal{G}_{cd} \]

\[ g^{cd} R_{ab} = R + \frac{1}{4} F_{bc} F_{cd} - \frac{1}{2} \mathcal{G}_{bc} \mathcal{G}_{cd} \]

\[ R_{ab} = D_a D_b - D_b D_a - D_a D_b \mathcal{G} + D_b D_a \mathcal{G} - D_a D_b \mathcal{G} - D_b D_a \mathcal{G} \]

\[ R_{ab} = D_a F_{bc} - D_b F_{ac} + D_c F_{ab} + D_a d_b - D_b d_a - D_c d_a + D_a d_b - D_b d_a - D_c d_a \]

\[ g^{cd} R_{cd} = R - \frac{1}{2} F_{ab} F^{ab} - \frac{1}{4} \mathcal{G}_{ab} (\mathcal{G}^{cd}) - \frac{1}{4} \mathcal{G}^{ab} \mathcal{G}^{cd} \mathcal{G}_{cd} \]
We thus have on spacetime, a field theory involving the spacetime metric, a gauge field $A^\mu dx^\mu$ and a Lie algebra inner product valued field with components $\mathfrak{g}_{ab}$. The end result should be identical to what Tabensky does without fiber bundles using an equivalent formalism living entirely on spacetime and not in a bundle over it.

I don't trust my computation completely (as it is 2:15 am) but neither do I trust either Tabensky or the typesetter. You have all the machinery to check every detail for yourself.

We have assumed the group $G$ to be unimodular for this generalization. (Existence of a bi-invariant metric is no longer relevant). This is a necessary assumption in order that the Lagrange equations are equivalent to $\text{Ricci}(\mathfrak{g}) = 0$. For a nonunimodular group the equations $\mathfrak{g}_{ab}$ are not equivalent to $\mathfrak{R}_{ab} = 0$. 
Dear Arthur,

As I promised, I spent my first "day at the office." Figuring out how Talbisky's junk relates to the principal bundle approach, and in so doing cleaned up my computational scheme for the latter approach. The results are sketched in a continuation of the notes I've already given you on principal fiber-bundle computations. I hope you can decipher them. I'm waiting to hear from you about the stuff you mentioned on the phone. I hope I can still be of some further use to you. I will think about the variational problem next.

Just a cute side remark. The splitting of the bundle into a four-parameter family of fibers is very similar to the slicing of spacetime into one-parameter family of hypersurfaces. In adapted coordinates a zero shift in the spacetime case corresponds to orthogonality of the horizontal and vertical subspaces (zero "supershift") in the bundle case.

\[ K_{ab} = -\frac{1}{2} \delta_a \delta_b \] 

corresponds to the extrinsic curvature

\[ \mathcal{K}_{ab} = -\frac{1}{2\lambda} \delta_a \delta_b \] 

in the spacetime case.

\[ \kappa \] 

to be induced metric, \( R \) to be hypersurface scalar curvature \( R^* \)

and the spacetime Lagrangian:

\[ L = R_{\mu
u} \gamma^\mu \gamma^\nu - 4R \eta_{\mu\nu} \eta^{\mu\nu} - \eta_{\mu\nu} \eta^{\mu\nu} (\kappa R - \kappa R^*) \]

- \[ 2 \kappa^2 \] 

- \[ \kappa (\kappa R - \kappa R^*) \]

The fiber part of the bundle Lagrangian:

\[ -2 \frac{c_0}{\lambda} \frac{\theta^a}{\theta^b} \left( \delta_a \delta_b \right)^2 + \frac{\kappa}{2} \left( -\kappa R + \kappa R^* \right) \]

etc.

I will be waiting patiently to hear from you again. I miss California weather. Among other things. But I've got an office in the basement next to the computers so I am assured the best airconditioning available.

Bob Jantzen
In this continuation we adopt a slight change in notation which is more consistent than above (just what you needed right) by introducing in a given local trivialization two frames: a barred "left invariant frame" \( \{ \bar{D}a \} = \{ D_a, \bar{e}_a \} \) whose elements' vertical elements are fundamental vector fields and an unbarred "right invariant frame" \( \{ Dx \} = \{ D_a, e_a \} \) whose vertical elements are invariant under the right action of the group \( G \) on the fiber \( \sim G \). In this local trivialization, \( \{ e_a \} \) is just a left invariant frame on the group \( G \) and \( \{ \bar{e}_a \} \) is the corresponding right invariant frame (the right action of \( G \) on the bundle becomes right translation on \( G \) in the local trivialization, and fundamental vector fields become left invariant vector fields on \( G \)).

The right invariant frame has the advantage that components of fields which telescope along the fibers (i.e., have no transformation properties under the right action of the group \( G \) on the bundle) depend only on the base point and are constant along the fibers.

So turn to page 3 of these notes and compare with our new notation. Again let \( \{ \bar{e}_a \} \) be a basis of \( T \bar{G}_e \), considered as the Lie algebra of \( \bar{G} \).

\[
\{ \bar{D}a \} = \{ D_a, \bar{e}_a \}
\]

\[
\{ \bar{W}^a \} = \{ \bar{d}x^a, \bar{R}^a = R^a_{\;b} \bar{A}^b + \bar{\omega}^a \}
\]

\[
\bar{D}a = \partial a - \bar{A}^a \bar{e}_a = \partial a - \bar{R}^a_{\;b} \bar{e}_b
\]

\[
\bar{A}^a = \bar{A}^a \bar{d}x^a, \quad \bar{A}^a = \bar{d} \bar{R}^a_{\;b} \bar{A}^b + \bar{\omega}^a
\]

\[
F = F^a \bar{e}_a = \partial \bar{A}^a + [\bar{A} \Lambda \bar{A}^a]
\]

\[
\bar{R}^a_{\;b} = \bar{R}^a_{\;b} \bar{e}_b
\]

\[
G = G_{ab} \bar{d}x^a \bar{d}x^b + g_{ab} \bar{A}^a \bar{A}^b
\]

\[
F = F^a \bar{e}_a = \partial \bar{A}^a + [\bar{A} \Lambda \bar{A}^a]
\]

\[
\bar{A}^a = \bar{R}^a_{\;b} \bar{e}_b
\]

Connection and curvature forms are \( T \bar{G}_e \)-valued forms.

\[
\partial a - \bar{R}^a_{\;b} \bar{e}_b
\]

\[
\bar{R}^a_{\;b} \bar{e}_b
\]

\[
\bar{A}^a \bar{d}x^a + \bar{G}_{ab} \bar{R}^a_{\;c} \bar{R}^c_{\;d} \bar{A}^d
\]

\[
\bar{A}^a \bar{d}x^a + \bar{G}_{ab} \bar{R}^a_{\;c} \bar{R}^c_{\;d} \bar{A}^d
\]

depend only on base point, as do \( \bar{A}^a \) and \( F^a \).
\[ \{ \mathbf{D}_a \}, \text{ dual frame } \{ \mathbf{W}^a \} \]
\[ \mathbf{Z}_a = E_{ab} \mathbf{D}_b \]

\[ \{ \mathbf{d}_a, \mathbf{A}^a \} \]

\[ \mathbf{Z}_a = E_{ab} \mathbf{d}_b \]

**LEFT INVARIANT FRAME**

**RIGHT INVARIANT FRAME**

**LIE BRACKETS**:
\[ [\mathbf{D}_a, \mathbf{D}_b] = -F_{ab} \mathbf{D}_c \]
\[ [\mathbf{D}_a, \mathbf{E}_b] = 0 \]
\[ [\mathbf{E}_a, \mathbf{E}_b] = C_{ab} \mathbf{E}_c \]

**STRUCTURE FUNCTIONS**:
\[ [\mathbf{D}_a, \mathbf{D}_b] = C_{ab} \mathbf{D}_c \]

**METRIC**
\[ g_{ab} = -F_{ab} \]
\[ C_{ab} = -C_{ba} \]

\[ g_{ab} = \frac{1}{2} F_{ab}^2 + \frac{1}{2} C_{ab}^2 \]

\[ g = g_{ab} \mathbf{d}_a \mathbf{d}_b \]

**METRIC CONNECTION COMPONENTS**
\[ \nabla_{\mathbf{d}_a} \mathbf{D}_b = \Gamma^c_{ab} \mathbf{D}_c \]

**EXAMPLES**
\[ \Gamma^a_{bg} = \frac{1}{2} (F_{bg} + F_{gb} - F_{bg} + F_{gb}) \]
A basis for his extended tangent space (an extended frame) is:
\[ \{ X_a, \text{ad}(e_a) \} \] acting on Lie algebra valued functions on space-time:
\[ \delta_a \left( x^d e_d \right) = (\partial_a x^d + \Lambda^k_{\alpha} C_{\alpha}^k x^k) e_d \]
\[ \text{ad}(e_a) \left( x^d e_d \right) = (C^d_{\alpha} e_{\alpha}) e_d \]

Commutators (or "extended Lie brackets")
\[ [\delta_a, \delta_b] = F^e_{ab} \text{ad}(e_e) \]
\[ [\delta_a, \text{ad}(e_b)] = (\Lambda^c_{\alpha} C^c_{\alpha} e_{\alpha}) \text{ad}(e_b) \]
\[ [\text{ad}(e_a), \text{ad}(e_b)] = C^c_{\alpha} \text{ad}(e_c) e_{\alpha} \]

Extended structure functions
\[ (\delta_a, \delta_b) = \tilde{C}^c_{\alpha c b} \]

These are the same as the structure functions for the right frame except for a sign:
\(-\)

This could be remedied by using instead the frame \( \{ \delta_a, -e_a \} \) in which case the correspondence would be exact or we just use the sign rule to go from the right frame to the extended frame.

Extended metric:
\[ g_{ab} = \tilde{g}_{ab} \]

Metric connection components:
\[ \Gamma_{\alpha \beta}^{\gamma} = \tilde{\Gamma}^{\gamma}_{\alpha \beta} \]

\( \Gamma_{\alpha \beta}^{\gamma} \): components of a spin connection
The right invariant frame has the advantage that the components of everything were interested in depend only on the base point (whereas the components of these guys in the left frame pick up adjacent matrices $R^a_b$ and $E^{a\circ}_b$ on all Latin indices). If we evaluate the Ricci tensor in the right invariant frame, then its components in the left invariant frame may be obtained by performing the adjoint transformation on all Latin indices while the components of Tabensky's extended Ricci tensor may be obtained by using the above sign correspondence. (For comparison with Tabensky's formulas one should note that his Barcovariant derivative is inconsistently defined in his paper, obscures the dependence on the individual fields and is just generally fucked.)

$$R_{a\circ b} = R^{a\circ b}_d - D_d R^{a\circ b}_e - D_d E^{a\circ}_e C_{fa} + P^a_{e\circ} P^{d\circ f}_{b\circ} - P^{a\circ e}_d P^{d\circ f}_{b\circ}$$

conventionally and suggestively define:

$$K_{a\circ b} = -\frac{1}{2} \nabla_a g_{b\circ} \quad TR_{K^a} = \frac{1}{8} K_a ^b K^b_a = \frac{1}{8} g^{ab} c_{ac} \nabla_d g_{a\circ} \nabla^d g_{b\circ}$$

$$TR_{K} = K_a ^b K^b_a = \frac{1}{8} g^{acd} g_{a\circ} \nabla_d g_{b\circ} \nabla^d g_{b\circ}$$

$$R_{a\circ b} = 4 R_{a\circ b} - \frac{1}{2} F_{a\circ} \star F^a \star g_{b\circ} - \frac{2}{3} \nabla_a \nabla_b \star g_{b\circ} - K_{a\circ b} \nabla_c g_{b\circ}$$

$$\text{stress energy tensor}$$

$$g^{a\circ b} R_{a\circ b} = \frac{1}{8} R - \frac{1}{2} F^2 - \frac{2}{3} \nabla_a \nabla_b \star g_{b\circ}^a = TR_{K}$$

$$R = 4 R - \frac{1}{2} F^2 - 2 \nabla_a \nabla_b \star g_{b\circ}^a + TR_{K} - TR_{K^a} - R_{a\circ b}$$

$$\nabla_a (g_{b\circ}^a) R = g_{b\circ}^a (\nabla^a R_{a\circ b} + \frac{1}{2} C^{ab}_c \nabla_d g_{b\circ}^a) = \frac{1}{2} \nabla_d g_{b\circ}^a$$

**equations of motion**

$$R_{a\circ b} = 0 \quad \text{gauge}$$

$$R_{a\circ b} = 0 \quad A_b$$

$$R_{d\circ 0} = 0 \quad g_{a\circ}^b$$

These follow from independent variation of these fields in the scalar curvature density Lagrangian $L$ on the bundle. (This is why we impose unimodularity.)
FIBER CURVATURE

The fiber Ricci tensor components $R_{ab}$ are given by the formula for $R_{ab}$ with all indices Latin and $D_a D_b = D_b D_a = 0$:

$$R_{ab} = -
\begin{array}{c}
\Gamma_{ac}^e C^e_{bd} + \Gamma_{be}^e C^e_{ad} - \Gamma_{ad}^e C^e_{be} \\
\frac{1}{2} \left( \Gamma_{ad}^{e} \Gamma_{be}^{e} - \Gamma_{be}^{e} \Gamma_{ad}^{e} \right)
\end{array}
$$

$$= -\Gamma_{ae}^{e} \Gamma_{de}^{e} - \Gamma_{de}^{e} \Gamma_{ae}^{e}$$

**Remark** $\Gamma^{ab} = \frac{1}{2} C_{ab}^{e} C_{eb}^{e}$ (note $\Gamma_{ae}^{e} \Gamma_{de}^{e} = C_{ae}^{e} C_{eb}^{e}$ by the contracted Bianchi identity $C_{abe}^{e} = 0$)

$$R_{ab} = -C_{ae}^{e} C_{be}^{e} + \frac{1}{4} C_{df}^{e} C_{fb}^{e} = C_{df}^{e} C_{fb}^{e}$$

$$R = -C_{ae}^{e} C_{be}^{e} + \frac{1}{4} C_{df}^{e} C_{fb}^{e} = C_{df}^{e} C_{fb}^{e}$$

These terms vanish for unimodular group.

In Tolstov's final line of section 3, in the formula for the fiber curvature in $R_{ab}$ the vertical-vertical part of $R_{ab}$ (for $\alpha$ in his notation), he has omitted the term $-C_{ae}^{e} C_{be}^{e}$ inside the parentheses.

All indices are raised and lowered with $g^{ef}$ and $g_{ab}$ of course.

This is formula 2 of § 1 of Jenson's paper The Scalar Curvature of Left-Invariant Riemannian Metrics, except for the fact that he assumes an orthonormal frame.
$M$ = spacetime manifold  \quad G = $\text{semi-simple Lie group with}$
\quad \text{Lie algebra } g$ with basis \{e_a\}
\quad [e_i, e_j] = C_{ij}^{k} e_k$

Tabensky introduces "extended functions" as elements of $L(M) = \text{Lie algebra-valued functions on } M$, and "extended vector fields" as derivations on $L(M)$. Since all derivations of a semi-simple Lie algebra are inner (and "ad" is an isomorphism), an extended vector field is just a pair $(X, \omega)$ consisting of a vector field $X$ and $\omega \in L(M)$ acting on $\theta \in L(M)$ as follows:

$$(X, \omega) \theta = (\mathcal{D}_X + \text{ad}(\omega)) \theta$$

$\mathcal{D}_X \theta = (X + \text{ad}(\lambda)) \theta$

$\theta = \theta \circ e_0, \quad \omega = \omega \circ e_0, \quad X = X \circ \omega$

$$(X, \omega)(\theta \circ e_0) = \left(\mathcal{D}_X \theta \right)_{\circ e_0} + (\omega \circ \theta \circ e_0) e_a$$

A basis for each extended tangent space is therefore:

$$\{ \mathcal{D}_A \} \quad \{ \frac{\partial}{\partial \theta} \circ e_0 \}, \quad \text{ad}(e_0) \}$$

The extended tangent space is isomorphic to the tangent space of a principal fiber bundle over spacetime with group $G$ and his entire extended Riemannian geometry is equivalent to putting a right-invariant metric on this bundle, but he only works in a single gauge patch so the bundle approach is the only natural way to globalize his approach.
Whenever a symmetry group acts simply transitively on its orbits, one can introduce a frame on the manifold on which it is acting, that is invariant under the action of the group; imposing symmetry on a tensor field simply kills the component derivatives along the orbits. One can then easily compare the Lagrange derivative of a Lagrangian before, and after the symmetry is imposed on the Lagrangian.

Invariance choice the frame \( B\mathbf{a} = \{ dx^a, dx^b \} \) with dual frame \( B\mathbf{e} = \{ dx^a, dx^b \} \)

\[ C^a = - \delta^a_b \delta^c_c C^b e_c \]

\[ \partial_a = \partial_a - \delta^a_b C^b \]

\[ \delta_a e_b = e_a - \delta^a_b C^b \]

\[ \delta_a B^b = \delta_a B^b \]

\[ \delta_a = \delta_a \]

\[ \delta_a B^b = \frac{1}{4} \sum_{b,c} (B_{ab} B_{bc} + \delta_b B_{ac}) + \frac{1}{2} \sum_{b,c} \delta_a (B_{bc} B_{bc} + \delta_c B_{bc}) + \frac{1}{2} \sum_{b,c} \delta_c (B_{ab} B_{bc} + \delta_a B_{bc}) \]

Imposing the bundle symmetry on \( \delta \) means \( \delta_a \delta_b = 0 \) = \( \delta_{ab} \delta_{bc} = \delta_{a} \delta_{bc} \).

The scalar curvature Lagrangian form:

\[ L = -\frac{1}{2} \frac{\sum_{a,b}}{\delta a \delta b} \sum_{a,b} \frac{\sum_{c,d} (B_{ab} B_{cd} + \delta_c B_{ab})}{\delta c \delta d} \]

The components of the Lagrange derivative of \( L \) with respect to \( \delta \) in this frame are:

\[ \delta_a L = \frac{\delta a}{\delta a} \delta b \delta b + \delta c \delta d \delta e \]

\[ \delta a \delta b = \delta a \delta b \]

The general form of the symmetrical Einsten tensor density (x \( \Omega^2 \)) of the symmetric metric minus the symmetrical derivative of the symmetrical Lagrangian is easily computed:

\[ \delta a \delta b = \delta a \delta b \]

\[ \delta a \delta b = \delta a \delta b \]

\[ \delta a \delta b = \delta a \delta b \]

\[ \delta a \delta b = \delta a \delta b \]

Thus unless \( \delta a = 0 \), the Lagrange derivative of the Lagrangian with the symmetry imposed does not equal the Einstein tensor density, i.e., one does not obtain the Einstein equations by varying the Lagrangian evaluated on the symmetric field.
LAGRANGE MULTIPLIER DILEMMA

If we want to extremize $\int R g^{1/2}$ subject to the constraints $L_\lambda g_{ab} = 0$, we just add the constraints to the Lagrangian with Lagrange multipliers and vary both $g$ and the Lagrange multipliers freely:

$$I = \int \left( R g^{1/2} + \lambda g_{ab} \frac{\partial L_\lambda}{\partial g^{ab}} \right)$$

$$I' = \int \left[ -\left( g^{1/2} \frac{\partial L_\lambda}{\partial g^{ab}} + \frac{\partial \lambda}{\partial g^{1/2}} \right) g_{ab} + \lambda \frac{\partial L_\lambda}{\partial \lambda} \right] + \int \frac{\partial}{\partial \lambda} \left( \lambda g_{ab} \frac{\partial L_\lambda}{\partial g^{ab}} \right) dV.$$

Divergence integrals - forget about since integrate to $\partial C$ where $g_{ab}$ and derivatives vanish.

Now what, usually one can solve for the Lagrange multipliers. Here one seems to be at a dead end.
Dear Arthur,

Thanks for getting in touch with me again. I've dashed off answers to all of your questions but got stuck on one term in the variation of the Lagrangian. Since I've only got a week left to prepare a new talk I'm giving in Waterloo, Canada, I'll have to leave it for now and get back to it later.

- Misner Thorne and Wheeler use a different (and I think unnatural) convention:
  \[ \Gamma^A_{BC} = \omega^A (\nabla_B \epsilon_C) = \nabla^A \Gamma^A_{BC} \]
  Since \( \Gamma^A_{BC} = \frac{1}{2} C^A_{BC} \) for a symmetric connection,
  \[ \nabla^A \Gamma^A_{BC} = \Gamma^A_{BC} - C^A_{BC} \]
  which accounts for the different sign \(-\frac{1}{2} C^A_{BC}\) instead of \(\frac{1}{2} C^A_{BC}\) in the connection formula.

- On page 14 it is noted that one may rewrite \( R_{\alpha\beta} \) (page 12) as:
  \[ R_{\alpha\beta} = R_{\alpha\beta}^{\text{local}} - \frac{1}{2} F_{\alpha}^{\gamma} F_{\beta}^{\gamma} - g_{\alpha\beta} \frac{1}{2} \nabla^\gamma \nabla_\gamma g_{\gamma\delta} \]
  \[ + Tr K_{\alpha} Tr K_{\beta} - 2 Tr K_{\alpha} \]
  \[ \text{this combination occurs all over the place} \]

  now its \( g^{\mu\nu} \) instead of \( \ln g^{\mu\nu} \)
  does that make you happier?

- Gauge derivatives aren't needed for \( g^{\mu\nu} \) which is a weight one scalar density under gauge transformations only since the groups we are considering are unimodular (i.e. no difference between gauge scalars and gauge scalar densities).

- I have no idea what \( \frac{\delta}{\delta \phi} \frac{\delta}{\delta \phi} g^{\mu\nu} \) means geometrically.

I'm looking forward to hearing from you again.

Best regards,

bob.

January 11, 1979
USEFUL IDENTITIES: \( V_a V_b \ln g_{ab}^{1/2} = -\text{Tr} K_a \text{Tr} K_b + g_{ab}^{1/2} V_a V_b g_{ab}^{1/2} \)

\( \text{if} \) expand in derivatives and use: \( \text{Tr} K_a = -V_a \ln g_{ab}^{1/2} \).

**EINSTEIN TENSOR COMPONENTS**

\[ G^{ab} = R_{ab} - \frac{1}{2} R g_{ab} = 4 G_{ab} - T^{ab}_{\delta \epsilon} \]

\[ + \text{Tr} K_a \text{Tr} K_b - \text{Tr} K_a K_b - \frac{1}{2} g_{ab} \left( \text{Tr} K^2 - \text{Tr} K^2 + R \right) 
- g_{ab}^{1/2} \left( g_{bc} \delta_{\delta \epsilon} - g_{ab} g_{\delta \epsilon} \right) \]

**DEFINITIONS:**

\[ T^{ab}_{\delta \epsilon} = \frac{1}{2} \left( F_a^{\delta \epsilon} F^{ab}_{\delta \epsilon} - \frac{1}{4} F^2 g_{ab} \right) \]

\[ G_{ab} = 4 g_{ab}^{1/2} \left( g_{bc} \delta_{\delta \epsilon} - g_{ab} g_{\delta \epsilon} \right) \]

**NOTE THAT USING THE USEFUL IDENTITY ONE CAN REWRITE** \( R_{ab} \) **AND** \( R_{db} \):

\[ R_{ab} = 4 R_{ab} - \frac{1}{2} F_{a \delta} F^{\delta \epsilon}_{ab} - g_{ab}^{1/2} V_a V_b g_{ab}^{1/2} + 2 K_a K_b - K_a K_b \]

\[ R_{db} = \frac{1}{4} F_{d \epsilon} F_{b \delta} - \frac{1}{2} V_d V_b g_{db} + 2 K_{ad} K_{cb} - 2 K_{ac} K_{db} \]

\[ G_{db} = R_{db} - \frac{1}{2} R g_{db} = g_{db} \left( F_{d \epsilon} F_{b \delta} + \frac{1}{2} V_d V_b g_{db} \right) - \frac{1}{2} V_d V_b g_{db} + \frac{1}{2} \left( F_{d \epsilon} F_{b \delta} + \frac{1}{2} V_d V_b g_{db} \right) \]

\[ + 2 K_{ad} K_{cb} - K_{ad} K_{cb} - \frac{1}{2} g_{db} \left( \text{Tr} K^2 - \text{Tr} K^2 \right) \]

\[ G_{db} = \frac{1}{2} R g_{db} \]
USEFUL VARIATION FORMULAS (for any metric g_{ab})

\[
\begin{align*}
(g/2)^2 &= g^{1/2} G^{ab} g_{ab} + g^{1/2} g^{ac} K_{ac} \\
g^{1/2} g^{a2} R_{ab} &= G^{ac} \nabla_a \nabla_c g_{ab} \\
R^a_{\nu a \alpha \beta} &= (G^{a2} \nabla_a \nu - N) g_{\nu a} + \xi \nu X^a
\end{align*}
\]

VARIATION OF BUNDLE LAGRANGIAN

\[
\mathcal{L} = g^{1/2} \sqrt{g} R - g^{1/2} \sqrt{g} \mathcal{L}_{EM} - g^{1/2} g^{a2} (Tr^b K' - Tr^b K) + \text{div} (\sqrt{g} \partial^b \mathcal{F}^a)
\]

\[
\frac{\delta}{\delta g_{ab}} (g^{1/2} \mathcal{L}_{EM}) = 3g^{1/2} g^{b2} (-\frac{1}{4} g^{cd} + Tr^a) + G^{ac} \nabla_a \nabla_c g_{ab} g^{1/2}
\]

\[
\frac{\delta}{\delta g_{ab}} (g^{1/2} g^{a2} R) = -g^{1/2} g^{b2} (Tr^b K' - Tr^b K) - \frac{1}{2} g^{cd} (Tr^a K - Tr^a K)
\]

compare with previous page.

\[
\frac{\delta}{\delta g_{ab}} \mathcal{L} = -g^{1/2} g^{a2} G^{ab}
\]

\[
\frac{\delta}{\delta A_a} (-\frac{1}{2} g^{b2} F_b^2) = 3g^{1/2} \sqrt{g} (g^{1/2} F_d^2)
\]

just as in usual computation

\[
\frac{\delta}{\delta A_a} \mathcal{L} = 9g^{1/2} \sqrt{g} (g^{1/2} F_d^2) - 2 g^{b2} g^{1/2} Tr K_d K^a
\]

\[
= -2g^{1/2} g^{b2} R_d^a = -2g^{1/2} g^{b2} G_d^a = \frac{\delta}{\delta A_d} \mathcal{L}
\]

notes by Robert J. Harten
The following computation of \( \frac{\partial}{\partial \phi} \mathcal{L} = -4 g^{ab} g^{cd} G_{ab} \) is a near miss. The variation of \( T^{\mu}_{\nu} \) isn’t coming out right and I’m temporarily puzzled. Perhaps you can figure it out. I’ll get back to it when I have more time.

\[
\left( g^{ab} g^{cd} \left( \frac{1}{2} R - \frac{1}{4} \phi \Box R - \frac{1}{4} \left( \frac{\partial}{\partial \phi} F^{ab} \right) F_{ab} + \frac{1}{2} F^{ab} F_{ab} \right) \right)_{\phi} = \\
g^{ab} g^{cd} \left( \frac{1}{2} R - \frac{1}{4} \phi \Box R - \frac{1}{4} \left( \frac{\partial}{\partial \phi} F^{ab} \right) F_{ab} + \frac{1}{2} F^{ab} F_{ab} \right) g_{ab} + \\
g^{ab} g^{cd} \left( -\varepsilon^{ab} g_{ab} \right)
\]

This uses the fact that \( \left( g^{\mu \nu} \Box R \right)_{\phi} = -g^{\mu \nu} g_{ab} g^{cd} g_{ab} \).

The divergence vanishes identically by unmodularity and right invariance.

\[
\left( g^{ab} g^{cd} \left( T^{\mu}_{\nu} - T \Box R \right) \right)_{\phi} = \\
g^{ab} g^{cd} \left( \frac{1}{2} g_{ab} \left( T^{\mu}_{\nu} - T \Box R \right) \right)_{\phi} + \text{div}
\]

\[
(\mathbf{T} R)_{\phi} = \left( \frac{\partial}{\partial \phi} g^{ab} g^{cd} K_{abcd} + \Box g_{abcd} \right)_{\phi} = \\
-2 K^{\alpha \beta} K_{\alpha \beta} g^{cd} + 2 K_{\alpha \beta} g_{\alpha \beta} \frac{\partial}{\partial \phi} g^{cd} + \text{div}
\]

\[
\left( \mathbf{T} R \right)_{\phi} = \left( \frac{\partial}{\partial \phi} g^{ab} g^{cd} K_{abcd} + \Box g_{abcd} \right)_{\phi} = \\
-2 K^{\alpha \beta} K_{\alpha \beta} g^{cd} + 2 K_{\alpha \beta} g_{\alpha \beta} \frac{\partial}{\partial \phi} g^{cd} + \text{div}
\]

\[
(\mathbf{T}^{\alpha \beta} T)_{\phi} = \left( g^{ab} g^{cd} K_{abcd} g^{\alpha \beta} \right)_{\phi} = 2 \Box K_{\alpha \beta} g^{cd} + \text{div}
\]

\[
-2 K^{\alpha \beta} K_{\alpha \beta} g^{cd} + 2 K_{\alpha \beta} g_{\alpha \beta} \frac{\partial}{\partial \phi} g^{cd} + \text{div}
\]

\[
= (\Box, T^{\alpha \beta}) g^{cd} g_{\alpha \beta} = (\Box, T R) g^{cd} + 2 T^{\alpha \beta} K_{\alpha \beta} g^{cd}
\]

\[
= (\Box, T^{\alpha \beta}) g^{cd} g_{\alpha \beta} = (\Box, T R) g^{cd} + 2 T^{\alpha \beta} K_{\alpha \beta} g^{cd}
\]

\[
= (\Box, T R) g^{cd} g_{\alpha \beta} = (\Box, T R) g^{cd} + 2 T^{\alpha \beta} K_{\alpha \beta} g^{cd}
\]
If the term \((\text{Tr} K)^{cd}\) had been instead \(\text{Tr} K^{a} K^{cd}\), we would have obtained

\[
\frac{\delta L}{\delta S_{ab}} = -4g_{\alpha \beta} g_{ab} \mathcal{G}^{\alpha \beta}
\]

Because all the other terms agree, I don't understand what I've done wrong yet.