BUNDLE OF FRAMES

M a manifold. The bundle of frames over M is

$$F(M) = \bigcup_{p\in M} \{\text{frames for } TM_p\} = \{ u=(p,e_p) ; e_p \text{ a frame at } p\in M\}$$

$\pi : F(M) \to M$ given by $\pi(p,e_p) = p$ is the natural projection, and $Gl(n\mathbb{R})$ acts on $F(M)$ on the right: $u' = u\cdot a$ ie $u\cdot a = (p, e_p\cdot a)$.

> Definition. A principal bundle over M with group G consists of a total space P, an action of G on P on the right, a C^∞ map $\pi : P \to M$ onto M such that: (1) $\pi(u\cdot a) = \pi(u)$.
> (2) (local triviality): every $p\in M$ has a nbhd U and a map $t : \pi^{-1}(U) \to U\times G$
> $u \mapsto (\pi(u), \varphi(u))$ which is a diffeo & satisfies $\varphi(u\cdot a) = \varphi(u)\cdot a$.
> A section is a function $s : M \to P$ with $\pi\circ s$ = identity. We define $u\cdot a \equiv R_a(u)$.

Thus $F(M)$ is a principal bundle over M with group $Gl(n\mathbb{R})$. A section $s: U\subset M \to F(M)$ is just a moving frame or simply frame over U. If we have $a : U \to Gl(n\mathbb{R})$ we get a new section $s' = s\cdot a$, ie a change of frame.

Note that there is a one-to-one correspondence between frames over U and the above local triviality maps φ. If φ is such a map let $s_\varphi(p) = \varphi^{-1}(I)$. Conversely if s is a section let $\varphi_s : F(M) \to GL(n\mathbb{R})$ by $\varphi_s(u) = a$ if $u = s(\pi(u))\cdot a$, ie a is the matrix which relates the frame u to the frame s. Thus $F(M) = [M \underset{\bullet}{\times} GL(n\mathbb{R})]_s$, ie $F(M)$ is just the product manifold $M\times GL(n\mathbb{R})$ locally with the specific cross-product determined by some section s.

Every section s over $U\subset M$ partially determines a coordinate system on $\pi^{-1}(U)$. Let (x, U) be a coordinate patch on M. Define $y^\mu = x^\mu\circ\pi$, $y^\mu{}_\nu = a^\mu{}_\nu \circ \varphi_s$ so that $u = (p, e_p) \in F(M)$ has coordinates $(y^\mu(u), y^\mu{}_\nu(u)) \equiv (x^\mu, a^\mu{}_\nu)$ if p has coordinates x^μ and $e_p = s(p)\cdot a$, ie $a^\beta{}_\alpha$ are the components of e_α at p in the frame $s(p)$.
In particular, taking $s = \frac{\partial}{\partial x}$ (where $\frac{\partial}{\partial x}$ is the frame associated with the coordinates x so that s is the coordinate section) yields a coordinate system on $F(M)$ adapted to the coordinates of M: if $u = (p, e)$ with $e_\alpha = e^\mu{}_\alpha(p) \frac{\partial}{\partial x^\mu}\big|_p$ then

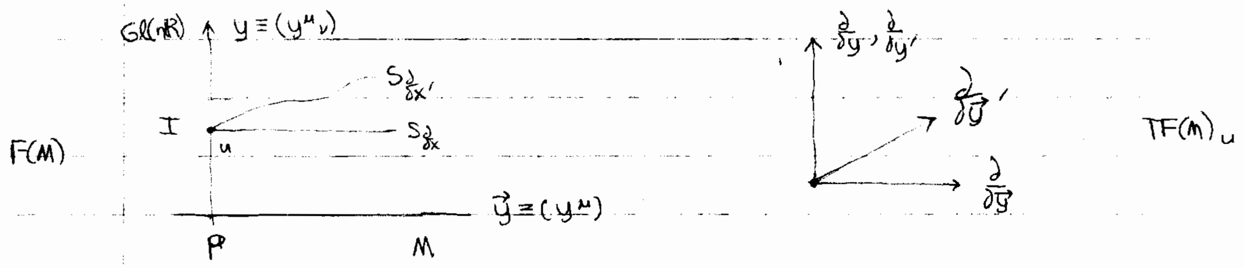
$$(y^\mu(u), y^\mu{}_\nu(u)) = (x^\mu(p), e^\mu{}_\nu(p)).$$

We shall always assume this coordinate system. Thus the coordinate frame for $TF(M)_u$ is $\left(\frac{\partial}{\partial y^\mu}\big|_u , \frac{\partial}{\partial y^\mu{}_\nu}\big|_u \right)$. Note the following:

$$\left.\begin{aligned} (\pi_* \tfrac{\partial}{\partial y^\mu}) f &= \tfrac{\partial}{\partial y^\mu} f\circ\pi = \tfrac{\partial}{\partial (x^\mu\circ\pi)} f\circ\pi = \tfrac{\partial f}{\partial x^\mu} \\ (\pi_* \tfrac{\partial}{\partial y^\mu{}_\nu}) f &= \tfrac{\partial}{\partial y^\mu{}_\nu} f\circ\pi = 0 \\ \pi^* dx^\mu &= d(x^\mu\circ\pi) = dy^\mu \end{aligned}\right\} \qquad \begin{aligned} \pi_* \tfrac{\partial}{\partial y^\mu}\Big|_u &= \tfrac{\partial}{\partial x^\mu}\Big|_{\pi(u)} \\ \pi_* \tfrac{\partial}{\partial y^\mu{}_\nu}\Big|_u &= 0 \\ \pi^* dx^\mu &= dy^\mu \end{aligned}$$

The coordinate frame reflects the fact that the tangent space $TF(M)_u$ is isomorphic to $TM_p \oplus TGl(n\mathbb{R})_a$. The subspace spanned by $\{\frac{\partial}{\partial y^\mu{}_\nu}\}$ is just the kernel of the map π_* and is called the vertical subspace V_u at u.

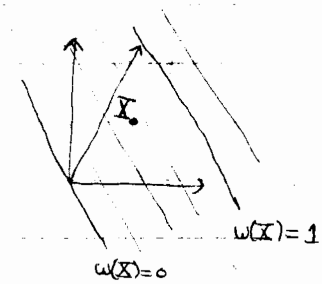
One might be tempted to call the subspace spanned by $\{\frac{\partial}{\partial y^\mu}\}$ a horizontal subspace but since the cross product $M \times GL(n\mathbb{R})$ depends on which section you take, no unique horizontal subspace is defined, but there is one for each coordinate system about the point u:



The elements of V_u, the vertical vectors, are just those vectors tangent to the fiber $\pi^{-1}(\pi(u))$ at u.

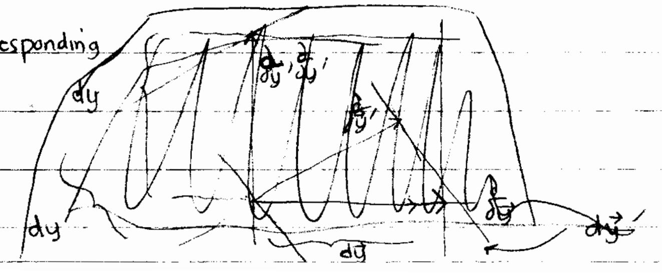
Correspondingly the only uniquely defined subspace of $TF(M)^*_u$ is the image of $TM^*_{\pi(u)}$ under the map $\pi^*: TM^*_{\pi(u)} \to TF(M)^*_u$, called the horizontal subspace H^*_u, which from above we see has the coordinate basis: $dy^\mu = \pi^* dx^\mu$. Without thinking one might think the subspace spanned by $\{dy^\mu{}_\nu\}$ was the invariantly defined (ie independent of coordinate system) subspace, but the fact that it is instead $\mathrm{span}\{dy^\mu\}$ is due to the dual nature of vectors and forms, which can be illustrated very nicely by a geometrical construction.

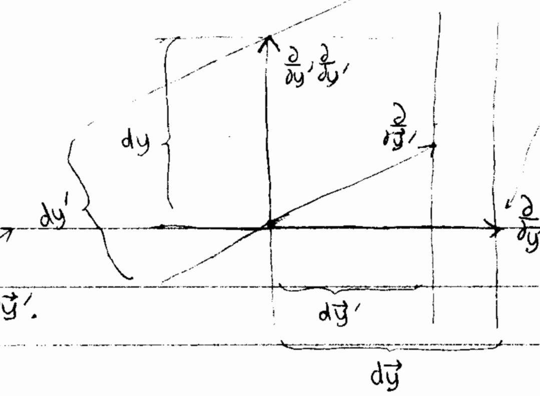
We identify a one-form ω at a point with the one-parameter family of planes $\omega(X) =$ constant ~~[illegible]~~ in the tangent space (X a vector) which are all parallel to the plane of vectors annihilated by ω ie $\omega(X) = 0$. In drawing these we will suppress all the planes except for the two $\omega(X)=0$, $\omega(X)=1$. If ω is an element of a dual basis then its dual vector lies on the second plane and all the other dual vectors lie on the first plane.



Now we shall consistently use 2-dimensional drawings to picture $TF(M)_u$, as if $\vec{y}, y$ were coordinates of one-dimensional manifolds, which is actually the case for $n=1$ in which locally M is the real line $\mathbb{R}$ and $GL(n=1,\mathbb{R})$ is just $\mathbb{R}-\{0\}$ so that $F(M) = [M \times GL(n\mathbb{R})]_{\frac{\partial}{\partial x}}$ has coordinates $\vec{y}, y$ and $\frac{\partial}{\partial \vec{y}}, \frac{\partial}{\partial y}$ span the 2-dimensional tangent space $TF(M)_u$. Thus we will picture the dual coordinate basis $(d\vec{y}, dy) \equiv (dy^\mu, dy^\mu{}_\nu)$ as one-dimensional:

Thus if we draw the dual bases corresponding to two coordinate systems:





For example

This says $d\vec y\left(\frac{\partial}{\partial \vec y}\right) = 1$

$d\vec y\left(\frac{\partial}{\partial y}\right) = 0$ since $\frac{\partial}{\partial y}$ is parallel to the planes of $d\vec y$

The dy' planes must be oblique to annihilate $\partial/\partial \vec y'$.

Thus geometrically it is clear that $d\vec y, d\vec y'$ span the same subspace since they are represented by the same family of planes.

$H_u^* \equiv \pi^*(TM^*_{\pi(u)})$ has a canonical basis $\bar\theta^\alpha(u)$ defined by $\bar\theta^\alpha(u) = \pi^*(\theta_p^\alpha)$ if θ_p is the dual frame to e_p where $u = (p, e_p)$. In other words for $X \in TF(M)_u$ we have:

$$\bar\theta^\alpha(u)(X) = \theta_p^\alpha(\pi_* X) \qquad u = (p, e_p) \qquad \theta_p(e_p) = I$$

In coordinates we have $\theta_p^\alpha = \underbrace{e_p^{-1\,\alpha}{}_\beta}_{\text{matrix inverse to}} dx^\beta$ if $e_p{}^\alpha = \underbrace{e_p{}^\alpha{}_\beta} \frac{\partial}{\partial x^\alpha}\Big|_p$, or $\theta^\alpha(x) = e^{-1\alpha}{}_\beta(x) dx^\beta$

But $[\pi^*(\theta^\alpha)]_u = \underbrace{e^{-1\alpha}{}_\beta(x\circ\pi)}_{\equiv\, y^{-1\alpha}{}_\beta(u)} dy^\beta = \boxed{y^{-1\alpha}{}_\beta\, dy^\beta \equiv \bar\theta^\alpha}$

Using matrix notation: $\boxed{\bar\theta = y^{-1} d\vec y}$

V_u also has a canonical basis. To define it we first define 2 maps.

(i) Given $X \in gl(n\mathbb{R})$ we have the curve $t \mapsto \exp tX$ in $Gl(n\mathbb{R})$. For each $u \in F(M)$ we have the curve lying in the fiber containing u: $c_u(t) = u\cdot \exp tX = R_{\exp tX}(u)$.

Then we get a map $\sigma : gl(n\mathbb{R}) \to TF(M)$ by: $\sigma(X)(u) = c_u'(0)$.

(ii) Alternatively define $\sigma_u : Gl(n\mathbb{R}) \to F(M)$ by $\sigma_u(a) = u\cdot a$. Clearly $\sigma(X)(u) = \sigma_{u*}(X)$, where ~~here X~~ on the right X is thought of as an element of $TGl(n\mathbb{R})_I$ which is identified with $gl(n\mathbb{R})$.

If $\{\hat e^\alpha{}_\beta\}$ is the standard basis of $gl(n\mathbb{R})$ define: $E^\alpha{}_\beta = \sigma(\hat e^\alpha{}_\beta)$.

In a coordinate system: $c(t) = u\cdot\exp(t\hat e^\alpha{}_\beta) \leftrightarrow y^\mu \cancel{c(t)}\; y^\mu{}_\gamma(\exp t\hat e^\alpha{}_\beta)^\gamma{}_\delta$

$$dy^\mu{}_\nu(c'(t)) = \frac{d}{dt}\Big|_{t=0} y^\mu{}_\nu \circ c(t) = y^\mu{}_\gamma \delta^\gamma_\beta \delta^\alpha_\delta = y^\mu{}_\beta \delta^\alpha_\nu$$

$$c'(0)(u) = y^\mu{}_\beta \delta^\alpha_\nu \frac{\partial}{\partial y^\mu{}_\nu}\Big|_u = \boxed{y^\mu{}_\beta \frac{\partial}{\partial y^\mu{}_\alpha}\Big|_u \equiv E^\alpha{}_\beta(u)}$$

Note that $\bar\theta^\alpha(E^\gamma{}_\beta) = 0$

The set $\sigma(gl(n\mathbb{R}))$ are called fundamental vector fields on $F(M)$

Using matrix notation $\boxed{E = \frac{\partial}{\partial y}\cdot y}$

Recall $F(M) = (M \times GL(n\mathbb{R}))_{\frac{\partial}{\partial x}}$ so that we recognize $E^\alpha{}_\beta$ as the images on the product manifold (as determined by the coordinate system) of the canonical basis of left invariant vector fields on $GL(n\mathbb{R})$. Since the right action of $GL(n\mathbb{R})$ on $F(M)$ becomes simply a right translation on the ~~copy~~ factor of $GL(n\mathbb{R})$ in the product manifold, and the left invariant vector fields generate the right translations, it is completely clear why the $E^\alpha{}_\beta$ are as they are.

In the given coordinate system we have the analogues of the other invariant fields as well

left invariant forms: $W = y^{-1} \cdot dy$ $\qquad W(E) = 1$

~~right~~ invariant vectorfields and forms: $\sigma = dy \cdot y^{-1}$ $\qquad \xi = y \cdot \frac{\partial}{\partial y}$ $\qquad \sigma(\xi) = 1$

But these are not invariantly defined. To see this we have:

COORDINATE TRANSFORMATION ASIDE.

A coordinate transformation $x^\mu = f^\mu(x') \equiv x^\mu(x')$ on M induces on $F(M)$ the transformation

$$y^\mu = f^\mu(\vec{y}\,') = y^\mu(\vec{y}\,') \quad , \quad y^\mu{}_\nu = \frac{\partial y^\mu}{\partial y'^\rho}(\vec{y}\,')\, y'^\rho{}_\nu \equiv y^\mu{}_\nu(\vec{y}\,'; y)$$

The last follows from: $e^\mu{}_\nu(x(x')) = \frac{\partial x^\mu}{\partial x'^\rho}(x')\, e'^\rho{}_\nu(x')$ and $\frac{\partial y^\mu}{\partial y'^\rho}(\vec{y}\,') = \frac{\partial x^\mu}{\partial x'^\rho}(x' \circ \pi)$.

Thus: $\frac{\partial y^\mu}{\partial y'^\rho{}_\sigma} = 0 \qquad \frac{\partial y^\mu{}_\nu}{\partial y'^\rho} = \frac{\partial^2 y^\mu}{\partial y'^\rho \partial y'^\gamma}\, y'^\gamma{}_\nu \qquad \frac{\partial y^\mu{}_\nu}{\partial y'^\rho{}_\sigma} = \frac{\partial y^\mu}{\partial y'^\rho}\, \delta^\sigma{}_\nu$

We can now see how vector fields and one-forms on $F(M)$ transform.

If $X = X^\mu \frac{\partial}{\partial y^\mu} + X^\mu{}_\nu \frac{\partial}{\partial y^\mu{}_\nu}$ and $\omega = \omega_\mu dy^\mu + \omega^\mu{}_\nu \, dy^\nu{}_\mu$ then the components transform in the following manner.

$$X^\mu = \frac{\partial y^\mu}{\partial y'^\rho} X'^\rho \qquad X^\mu{}_\nu = \frac{\partial y^\mu{}_\nu}{\partial y'^\rho} X'^\rho + \frac{\partial y^\mu{}_\nu}{\partial y'^\rho{}_\sigma} X'^\rho{}_\sigma = \left(\frac{\partial^2 y^\mu}{\partial y'^\rho \partial y'^\gamma} y'^\gamma{}_\nu\right) X'^\rho + \frac{\partial y^\mu}{\partial y'^\rho} X'^\rho{}_\nu$$

$$\omega'_\rho = \frac{\partial y^\mu}{\partial y'^\rho}\omega_\mu + \frac{\partial y^\mu{}_\nu}{\partial y'^\rho}\omega^\nu{}_\mu \qquad \omega'^\sigma{}_\rho = \cancel{\frac{\partial y^\mu}{\partial y'^\rho{}_\sigma}\omega_\mu} + \frac{\partial y^\mu{}_\nu}{\partial y'^\rho{}_\sigma}\omega^\nu{}_\mu = \frac{\partial y^\mu}{\partial y'^\rho}\omega^\sigma{}_\mu$$

$E^\alpha{}_\beta = (\delta^\alpha_\nu y^\mu{}_\beta) \frac{\partial}{\partial y^\mu{}_\nu}$, ie $(E^\alpha{}_\beta)^\mu{}_\nu = \delta^\alpha_\nu y^\mu{}_\beta$. Let $(E^\alpha{}_\beta)'^\mu{}_\nu = \delta^\alpha_\nu y'^\mu{}_\beta$ then

$$(E^\alpha{}_\beta)^\mu{}_\nu = \frac{\partial y^\mu}{\partial y'^\rho}(E^\alpha{}_\beta)'^\rho{}_\nu = \delta^\alpha_\nu\left(\frac{\partial y^\mu}{\partial y'^\rho} y'^\rho{}_\beta\right) = \delta^\alpha_\nu y^\mu{}_\beta$$

which shows that

$E^\alpha{}_\beta$ is an "invariantly defined set of vectorfields". But suppose we let $(\xi^\alpha{}_\beta)'^\mu{}_\nu = y'^\alpha_\nu \delta^\mu_\beta$ which is how we would define the components of $\xi^\alpha{}_\beta$ in the primed coordinate system. Then

$$(\xi^\alpha{}_\beta)^\mu{}_\nu = \frac{\partial y^\mu}{\partial y'^\rho}(\xi^\alpha{}_\beta)'^\rho{}_\nu = \frac{\partial y^\mu}{\partial y'^\rho} y'^\alpha_\nu \delta^\rho_\beta = y'^\alpha{}_\nu \frac{\partial y^\mu}{\partial y'^\beta} = \frac{\partial y'^\alpha}{\partial y^\rho} y^\rho{}_\nu \frac{\partial y^\mu}{\partial y'^\beta} = \frac{\partial y'^\alpha}{\partial y^\rho}\left(y^\rho{}_\nu \delta^\sigma_\beta\right)\frac{\partial y^\mu}{\partial y'^\sigma}$$

These are not the components of what we have defined $\xi^\alpha{}_\beta$ to be in the original coordinate system, ie $\xi^\alpha{}_\beta$ is not invariantly defined. Similarly computations show that $W^\alpha{}_\beta, \sigma^\alpha{}_\beta$ are also coordinate dependent objects. However a computation for $\bar{\Theta}^\alpha$ would ~~show that it is~~ verify that it is invariantly defined.

Suppose we see what $X = \partial/\partial y'^\alpha$ is in the unprimed coordinate system. It has components $X'^\mu = \delta^\mu_\alpha$, $X'^\mu{}_\nu = 0$.

A straightforward computation shows that:

$$X = \frac{\partial y^\beta}{\partial y'^\alpha}\left(\frac{\partial}{\partial y^\beta} - \Gamma^\mu{}_{\beta\rho} y^\rho{}_\nu \frac{\partial}{\partial y^\mu{}_\nu}\right) \qquad \Gamma^\mu{}_{\beta\rho}(\vec{y}) = \frac{\partial y^\mu}{\partial y'^\alpha}\left[\frac{\partial^2 y'^\alpha}{\partial y^\beta \partial y^\rho} + \frac{\partial y'^\gamma \partial y'^\delta}{\partial y^\beta \partial y^\rho}(0)^\alpha{}_{\gamma\delta}\right]$$

Thus if we define a distribution of subspaces = span$\left\{\frac{\partial}{\partial y'^\alpha}\right\}$ in some coordinate system and want ~~an~~ ~~a~~ invariant definition, we must define a geometric object $\Gamma'^\alpha{}_{\beta\gamma}(\vec{y})$ on $F(M)$

depending only on the coordinates y^μ in any coordinate system which transforms like a connection and vanishes in the primed coordinate system. Then in any coordinate system the subspaces are given by: span$\{D_\alpha\}$ where $D_\alpha \equiv \frac{\partial}{\partial y^\alpha} - \Gamma^\mu_{\alpha\rho} y^\rho_\nu \frac{\partial}{\partial y^\mu{}_\nu}$ and $\Gamma^\mu_{\alpha\rho}(\vec{y})$ is defined in ~~any coordinate system by the transformation law for the components of a connection which vanishes in the fixed primed system.~~

In short $F(M)$ is begging for additional structure.

We assume M is gifted with a connection, ie in the coordinate frame $\frac{\partial}{\partial x}$ over $U \subset M$:

$\nabla_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^\beta} = \Gamma^\gamma_{\alpha\beta}(x) \frac{\partial}{\partial x^\gamma}$. $\omega^\alpha{}_\beta = \Gamma^\alpha_{\gamma\beta}(x) dx^\gamma$ defines the connection forms for the coordinate frame. In any other frame: $e = \frac{\partial}{\partial x} \cdot a$ we therefore have

$$\omega_{(e)} = a^{-1}(da + \omega a).$$

Let $c(t)$ be a curve in M with tangent $C'^\mu(t) \equiv C'^\mu(t) \frac{\partial}{\partial x^\mu}$ and $c(0) = p$.

Let $\left\{ a^\mu{}_\nu(t) \frac{\partial}{\partial x^\mu}\Big|_{c(t)} \right\} \equiv \frac{\partial}{\partial x}\Big|_{c(t)} \cdot a(t)$ be a frame which is parallel transported along $c(t)$ and is equal to $\frac{\partial}{\partial x}\Big|_p \cdot a(0)$ at p. thru the point $(p, \frac{\partial}{\partial x}\big|_p \cdot a(0))$

Define the unique horizontal lift $\bar{C}(t)$ of $C(t)$ to $F(M)$ by:

$$\bar{C}(t) = \left(c(t), \frac{\partial}{\partial x}\Big|_{c(t)} \cdot a(t) \right)$$

$\bar{C}'(t) \in TF(M)_{\bar{C}(t)}$ is called a horizontal vector. The set of tangents $\bar{C}'(t)$ of all such curves passing through $u = \bar{C}(t)$ defines a "horizontal subspace" H_u of $TF(M)_u$.

In coordinates: $\left(y^\mu \circ \bar{C}(t), y^\mu{}_\nu \circ \bar{C}(t) \right) = \left(x^\mu \circ c(t), a^\mu{}_\nu(t) \right)$

so: $$\bar{C}'(t) = \frac{dx^\mu}{dt} \frac{\partial}{\partial y^\mu}\Big|_{\bar{C}(t)} + \frac{da^\mu{}_\nu}{dt} \Big| \frac{\partial}{\partial y^\mu{}_\nu}\Big|_{\bar{C}(t)}$$

Parallel transport condition:

$$0 = C'^\alpha \nabla_\alpha \left(a^\mu{}_\nu(t) \frac{\partial}{\partial x^\mu} \right) = \Big[\underbrace{C'^\alpha \nabla_\alpha a^\mu{}_\nu}_{\equiv \frac{da^\mu{}_\nu}{dt}} + \Gamma^\mu_{\alpha\beta} C'^\alpha a^\beta{}_\nu \Big] \frac{\partial}{\partial x^\mu}$$

Also $C'^\mu(t) = \frac{dx^\mu \circ c(t)}{dt}$

Thus: $$\bar{C}'(t) = C'^\alpha(t) \left\{ \frac{\partial}{\partial y^\alpha}\Big|_u - \Gamma^\mu_{\alpha\beta}(\vec{y}) y^\beta{}_\nu \frac{\partial}{\partial y^\mu{}_\nu} \right\}_{u = \bar{C}(t)} \equiv C'^\alpha(t) D_\alpha\big|_{\bar{C}(t)}$$

$$\boxed{D_\alpha \equiv \frac{\partial}{\partial y^\alpha} - (\Gamma^\mu_{\alpha\beta} \circ \pi) y^\beta{}_\nu \frac{\partial}{\partial y^\mu{}_\nu}}$$

These vector fields at $TF(M)_u$ are a basis of the horizontal subspace H_u since all tangents at u to horizontal lifts of curves through $\pi(u)$ are of the form $T^\alpha D_\alpha(u)$ and these vectors are obviously linearly independent. *The horizontal subspaces are simply the tangents to all curves representing parallel transported frames along the projection of the curve in M.*

But the D_α are not coordinate independent because the index α "transforms." It should be clear from the above computation (or another computation will back me up) that using the transformation law for the connection coefficients, we have:

$D_{\alpha'} \equiv$ transform of D_α defined in coordinates $(\vec{y}, y)$ $= \dfrac{\partial y^\beta}{\partial y^{\alpha'}} D_\beta$.

To get a coordinate independent basis all we need do is contract the index α on something which transforms oppositely; so define $E_\alpha \equiv y^\beta{}_\alpha D_\beta$

$$\boxed{E_\alpha \equiv y^\beta{}_\alpha \left(\frac{\partial}{\partial y^\beta} - \Gamma^\mu{}_{\beta\gamma} y^\gamma{}_\nu \frac{\partial}{\partial y^\mu{}_\nu} \right)}$$

Note $\pi_* E_\alpha(u) = y^\beta{}_\alpha(u) \dfrac{\partial}{\partial x^\beta}\Big|_{\pi(u)} = e_{\pi(u)}$ if $u = (\pi(u), e_{\pi(u)})$.

E_α is the unique horizonal vector at $u = (p, e)$ which projects down to the α^{th} vector of the frame e at p.

Notice that the vertical part of D_α is just $-\Gamma^\gamma{}_{\alpha\beta}(\vec{y})\, \xi^\beta{}_\gamma$ where $\xi^\beta{}_\gamma$ are the (coordinate dependent) right invariant vector fields on the copy of $GL(n,\mathbb{R})$ which is the factor in the product manifold $F(M) = M \times GL(n,\mathbb{R})$ determined by the coordinate system. A right action of $GL(n,\mathbb{R})$ on $F(M)$ by a is just a right translation on this copy of $GL(n,\mathbb{R})$ which doesn't affect the coordinates y^μ (ie $u' = R_a(u) = u \cdot a$ is given by $y^{\mu\prime} = y^\mu$, $y'^\mu{}_\nu = y^\mu{}_\gamma a^\gamma{}_\nu$) so D_α is right invariant under the action of $GL(n,\mathbb{R})$, ie *the horizontal subspaces are right invariant on the fibers under the action of $GL(n,\mathbb{R})$ on $F(M)$.*

$R_{a*} H_u = H_{u \cdot a}$ $\rightarrow$

If this handwaving is too much, a trivial computation backs it up: $\dfrac{\partial y'^\mu}{\partial y^\nu} = \delta^\mu_\nu$ $\quad \dfrac{\partial y'^\mu{}_\nu}{\partial y^\alpha{}_\beta} = \delta^\mu_\alpha a^\beta{}_\nu$

$$(R_{a*} D_\alpha)(u') \text{“}=\text{”} \left(\delta^\mu_\alpha, -\Gamma^\sigma{}_{\alpha\gamma} y^\gamma{}_\rho \frac{\partial y'^\mu{}_\nu}{\partial y^\sigma{}_\rho}\right) = \left(\delta^\mu_\alpha, -\Gamma^\mu{}_{\alpha\gamma} y^\gamma{}_\rho a^\rho{}_\nu\right) = \left(\delta^\mu_\alpha, -\Gamma^\mu{}_{\alpha\gamma} y'^\gamma{}_\nu\right) = D_\alpha(u')$$

Thus $R_{a*} E_\alpha(u') = y^\beta{}_\alpha D_\beta = a^{-1\gamma}{}_\alpha a^\delta{}_\gamma y^\beta{}_\delta D_\beta = D_\beta \cdot y'^\beta{}_\gamma a^{-1\gamma}{}_a = E_\gamma(u') \cdot a^{-1\gamma}{}_a$

$$\boxed{R_{a*} \vec{E} = \vec{E} \cdot a^{-1}}$$

Recall $\bar{\theta}^\alpha = y^{-1\alpha}{}_\gamma dy^\gamma$. So $\bar{\theta}^\alpha(E_\beta) = y^{-1\alpha}{}_\gamma y^\delta{}_\beta dy^\gamma\left(\frac{\partial}{\partial y^\delta}\right) = \delta^\alpha_\beta$

Also $(R_a^* \bar{\theta}^\alpha)_u = y^{-1\alpha}{}_\gamma(u \cdot a)\, dy^\gamma = (y \cdot a)^{-1\alpha}{}_\gamma dy^\gamma = a^{-1\alpha}{}_\beta y^{-1\beta}{}_\gamma dy^\gamma = a^{-1\alpha}{}_\beta \bar{\theta}^\beta|_u$

$$\boxed{R_a^* \bar{\theta} = a^{-1} \bar{\theta}}$$

So we have defined horizontal fields E_α, vertical fields $E^\alpha{}_\beta$ spanning H_u and V_u respectively, and one-forms $\bar{\theta}^\alpha$ satisfying $\bar{\theta}^\alpha(E_\beta) = \delta^\alpha{}_\beta$, $\bar{\theta}^\alpha(E^\gamma{}_\beta) = 0$. We therefore complete $\bar{\theta}^\alpha$ uniquely to a dual basis to $(E_\alpha, E^\alpha{}_\beta)$ by defining:

$$\boxed{\bar{\omega}^\alpha{}_\beta = y^{-1\alpha}{}_\gamma(dy^\gamma{}_\beta + \Gamma^\gamma{}_{\delta\rho}\, y^\rho{}_\beta\, dy^\delta) \qquad \bar{\omega} = y^{-1}(dy + \omega y)}$$

where by ω we mean $(\Gamma^\gamma{}_{\delta\rho} \circ \pi) dy^\delta = \pi^* \omega^\gamma{}_\rho$, $\omega^\gamma{}_\rho$ the connection forms for the coordinate system (x, U).

By inspection $\bar{\omega}^\alpha{}_\beta(E_\gamma) = 0$, $\bar{\omega}^\alpha{}_\beta(E^\gamma{}_\delta) = \delta^\alpha_\delta \delta^\gamma_\beta$ which is what we require. We can thus write the identity operator on $TF(M)_u$ as: $I = E_\alpha \otimes \bar{\theta}^\alpha + E^\alpha{}_\beta \otimes \bar{\omega}^\beta{}_\alpha \equiv I_H + I_V$ where each term is the identity operator on the corresponding subspace. Thus we can decompose any vector $X \in TF(M)_u$ as: $X = \bar{\theta}^\alpha(X) E_\alpha + \bar{\omega}^\beta{}_\alpha(X) E^\alpha{}_\beta \equiv hX + vX$.

$hX \equiv \bar{\theta}^\alpha(X) E_\alpha \equiv$ *horizontal component of X* [Note $\bar{\theta}^\alpha \otimes E_\alpha = dy^\mu \otimes D_\mu$ in a coord system]

$vX \equiv \bar{\omega}^\alpha{}_\beta(X) E^\beta{}_\alpha \equiv$ *vertical component of X.*

Note that $\bar{\omega}^\alpha{}_\beta$ is a basis for the one-forms which annihilate H_u. So one can specify the distribution of horizontal subspaces by either specifying a local basis E_γ of n vector fields or a set of n^2 1-forms which annihilate this basis.

Notice that the "vertical" part of $\bar{\omega}^\alpha{}_\beta$ (that part involving $dy^\alpha{}_\beta$) is just $W^\alpha{}_\beta$ the "left invariant" vector fields for this coordinate system. It is this part which insures the equality $\bar{\omega}^\alpha{}_\beta(E^\gamma{}_\delta) = \delta^\alpha_\delta \delta^\gamma_\beta$ since $W^\alpha{}_\beta(E^\gamma{}_\delta) = \delta^\alpha_\delta \delta^\gamma_\beta$ and $dy^\rho(E^\gamma{}_\delta) = 0$. (The extra part involving the connection insures that $\bar{\omega}^\alpha{}_\beta(E_\gamma) = 0$.)

Recall that we defined $E^\alpha{}_\beta = \sigma(\hat{e}^\alpha{}_\beta)$. The dual nature of $\bar{\omega}^\alpha{}_\beta$ says that for $X = X^\beta{}_\alpha \hat{e}^\alpha{}_\beta \in gl(n\mathbb{R})$:

$$\boxed{\bar{\omega}^\alpha{}_\beta(\sigma(X)) = X^\alpha{}_\beta}$$

and this just reflects the fact that in a coordinate system, the only part of $\bar{\omega}$ which is important when it is evaluated on ~~the~~ $\sigma(X)$ is just the canonical basis of left invariant forms for that coordinate system.

Now $W = y^{-1}\sigma y$, where σ is the matrix of right invariant one-forms for the coordinate system. hence $R_a{}^* W_{u} = (y \cdot a)^{-1} (R_a{}^*\sigma)_{u \cdot a} (y \cdot a) = a^{-1} \cdot y^{-1} \sigma_u \cdot y \cdot a = a^{-1} W \cdot a$ which shows how the first part of $\bar{\omega}$ transforms under a right action of $GL(n\mathbb{R})$. [we have used the fact $R_a{}^*(f\alpha) = f^* R_a{}^*\alpha = (f \circ R_a) R_a{}^*\alpha$]. The second part of $\bar{\omega}$ is $y^{-1}\omega y$:

$R_a{}^*(y^{-1}\omega y) = a^{-1}(y^{-1}\omega y)\, a$ since $\omega = \omega_\mu(\vec{y}) d\vec{y}^\mu$ is invariant under the right action of $GL(n\mathbb{R})$.

Hence $\boxed{R_a{}^* \bar{\omega} = a^{-1}\bar{\omega} a = ad(a^{-1})\bar{\omega}}$

This statement is equivalent to the right invariance of the horizontal subspaces H_u under the action of H_u, since $\bar{\omega}^\alpha{}_\beta$ determines H_u as the subspace of $TF(M)_u$ which it annihilates.

Thus a connection determines a unique decomposition: $TF(M)_u = H_u \oplus V_u$ where H_u is isomorphic to $TM_{\pi(u)}$ (V_u isomorphic to $GL(n\mathbb{R})$) and right invariant under the action of $GL(n\mathbb{R})$ on $F(M)$. Conversely suppose we are given a right invariant distribution of ~~n-dimensional~~ subspaces H_u such that $TF(M)_u = H_u \oplus V_u$. Then pick a coordinate system $(\vec{y}, y)$ and consider the point $(\vec{y}, I)$. Since $H_u \cap V_u = 0$ we can choose a basis $D_\alpha = \frac{\partial}{\partial y^\alpha} - \Gamma_\alpha{}^\mu{}_\nu(\vec{y}) \frac{\partial}{\partial y^\mu{}_\nu}$ of $H_{(\vec{y}, I)}$. But the right invariant fields equal to D_α at $(\vec{y}, I)$ are just: $D_\alpha = \frac{\partial}{\partial y^\alpha} - \Gamma_{\alpha}{}^{\mu}{}_{\rho} y^\rho{}_\nu \frac{\partial}{\partial y^\mu{}_\nu}$

Under coordinate transformations on $F(M)$, the $\Gamma_\alpha{}^\mu{}_\rho(\vec{y})$ transform correctly so $\Gamma^\alpha_{\beta\gamma}(X)$ define a connection on M.

This defines the coefficients $\Gamma_\alpha{}^\mu{}_\nu(\vec{y})$

Now suppose s is a section over $U \subset M$: $x^\mu \mapsto (y^\mu(x), y^\mu{}_\nu(x)) \equiv (x^\mu, y_0{}^\mu{}_\nu(x))$.

Recall $\bar{\theta}(u) = \pi^* \theta_{\pi(u)}$ where $\theta_{\pi(u)}$ is the dual frame to $e_{\pi(u)}$ and $u = (\pi(u), e_{\pi(u)})$.

Thus $s^*(\bar{\theta}(s(p))) = s^*(\pi^* \theta_p)_{s(p)} = (\pi \circ s)^* \theta_p = \theta_p$ where θ_p is the dual frame to $s(p)$.

$$\boxed{\text{" } s^*\bar{\theta} = \text{dual frame to } s \text{ "}}$$

Suppose α is a one-form on $F(M)$: $\alpha = \alpha_\mu dy^\mu + \alpha^\mu{}_\nu dy^\nu{}_\mu$

$$(s^*\alpha)_p = \alpha_\nu(s(p)) \underbrace{\frac{dy^\nu}{\partial x^\mu}}_{\delta^\nu_\mu} dx^\mu|_p + \underbrace{\frac{\partial y^\nu{}_\rho}{\partial x^\mu}}_{\frac{\partial y_0{}^\nu{}_\rho}{\partial x^\mu}} \sigma^\rho{}_\nu \, dx^\mu|_p = \left[\alpha_\mu(s(p)) + \frac{\partial y_0{}^\nu{}_\rho}{\partial x^\mu} \alpha^\rho{}_\nu(s(p))\right] dx^\mu|_p$$

For $\alpha_\mu = y^{-1\alpha}{}_\mu$, $\alpha^\mu{}_\nu = 0$ we get $s^*\theta^\alpha = y_0^{-1\alpha}{}_\mu(x)\, dx^\mu$ = dual frame to frame $y_0{}^\alpha{}_\alpha(x) \frac{\partial}{\partial x^\alpha}\Big|_p$ verifying the above computation.

Dumb The easiest way to compute s^* is just by substituting the map $y^\mu = x^\mu$, $y^\mu{}_\nu = y_0{}^\mu{}_\nu(x)$ into the forms:

$$\begin{cases} s^*\bar{\theta}^\alpha = y_0^{-1\alpha}{}_\nu dx^\nu = \text{dual frame to frame } y_0{}^\nu{}_\alpha \frac{\partial}{\partial x^\nu} \\ s^*\bar{\omega} = y_0^{-1}(dy_0 + \omega y_0) = \text{connection forms for the frame } \frac{\partial}{\partial x} \cdot y_0 \end{cases}$$

In other words if $s: U \to F(M)$ is a section, and $a: U \to GL(n\mathbb{R})$ then $s \cdot a$ is another section and $\bar{\omega}$ satisfies
$$\boxed{(s \cdot a)^* \bar{\omega} = a^{-1}(da + s^*\bar{\omega} \cdot a)}$$

LIFTING

The connection has determined a unique decomposition $TF(M)_u = H_u \oplus V_u$ with H_u isomorphic to $TM_{\pi(u)}$. (As well as $TF(M)_u^* = H_u^* \oplus V_u^*$ with H_u^* isomorphic to $TM^*_{\pi(u)}$ and dual to H_u).
In fact it gave us a nice basis E_α of H_u and dual basis $\bar\Theta_\alpha$ of H_u^*: $\bar\Theta^\alpha(E_\beta) = \delta^\alpha{}_\beta$
Recall $\pi_* E_\alpha(p, e_p) = e_p$, $\pi^* \Theta_p = \bar\Theta(p, e_p)$ where $\Theta_p(e_p) = I$

Given any tensor field S on M we can lift it up to $F(M)$ in an invariant way using this isomorphism. Namely define $\bar S(p, e_p) = \bar S^{\alpha\cdots}{}_{\beta\cdots}(p,e)\, E_\alpha \otimes \cdots \otimes \bar\Theta^\beta \otimes \cdots$

where $\bar S^{\alpha\cdots}{}_{\beta\cdots}(p, \mathbf{e}) = S(p)(\theta^\alpha, \ldots, e_\beta, \ldots) =$ components of $S(p)$ in frame e at p.

If S is a tensor-valued form we can either lift the collection of forms up to $F(M)$ or lift the tensor-valued form up. For example if $\Omega(p) = \Theta_\alpha \otimes \theta^\beta \otimes \Omega^\alpha{}_\alpha(p)$ then we have both the set of forms $\bar\Omega^\beta{}_\alpha(p,e)$ and the tensor-valued form $\bar\Omega(p,e) = E_\alpha \otimes \bar\Theta^\beta \otimes \bar\Omega^\beta{}_\alpha(p, E)$ and again we will tend to ignore the distinction.

If $S = S^{\alpha\cdots}{}_{\beta\cdots}(x)\, \frac{\partial}{\partial x^\alpha} \otimes \cdots \otimes dx^\beta \otimes \cdots$, then the above recipe says:

$$\bar S = \underbrace{(y^{-1\alpha}{}_\gamma \cdots S^{\gamma\cdots}{}_{\delta\cdots}(\vec y)\, y^\delta{}_\beta \cdots)}_{\bar S^{\alpha\cdots}{}_{\beta\cdots}(\vec y, y)} E_\alpha \otimes \cdots \otimes \bar\Theta^\beta \cdots = \underbrace{S^{\alpha\cdots}{}_{\beta\cdots}(\vec y)}_{S^{\alpha\cdots}{}_{\beta\cdots} \circ \pi} D_\alpha \otimes \cdots \otimes dy^\beta \otimes \cdots$$

For vector fields, if $X = X^\alpha(x) \frac{\partial}{\partial x^\alpha}$ then $\bar X = X^\alpha(\vec y) D_\alpha = y^{-1\alpha}{}_\beta X^\beta(\vec y)\, E_\alpha = \bar X^\alpha(\vec y, y) E_\alpha$

We shall call all such fields horizontal lifts. Note that these lifts are determined on $F(M)$ by their transformation law, ie $y^{-1\alpha}{}_\gamma \cdots S^\gamma{}_\delta \cdots (\vec y)\, y^\delta{}_\beta \cdots$ is just the transform of the components of S from the coordinate frame to the frame $\frac{\partial}{\partial x} \cdot y$. This suggests that we can lift other geometric objects up to $F(M)$. For instance if $\omega^\alpha{}_\beta(x)$ are the one forms for the coordinate frame, then ~~the~~ in the frame $\frac{\partial}{\partial x} \cdot y$ the new forms are $y^{-1}(dy + \omega y)$

so define on $F(M)$: $\bar\omega = y^{-1}(dy + \omega y)$ where here "ω" $= \pi^*\omega$

the set of forms or $\bar\omega = E_\alpha \otimes \bar\Theta^\beta \otimes \bar\omega^\alpha{}_\beta$ the tensor-valued form.

Thus $\bar\omega$ is what we have already defined for the given connection ~~and in fact~~ ie $\bar\omega$ is just the natural horizontal lift of the connection on M. It should be clear how to lift other geometric objects up to $F(M)$ using the law of transformation under change of frame.

Because $S^{\alpha}{}_{\beta\cdots}(\vec y)$, D_α, dy^β are right invariant, if we have a tensor-valued form $\bar S$ it is right invariant but then the set of forms $\bar S^{\alpha\cdots}{}_{\beta\cdots}$ are not and clearly satisfy:

$$R_a{}^* \bar S^{\alpha\cdots}{}_{\beta\cdots} = a^{-1\alpha}{}_\gamma \cdots \bar S^{\gamma\cdots}{}_{\delta\cdots}\, a^\delta{}_\beta \cdots$$

Also since ω is right invariant: $R_a{}^*\bar\omega = (ya)^{-1}(d(ya) + \omega y a) = a^{-1}\bar\omega a$

These just reflect that on a fiber, $\bar S^\alpha{}_{\beta\cdots}(p,e)$ are just the components of $\bar S$ wrt the frame e and if $e' = e\cdot a$, $a \in GL(n,\mathbb{R})$ is another point on the fiber, they must transform correctly. And if S is a section, and $\bar S^{\alpha\dots}_{\beta\dots}$ the "components" of a tensor-valued form then it should be clear that $S^*\bar S^{\alpha\dots}_{\beta}$ are just "components" of the tensor-valued form on M in the frame S.

So we can lift up the curvature and torsion forms to F(M) in our coordinate system:

$\bar\Theta = y^{-1}(\pi^*\Theta)$ $\qquad \pi^*\Theta^\alpha = \Theta^\alpha_{\beta\gamma}(\vec y)\, dy^\beta\wedge dy^\gamma$, etc.

$\bar\Omega = y^{-1}(\pi^*\Omega)\, y$ $\qquad$ (Note we have already lifted $\bar\theta = y^{-1}(\pi^*\theta)$ where $\theta^\alpha = dx^\alpha$)

Alternatively we could simply define $\bar\Theta, \bar\Omega$ on F(M) using the analogues of the Cartan Structural equations:

CARTAN STRUCTURAL EQUATIONS ON F(M)

$$\boxed{\begin{aligned}\bar\Theta &= d\bar\theta + \bar\omega\wedge\bar\theta\\ \bar\Omega &= d\bar\omega + \bar\omega\wedge\bar\omega\end{aligned}}$$

Note that since $S^*\bar\theta = \theta$, $S^*\bar\omega = \omega$, where θ, ω are the dual frame and connection forms with respect to the frame S, and since S^* commutes with d & is distributive wrt $\wedge$, we have immediately that

$$\left.\begin{aligned} S^*\bar\Theta &= d\theta + \omega\wedge\theta = \Theta\\ S^*\bar\Omega &= d\omega + \omega\wedge\omega = \Omega\end{aligned}\right\}$$ torsion and curvature forms for frame S.

We can also lift D to F(M) by defining it using $\bar\omega$ in the same way as on M:

Thus $D\bar\theta \equiv d\bar\theta + \bar\omega\wedge\bar\theta = \bar\Theta$

$$\left.\begin{aligned} D\bar\Theta &= d\bar\Theta + \bar\omega\wedge\bar\Theta\\ D\bar\Omega &= d\bar\Omega + \bar\omega\wedge\bar\Omega - \bar\Omega\wedge\bar\omega\end{aligned}\right\}$$

Two comments. First the Bianchi Identities follow immediately from the Cartan Structural equations. Second $S^*(D\bar S) = D(S^*S) = DS$ on M, so

(1) $$\boxed{\begin{aligned} D\bar\Theta &= \bar\Omega\wedge\bar\Theta\\ D\bar\Omega &= 0\end{aligned}}$$ Bianchi identities on F(M)

(2) $\Rightarrow$ $$\left.\begin{aligned} S^*(D\bar\Theta) &= D\Theta \qquad\qquad S^*(D\bar\Omega) = D\Omega = 0\\ S^*(\bar\Omega\wedge\bar\Theta) &= \Omega\wedge\Theta\end{aligned}\right\}$$ Bianchi identities on M in the frame S.

We should show that $\bar\Theta, \bar\Omega$ are the lifts as we have defined them.

$\bar\theta = y^{-1}d\vec y \qquad d\bar\theta = -y^{-1}dy\,y^{-1}\wedge d\vec y = -W\wedge\bar\theta =$

$\bar\omega\wedge\bar\theta = W\wedge\bar\theta + y^{-1}\omega y\wedge y^{-1}\theta$

$\bar\Theta = d\bar\theta + \bar\omega\wedge\bar\theta = y^{-1}(\omega\wedge\theta) = y^{-1}\Theta$. $\qquad$ ($\theta = d\vec y = \pi^* dx$, $\Theta = \pi^*\Theta$, ... we've been a little careless about notation)

$d\bar\omega = -W\wedge\bar\omega + y^{-1}d\omega\, y + y^{-1}\omega y\wedge W$

$\bar\omega\wedge\bar\omega = (W + y^{-1}\omega y)\wedge(W + y^{-1}\omega y) = W\wedge W + y^{-1}\omega\wedge dy + y^{-1}dy\wedge y^{-1}\omega y + y^{-1}\omega\wedge\omega y$

$\bar\Omega = d\bar\omega + \bar\omega\wedge\bar\omega = y^{-1}(d\omega + \omega\wedge\omega)y = y^{-1}\Omega y$ $\qquad$ (ie $\Omega = \pi^*\Omega$ etc)

Lie Brackets of ~~fundamental~~ basic vector fields

Compute $[D_\alpha, D_\beta] = \left[\frac{\partial}{\partial y^\alpha} - \Gamma_\alpha{}^\sigma{}_\delta y^\delta{}_\gamma \frac{\partial}{\partial y^\sigma{}_\gamma}, \frac{\partial}{\partial y^\beta} - \Gamma_\beta{}^m{}_e y^e{}_n \frac{\partial}{\partial y^m{}_n} \right]$

$$= -\frac{\partial \Gamma^m_{\beta e}}{\partial y^\alpha} y^e{}_n \frac{\partial}{\partial y^m{}_n} + \frac{\partial}{\partial y^\beta}(\Gamma^\sigma_{\alpha\delta}) y^\delta{}_\delta \frac{\partial}{\partial y^\sigma{}_\gamma} + \Gamma^\sigma_{\alpha\delta} y^\delta{}_\gamma \Gamma^m_{\beta\sigma} \frac{\partial}{\partial y^m{}_\gamma} - \Gamma^m_{\beta e} y^e{}_n \Gamma^\sigma_{\alpha m} \frac{\partial}{\partial y^\sigma{}_n}$$

$$= -2\,\partial_{[\alpha}\Gamma_{\beta]}{}^\delta{}_\gamma y^\gamma{}_\sigma \frac{\partial}{\partial y^\delta{}_\sigma} - \Gamma^\delta_{\alpha\rho}\Gamma^\rho_{\beta\gamma} y^\gamma{}_\sigma \frac{\partial}{\partial y^\delta{}_\sigma} + \Gamma^\delta_{\beta\rho}\Gamma_{\alpha\gamma}{}^\rho y^\gamma{}_\sigma \frac{\partial}{\partial y^\delta{}_\sigma}$$

$$= -\left\{ 2\,\partial_{[\alpha}\Gamma_{\beta]}{}^\delta{}_\gamma + 2\Gamma^\delta_{[\alpha|\rho|}\Gamma^\rho_{\beta]\gamma} \right\} \xi^\gamma{}_\delta$$

$$\boxed{[D_\alpha, D_\beta] = -\underbrace{R^\delta{}_{\gamma\alpha\beta}}\, \xi^\gamma{}_\delta = \text{vertical field}}$$

$R^\delta{}_{\gamma\alpha\beta}(\vec{y}) = R^\delta{}_{\gamma\alpha\beta} \circ \pi$, $R^\delta{}_{\gamma\alpha\beta}$ Riemann in (x, U) system.

Since $\xi^\gamma{}_\delta$ is vertical and D_α horizontal, the distribution of horizontal subspaces will be involutive iff the Riemann tensor vanishes identically. In this case the integral submanifolds will be generated by the horizontal lifts of all curves in M. Suppose p and q are two points of M joined by the curve $C: [0, t_0] \to M$. Let e_p be a fixed frame at p and let e_q be its parallel transport along C to q. Then (p, e_p) and (q, e_q) lie on the same integral submanifold generated by the horizontal distribution as does the lift of C to $F(M)$ joining these two points. By definition any curve connecting p and q will lift to a curve joining (p, e_p) and (q, e_q) which just says that parallel transport is ~~in~~dependent of the curve, ie "completely integrable". Thus we can define a covariant constant frame by parallel transporting a fixed frame at a fixed point all over M, from which we can get a local cartesian coordinate system out.

Recall that the lift of a vector field $X = X^\alpha \frac{\partial}{\partial x^\alpha}$ is just $\bar{X} = X^\alpha D_\alpha$ where by X^α here we really mean $X^\alpha \circ \pi$. Compute:

$$[\bar{X}, \bar{Y}] = [X^\alpha D_\alpha, Y^\alpha D_\beta] = X^\alpha Y^\alpha [D_\alpha, D_\beta] + X^\alpha D_\alpha(Y^\beta) D_\beta - Y^\beta D_\beta(X^\alpha) D_\alpha$$

$$= \underbrace{-R^\alpha{}_\beta(X, Y)\, \xi^\beta{}_\alpha}_{\text{vertical}} + \underbrace{[X, Y]^\alpha D_\alpha}_{\text{horizontal}}.$$

Thus $\boxed{\overline{[X, Y]} = h[\bar{X}, \bar{Y}]}$

Compute: $E_\alpha = y^\gamma{}_\alpha D_\gamma$

$$[E_\alpha, E_\beta] = y^\gamma{}_\alpha y^\delta{}_\beta \underbrace{[D_\gamma, D_\delta]}_{-R^d{}_{c\gamma\delta}\xi^c{}_d} + y^\gamma{}_\alpha \underbrace{D_\gamma(y^\delta{}_\beta)}_{-\Gamma^\delta_{\gamma e} y^e{}_\beta} D_\delta - y^\delta{}_\beta \underbrace{D_\delta(y^\gamma{}_\alpha)}_{-\Gamma^\gamma_{\delta e} y^e{}_\alpha} D_\gamma$$

$$-(y^\gamma{}_\alpha \Gamma^\sigma_{\gamma\delta} y^\delta{}_\beta - y^\gamma{}_\alpha \Gamma^\sigma_{\delta\gamma} y^\delta{}_\beta) D_\sigma$$

$$= -y^\gamma{}_\alpha \Big\{ \underbrace{R^d{}_{c\gamma\delta}\xi^c{}_d}_{\text{vertical}} + 2\underbrace{\Gamma^\sigma_{[\gamma\delta]}}_{T^\sigma{}_{\gamma\delta}} D_\sigma \Big\} y^\delta{}_\beta$$

horizontal.

or using matrix notation:

$$[\vec{E},\vec{E}] = -\mathrm{TR}[\Omega(\vec{E},\vec{E})\cdot\xi] - \vec{D}\cdot\Theta(\vec{E},\vec{E})$$

where by Ω^α_β we mean $\frac{1}{2}R^\alpha{}_{\beta\gamma\delta}\,dy^\gamma dy^\delta \equiv \pi^*\Omega^\alpha_\beta$, Ω^α_β the curvature forms with respect to the coordinates (x,U) and similarly for Θ.

We have used the fact that $dy^\gamma(E_\alpha) = dy^\gamma(y^\beta_\alpha \frac{\partial}{\partial y^\beta}) = y^\gamma_\alpha$.

But recall $\xi \equiv y\cdot\frac{\partial}{\partial y}$ and $E \equiv \frac{\partial}{\partial y}\cdot y$ so $\xi = y\cdot E\cdot y^{-1}$

$$\mathrm{TR}[\Omega\cdot\xi] = \mathrm{TR}[\Omega\cdot y\cdot E\cdot y^{-1}] = \mathrm{TR}[y^{-1}\Omega\cdot y\cdot E] = \mathrm{TR}(\bar{\Omega}\cdot E)$$

Also $\vec{D} = \vec{E}\cdot y^{-1}$, $\Theta = y\cdot\bar{\Theta}$ $\quad \vec{D}\cdot\Theta = \vec{E}\cdot\bar{\Theta}$.

$$\boxed{[\vec{E},\vec{E}] = -\mathrm{TR}(\bar{\Omega}(\vec{E},\vec{E})\cdot E) - \vec{E}\cdot\bar{\Theta}(\vec{E},\vec{E})}$$

$$\boxed{\begin{aligned}\bar{\theta}([\vec{E},\vec{E}]) &= -\bar{\Theta}(\vec{E},\vec{E})\\ \bar{\omega}([\vec{E},\vec{E}]) &= -\bar{\Omega}(\vec{E},\vec{E})\end{aligned}}$$

factor of 2 from Nomizu who uses different wedge normalization.

The E_α are a basis for what Nomizu, Spivak call ~~fundamental~~ basic vector fields. $(\{c^\alpha E_\alpha;\ c^\alpha\in\mathbb{R}\})$

Lifting ∇

It is not surprising that we can lift up ∇ to $F(M)$ as an operator on horizontal lift fields.

If $\bar{S} = \bar{S}^{\alpha\cdots}_{\beta\cdots}\, E_\alpha\otimes\cdots\otimes\bar{\theta}^\beta$ is the lift of S down on M, define

$$\bar{\nabla}\bar{S} \equiv E_\rho(\bar{S}^{\alpha\cdots}_{\beta\cdots})\, E_\alpha\otimes\cdots\otimes\bar{\theta}^\beta\otimes\cdots\otimes\bar{\theta}^\rho$$

A straightforward computation shows that: $\bar{\nabla}\bar{S} = \overline{\nabla S}$.

$$D_\rho(y^\delta{}_\beta) = -\Gamma^\mu_{\rho\sigma}y^\sigma{}_\nu\frac{\partial y^\delta{}_\beta}{\partial y^\mu{}_\nu} = -\Gamma^\mu_{\rho\sigma}y^\sigma{}_\nu\delta^\delta_\mu\delta^\nu_\beta = -\Gamma^\delta_{\rho\sigma}y^\sigma{}_\beta$$

$$D_\rho(y^{-1\alpha}{}_\gamma) = -y^{-1\alpha}{}_m D_\rho(y^m{}_n)y^{-1n}{}_\gamma = -y^{-1\alpha}{}_m(-\Gamma^m_{\rho\sigma}y^\sigma{}_n)y^{-1n}{}_\gamma = -y^{-1\alpha}{}_m\Gamma^m{}_{\sigma\gamma}$$

$$D_\rho(\bar{S}^{\alpha\cdots}_{\beta\cdots}) = D_\rho(y^{-1\alpha}{}_\gamma\cdots S^{\gamma\cdots}_{\delta\cdots}y^\delta{}_\beta\cdots) = y^{-1\alpha}{}_\gamma\cdots\frac{\partial S^{\gamma\cdots}_{\delta\cdots}}{\partial y^\rho}y^\delta{}_\beta\cdots$$

$$+\, y^{-1\alpha}{}_\gamma\cdots\Gamma^\gamma_{\rho\sigma}S^{\sigma}{}_{\delta\cdots}+\cdots - y^{-1\alpha}{}_\gamma\cdots S^\gamma{}_{\sigma\cdots}\Gamma^\sigma_{\rho\delta}y^\delta{}_\beta - \cdots$$

$$= y^{-1\alpha\cdots}{}_\gamma\{(\nabla_\rho S^{\gamma\cdots}_{\delta\cdots})\circ\pi\}y^\delta{}_\beta\cdots$$

$$E_\sigma(\bar{S}^{\alpha\cdots}_{\beta\cdots}) = y^{-1\alpha}{}_\gamma\cdots\{(\nabla_\rho S^{\gamma\cdots}_{\delta\cdots})\circ\pi\}y^\delta{}_\beta\cdots y^\rho{}_\sigma = [\overline{\nabla S}]^{\alpha\cdots}_{\beta\cdots\sigma}$$

$$\equiv [\bar{\nabla}\bar{S}]^{\alpha\cdots}_{\beta\cdots\sigma}$$

which is what we wished to show.

Then for any X, $\bar{\nabla}_X\bar{S} \equiv \{E_\sigma(\bar{S}^{\alpha\cdots}_{\beta\cdots})E_\alpha\otimes\cdots\otimes\bar{\theta}^\beta\otimes\cdots\otimes\bar{\theta}^\sigma\}\lrcorner X = \bar{\theta}^\sigma(X)E_\sigma(\bar{S}^{\alpha}_{\beta\cdots})E_\alpha\otimes\cdots\otimes\bar{\theta}^\beta\otimes\cdots$

ie $\bar{\nabla}_X \equiv \bar{\nabla}_{hX}$.

If we define: $\begin{cases} \bar{R}(\bar{X},\bar{Y})\bar{Z} = \{\bar{\nabla}_{\bar{X}}\bar{\nabla}_{\bar{Y}} - \bar{\nabla}_{\bar{Y}}\bar{\nabla}_{\bar{X}} - \bar{\nabla}_{[\bar{X},\bar{Y}]}\}\bar{Z} \\ \bar{T}(XY) = \bar{\nabla}_{\bar{X}}\bar{Y} - \bar{\nabla}_{\bar{Y}}\bar{X} - h[\bar{X},\bar{Y}] \end{cases}$

then from the property $\bar{\nabla}\bar{S} = \overline{\nabla S}$ it is clear that $\bar{R}, \bar{T}$ are the lifts of the Riemann and torsion tensors on M and so $\begin{cases} \bar{R} = E_\alpha \otimes \bar{\theta}^\beta \otimes \bar{\Omega}^\alpha{}_\beta \\ \bar{T} = E_\alpha \otimes \bar{\Theta}^\alpha \end{cases}$

We need to include $h[\bar{X},\bar{Y}]$ in the torsion because $[\bar{X},\bar{Y}]$ has a nonhorizontal part. This doesnt enter in the definition of $\bar{R}$ since $\bar{\nabla}_X$ only depends on hX.

We can define a map $u: H_u \otimes \cdots \otimes H_u^* \longrightarrow TM_{\pi(u)} \otimes \cdots \otimes TM^*_{\pi(u)}$

$$S^{\alpha\cdots}_{\beta\cdots}(u)\, E_\alpha(u) \otimes \cdots \otimes \bar{\theta}^\beta(u) \longmapsto S^{\alpha\cdots}_{\beta\cdots}(u)\, e_\alpha(\pi(u)) \otimes \cdots \otimes \theta^\beta(\pi(u))$$

if $u = (\pi(u), e_{\pi(u)})$ and $\theta(\pi(u))\,(e_{\pi(u)}) = I$.

Then given a connection on $F(M)$ we can define covariant differentiation on M by

$$\nabla_X S = u((\bar{\nabla}_{\bar{X}} \bar{S})(u)) = \text{independent of } u.$$

$$\text{or } \nabla_X S = u(\bar{X}(\bar{S}^{\alpha\cdots}_{\beta\cdots})(u)) \qquad (\bar{S}^{\alpha}_{\beta\cdots} = \text{components of } S \text{ in frame } e_{\pi(u)})$$

Suppose C is a geodesic on M: $\nabla_{C'(t)} C'(t) = 0$ (we can always differentiate fields defined on a curve along its tangential direction)

Then $\bar{\nabla}_{\bar{C}'(t)} \bar{C}'(t) = \overline{\nabla_{C'(t)} C'(t)} = 0$

ie $\bar{C}(t)$ is a "geodesic" on $F(M)$ and just corresponds to a parallel transported frame along a geodesic in M. Alternatively $0 = \bar{C}'(\bar{C}'^\alpha) = \bar{C}'(\bar{\theta}(\bar{C}'))$

$$0 = \bar{C}'(\bar{\theta}(\bar{C}'))$$

Nomizu and Spivak define another ~~def~~ derivation operator on $F(M)$ by

$$DS(X_1 \ldots X_{p+1}) = dS(hX_1 \ldots hX_{p+1})$$

for a tensor-valued p-form (interpreted as a set of p-forms). We show that this coincides with our covariant exterior derivative when applied to horizontal lift tensor-valued p-forms.

Preliminary details

(i) $dy^{-1} = -y^{-1}dy\,y^{-1} = -\bar{W}y^{-1}$ (iii) $\bar{W}(vX) = \bar{\omega}(X)$:

(ii) $dy = y\,y^{-1}dy = y\bar{W}$ $\qquad \bar{W}^\gamma{}_\delta(vX) = \bar{W}^\gamma{}_\delta(\bar{\omega}^\beta{}_\alpha(X)\, E^\alpha{}_\beta) = \bar{\omega}^\beta{}_\alpha(X)\,\delta^\alpha_\beta\delta^\gamma_\delta = \bar{\omega}^\gamma{}_\delta(X)$

Rather than go through an involved proof for a general tensor-valued form $S^\alpha{}_{\beta\cdots}$ we show the equivalence for $(\frac{1}{1})$ and $(\frac{1}{0})$ tensor-valued 2-forms, from which it will be clear that the equivalence holds for all tensor-valued forms.

$\bar{\Omega} = y^{-1}\Omega y$ where by Ω we mean $\pi^*\Omega^\alpha_\beta = \Omega^\alpha{}_{\beta\gamma\delta}(\vec{y})\,dy^\gamma \wedge dy^\delta/2$

$\bar{\Theta} = y^{-1}\Theta$ ditto for Θ.

$\bar{\theta} = y^{-1}\theta$ ditto for θ

The unbarred forms involve only dy^α and not $dy^\alpha{}_\beta$ and so annihilate vertical ~~forms~~ vectors. So $d\bar{S}$ for any such $\bar{S}$ will only involve one factor of $dy^\alpha{}_\beta$ and will thus accept only one vertical argument.

(1) $\bar{\Omega} = y^{-1}\Omega y \qquad d\bar{\Omega} = -\bar{W}\wedge\bar{\Omega} + \bar{\Omega}\wedge\bar{W} + y^{-1}d\Omega y$

zero when one vertical argument present.
only $\bar{W}$ accepts a vertical argument & $\bar{W}(vX) = \bar{\omega}(X)$
$\bar{\Omega}(hX) = \bar{\Omega}(X)$ since annihilates vertical part.

$D\bar{\Omega}(X_1X_2X_3) \equiv d\bar{\Omega}(hX_1hX_2hX_3)$

$d\bar{\Omega}(X_1X_2X_3) = d\bar{\Omega}(hX_1+vX_1, hX_2+vX_2, hX_3+vX_3)$

$= d\bar{\Omega}(hX_1hX_2hX_3) + \{d\bar{\Omega}(vX_1hX_2hX_3) + d\bar{\Omega}(hX_1vX_2hX_3) + d\bar{\Omega}(hX_1hX_2vX_3)\} + \{$ terms with more than one vertical argument, therefore zero $\}$

But $d\bar{\Omega}(vX_1\, hX_2\, hX_3) = -\bar{\omega}(X_1)\bar{\Omega}(X_2X_3) + \bar{\Omega}(X_2,X_3)\bar{\omega}(X_1)$

$d\bar{\Omega}(hX_1\, vX_2\, hX_3) = -\bar{\omega}(X_2)\bar{\Omega}(X_3X_1) + \bar{\Omega}(X_3X_1)\bar{\omega}(X_2)$

$d\bar{\Omega}(hX_1\, hX_2\, vX_3) = -\bar{\omega}(X_3)\bar{\Omega}(X_1X_2) + \bar{\Omega}(X_1X_2)\bar{\omega}(X_3)$

$\Sigma(\,) = -\bar{\omega}\wedge\bar{\Omega}\,(X_1X_2X_3) + \bar{\Omega}\wedge\bar{\omega}\,(X_1X_2X_3)$

$D\bar{\Omega}(X_1X_2X_3) = d\bar{\Omega}(X_1X_2X_3) - \Sigma(\,) \quad\rightarrow\quad D\bar{\Omega} = d\bar{\Omega} + \bar{\omega}\wedge\bar{\Omega} - \bar{\Omega}\wedge\bar{\omega}$

(2) $\bar{\Theta} = y^{-1}\Theta \qquad d\bar{\Theta} = -\bar{W}\wedge\bar{\Theta} + y^{-1}d\Theta$

zero when one vertical argument present.
$\bar{\Theta}(hX) = \bar{\Theta}(X)$ since annihilates vertical part.
only $\bar{W}$ accepts a vertical argument & $\bar{W}(vX) = \bar{\omega}(X)$.

$D\bar{\Theta}(X_1X_2X_3) = d\bar{\Theta}(hX_1hX_2hX_3)$.

$d\bar{\Theta}(X_1X_2X_3) = d\bar{\Theta}(hX_1+vX_1, hX_2+vX_2, hX_3+vX_3)$

$= d\bar{\Theta}(hX_1hX_2hX_3) + \{d\bar{\Theta}(vX_1hX_2hX_3) + d\bar{\Theta}(hX_1vX_2hX_3) + d\bar{\Theta}(hX_1hX_2vX_3)\} + \{\text{zero}\}$

But $d\bar{\Theta}(vX_1, hX_2\, hX_3) = -\bar{\omega}(X_1)\bar{\Theta}(X_2X_3)$

$d\bar{\Theta}(hX_1\, vX_2\, hX_3) = -\bar{\omega}(X_2)\bar{\Theta}(X_3X_1)$

$d\bar{\Theta}(hX_1\, hX_2\, vX_3) = -\bar{\omega}(X_3)\bar{\Theta}(X_1X_2)$

$\Sigma(\,) = -\bar{\omega}\wedge\bar{\Theta}\,(X_1X_2X_3)$

$D\bar{\Theta}(X_1X_2X_3) = d\bar{\Theta}(hX_1hX_2hX_3) = d\bar{\Theta}(X_1X_2X_3) - \Sigma(\,) \quad\rightarrow\quad D\bar{\Theta} = d\bar{\Theta} + \bar{\omega}\wedge\bar{\Theta}$

A little effort would produce a general argument applying to all tensor-valued forms but it should be clear what is happening.

This D is also applied to $\bar{\omega}$ which is not a tensor-valued form. But we shall evaluate it.

$\bar{\omega} = y^{-1}(dy + \omega y) \qquad d\bar{\omega} = -\bar{W}\wedge\bar{\omega} + y^{-1}d\omega y + y^{-1}\omega y\wedge\bar{W} = \underbrace{-\bar{W}\wedge\bar{\omega} + \bar{\omega}\wedge\bar{W} + y^{-1}d\omega y}_{\text{usual terms as above for } \Omega} \underbrace{-\bar{W}\wedge\bar{W}}_{\text{extra term because of "non-tensor" property of } \bar{\omega}}$

$d\bar{\omega}(X_1X_2) = d\bar{\omega}(hX_1+vX_1, hX_2+vX_2)$

$= d\bar{\omega}(hX_1hX_2) + d\bar{\omega}(vX_1, hX_2) + d\bar{\omega}(hX_1vX_2) + d\bar{\omega}(vX_1vX_2)$

$D\bar{\omega}(X_1X_2) = d\bar{\omega}(hX_1hX_2) = d\bar{\omega}(X_1X_2) - \underbrace{d\bar{\omega}(vX_1\,hX_2) - d\bar{\omega}(hX_1\,vX_2)}_{\text{lead to: } (d\bar{\omega} + \underbrace{\bar{\omega}\wedge\bar{\omega} - \bar{\omega}\wedge\bar{\omega}}_{0})(X_1X_2) \text{ as above.}} - \underbrace{d\bar{\omega}(vX_1vX_2)}_{\bar{\omega}\wedge\bar{\omega}\,(X_1X_2)}$

$$D\bar{\omega} = d\bar{\omega} + \bar{\omega}\wedge\bar{\omega} = \bar{\Omega}$$

This is how Nomizu, Spivak define $\bar{\Omega}$; also they define $\bar{\Theta} = D\bar{\theta}$.