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On Fermi's resolution of the "4/3 problem" in the classical theory of the electron

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Abstract

We discuss the solution proposed by Fermi to the so called "4/3 problem" in the classical theory of the electron, a problem which puzzled the physics community for many decades before and after his contribution to the discussion. Unfortunately his early resolution of the problem in 1922–1923 published in three articles in Italian and German journals went largely unnoticed, and even recent texts devoted to classical electron theory still do not present his argument or acknowledge the actual content of the article. Although another way to resolve the problem of defining the mass of the unaccelerated electron through an integral over its Coloumb field was independently discovered much later by others, including Kwal in 1949 and Rohrlich in 1960, Fermi's argument was completely different, analyzing the equations of motion of a rigid but accelerated electron model with vanishing bare mass to evaluate the inertial mass as the coefficient of the acceleration in the resulting force law after isolating the self-forces, correcting Lorentz's calculation of 1906. However, both ideas share the common thread of relying on time slices in the rest frame of the electron, and in fact by agreeing on the proper definition of the total momentum on a given rest frame time slice, its time derivative can be re-expressed in terms of a Gauss's law application which leads to the starting point spatial integral for Fermi's argument. Although many cite Fermi's article, until now apparently no one has explicitly reconciled the two approaches.

Introduction

The simplest classical model of the electron consists of a static spherically symmetric distribution of electric charge e over the surface of a rigid sphere of radius r_0, as measured by an observer at rest with respect to the sphere. This model was first developed by Abraham [1], Lorentz [2] and Poincaré [3], based entirely on Maxwell's theory of electromagnetism. For an unaccelerated electron, the rest frame integral of the local energy density of the Coulomb field over the exterior of the electron sphere representing the total energy $e^2/(2r_0)$ stored in that field, where e is the charge of the electron, is then assumed to be responsible for the entire rest inertial mass of the electron $m_e c^2$ (i.e., the so-called "bare mass" is zero), while the Poynting vector representing the local density of momentum in the field is zero, resulting in zero total momentum. Equating this rest mass to the observed mass of the electron then determines the radius r_0 of this model. Neglecting the factor of 2 in this relation defines what is known as the classical radius of the electron $r_e = e^2/(m_e c^2)$.

The factor of 2 in the energy formula is a geometric factor which is replaced by 5/3 if the model of the electron is a constant charge density solid sphere rather than a constant density spherical surface charge distribution and one also considers the contribution to the electromagnetic field energy inside the sphere (zero in the surface distribution case): $1/2 + 1/10 = 3/5$. Dropping these factors gives the formula for the radius r_e that pure dimensional analysis would lead to. It should be remarked that in either case this energy is equal to the work needed to assemble the charge configuration by slowly bringing the charges in from spatial infinity.

The problem is that repeating these energy and momentum density integrals in a frame in relative motion over the exterior of the electron charge distribution in that frame, even at a nonrelativistic relative velocity v of the electron in that frame, leads to a total momentum of the field which is 4/3 times the inertial mass m_e times the electron velocity v rather than simply the product of the latter two quantities (at relativistic velocity the appropriate Lorentz gamma factor γ appears). This became known as the 4/3 problem. The computed value of the energy and momentum in the Coulomb field of the electron obtained by integration of the local energy and momentum densities is simply not Lorentz invariant.

This led incorrectly to the conclusion that part of the electron's self-energy must be of non-electromagnetic nature. Furthermore, the nonrelativistic theory was also unsatisfactory, since the various parts of the charged sphere must repel one another according to Coulomb's law, giving rise to an unstable electron. An ingenious solution was suggested by Poincaré: the Coulomb repulsion, which is responsible for the instability of the electron, can be compensated for by non-electromagnetic cohesive forces ("stresses"), a kind of negative pressure. Although such forces make the electron stable and bring the theory into agreement with the special theory of relativity, they must be postulated ad hoc, and hence this was not a very satisfactory resolution, essentially sweeping the problem into another one, that of these unexplained stresses. In fact by implying that non-electromagnetic forces were needed to reconcile the theory with special relativity, it only confused the issue.

Using a completely different approach the problem of restoring Lorentz invariance without introducing Poincaré stresses was solved by Fermi [4]. He considered a spherically symmetric distribution of accelerated charges (with zero bare mass) held in a rigid configuration by some external force and applied the Lagrangian variational principle to compute the time rate of change of the

4-momentum in the force law. The problem is that rigid motion does not allow one to fix the variations at the beginning and ending time hypersurface while allowing arbitrary variations of the paths in between, which is necessary for being able to conclude the Lagrangian equations of motion follow from the variation. Imposing a symmetry like rigid motion on the world lines of the electron charge elements therefore requires some care in order not to invalidate the resulting Lagrangian equations. Indeed fixing the variations as usual between two slices of the same inertial time function is incompatible with special relativity as Fermi noted. In fact if the electron is accelerated as Fermi assumes, it must be changing its shape due to the time-varying Lorentz contraction which occurs within the laboratory frame. Thus imposing rigidity at successive laboratory times is doomed to fail, and this is exactly what produces the incorrect factor of 4/3.

In his discussion Fermi compared the results obtained by using two different kinds of variations of the world lines of the elements of charge in the electron: the incorrect one, in which the variation is performed between two hyperplanes of constant inertial coordinate time and the correct one, where the hyperplanes are instead chosen to be orthogonal to the particle 4-velocity, a covariant condition which does not rely on any particular choice of reference frame. The latter method is the one that agrees with the relativity principle and is compatible with the rigidity conditions of the system as formulated by Born [5]: the shape of the system is the same on any of the orthogonal hypersurfaces cutting the world tube it describes in spacetime. Operationally, a congruence of world lines is said to be Born-rigid if it has vanishing expansion. The former method is clearly not Lorentz invariant, and hence suspect from the very beginning. Fermi then showed that such a covariant application of the Lagrangian principle leads to an appropriate modification of the force so that the unwanted factor of 4/3 is eliminated, as expected. Therefore, there is no relation between the need for cohesive forces and the factor 4/3. Moreover, Fermi showed that the incorrect variation leads to the usual starting point for separating the self-force from the external force reproduced in the current leading texts devoted to this issue, an argument which none of the authors of those texts appears to have seen.

Although Wilson [6] discussed this problem from a different point of view in 1936 with no citations, the relation of his Poynting vector discussion to the subsequent approaches is not straightforward. These latter approaches provide an alternative resolution of the somewhat different problem of defining the total energy and momentum of the field configuration of the classical electron model. In 1949 Kwal [7] showed that Abraham's original integral definition of the electromagnetic energy and momentum does not lead to an electromagnetic 4-momentum endowed with the correct transformation properties, but that a slight modification of the definition does so. Even later Rohrlich [8] in 1960 came to the same conclusion without being aware of previous work. They both explained that the correct result can only be obtained from the usual special relativistic integrals over a hypersurface of constant inertial time if that hypersurface represents a time slice in the rest frame of the electron. The classical electron model has continued to intrigue people ever since, see for example, Feynman [9], Teitelboim [10, 11], Boyer [12], Rohrlich [13], Campos and Jimenéz [14], Cohen and Mustafa [15], Comay [16], Moylan [17], Kolbenstvedt [18], Rohrlich [19], and de Leon [20] (see also [21]). At least three entire books are devoted to the topic of the classical theory of the electron, those by Rohrlich [22], Yaghjian [23], and Spohn [24], and the model is described in detail by Jackson [26], the universally accepted reference textbook on classical electrodynamics (see also Chapter 8 of Anderson [27]). Some interesting historical details may be found in the recent article of Janssen and Mecklenburg [25]. None of these references seem to take into account Fermi's argument nor connect it to that of Kwal and Rohrlich even though most of them cite Fermi's original article.

An important element of this discussion is the conserved nature of the integrals of the local densities of energy and momentum associated with the divergence-free stress-energy tensor of the sourcefree electromagnetic field when integrated over an entire spacelike hyperplane of Minkowski spacetime due to Gauss's law. The appendix shows how the usual reasoning fails to apply to the case in which a world tube is excluded from the integral, leading to an internal boundary that must be taken into account in Gauss's law, and explains exactly how the incorrect factor of $4/3$ arises from the usual integrals due to this internal boundary. On the other hand, when such an internal boundary enclosing sources is not present, the same analysis with intersecting spacelike hyperplane boundaries shows why the integrals lead to the same "conserved" 4-momentum on any spacelike hyperplane, a situation that is never discussed in the usual textbook discussions of this topic. Finally, by extending the integrals over the entire spacelike hyperplanes so that one must take into account the nonzero divergence of the electromagnetic stress-energy tensor, one can tie together this discussion with that of Fermi.

First the Abraham-Lorentz calculation for the unaccelerated electron is reproduced to set the stage for Fermi's completely different approach, which is then discussed at length to clarify the assumptions that are made about the motion of the individual charge elements in the electron distribution. Finally the relatively simple fix of the Abraham-Lorentz integration by Kwal and Rohrlich is explained. Then the appendix shows how nonstandard applications of Gauss's law are required to handle the complications of the world tube of the electron charge distribution and to tie together the Kwal-Rohrlich integral definition with the original Fermi discussion.

The Abraham-Lorentz paradox

According to the earliest classical electron theory proposed by Abraham [1] and Lorentz [2] the electron can be modeled as a spherically symmetric distribution of electric charge e over the surface of a rigid sphere of radius r_0 in its rest frame. The energy and momentum of the electron itself is then equated to the energy and momentum of the electromagnetic field in the exterior of that sphere, which can be evaluated by suitably integrating the normal components of the stress-energy tensor of the electromagnetic field over a spacelike hyperplane representing a moment of time in an inertial reference frame. Letting $c = 1$ in this section, the stress-energy tensor

$$T_{\rm em}^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) \tag{1}$$

has the following explicit components in an inertial system of Cartesian coordinates $(x^\mu) = (t = x^0, x^1, x^2, x^3)$ associated with an inertial reference frame in Minkowski spacetime with signature $(-+++)$ following the conventions of Misner, Thorne and Wheeler [28], which is sign-reversed compared to Rohrlich's definition in his Eq. (4–114)[22]

$$\begin{aligned} T_{\rm em}^{00} &= \frac{1}{8\pi}(E^2 + B^2) = U_{\rm em}\,, \\ T_{\rm em}^{0i} &= \frac{1}{8\pi}(E \times B)^i = S^i\,, \\ T_{\rm em}^{ij} &= \frac{1}{8\pi}[-E^iE^j - B^iB^j + \frac{1}{2}g^{ij}(E^2 + B^2)]\,, \end{aligned} \tag{2}$$

where $U_{\rm em}$ and S are the electromagnetic energy density and the Poynting vector respectively, and of course E and B are the usual electric and magnetic fields observed in the associated reference frame. This stress-energy tensor is divergence-free in the absence of charge and current sources, namely vanishing 4-current $J^\mu = 0$. In inertial coordinates this condition is

$$T_{\rm em}^{\mu\nu}{}_{,\nu} = -F^{\mu}{}_{\nu}J^{\nu} = 0\,, \tag{3}$$

as shown by Exercise 3.18 of Misner, Thorne and Wheeler [28].

If the electron sphere is at rest at the origin of a corresponding system of spherical coordinates (t, r, θ, ϕ) with metric

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \tag{4}$$

and 4-velocity $U = \partial_t$ characterizing the rest frame K, then the exterior field is

$$F = -\frac{e}{r^2}dt \wedge dr\,, \qquad E = \frac{e}{r^2}\,\partial_r\,, \qquad B = 0 \quad (r > r_0)\,. \tag{5}$$

The nonvanishing orthonormal components ("overhatted" indices) of its stress-energy tensor in the normalized spherical frame are then

$$T_{\rm em}^{00} = -T_{\rm em}^{rr} = T_{\rm em}^{\hat\theta\hat\theta} = T_{\rm em}^{\hat\phi\hat\phi} = \frac{1}{8\pi}E^2 = \frac{e^2}{8\pi r^4} = U_{\rm em}\,, \tag{6}$$

while $S = 0$. Its divergence is zero in the exterior of the electron sphere.

In the Abraham-Lorentz theory the electromagnetic energy and momentum of the electron are given by the following expressions in inertial Cartesian coordinates

$$W = \int_\Sigma T_{\rm em}^{00}dV = \int_\Sigma U_{\rm em}dV\,, \qquad p^k = \int_\Sigma T_{\rm em}^{0k}dV = \int_\Sigma S^k dV\,, \tag{7}$$

where the integration is carried out at a fixed inertial time t over the whole region Σ outside the electron sphere, i.e., $r > r_0$, and dV denotes the spatial 3-volume element. For the Coulomb field of the electron in its rest frame K this gives

$$W = \frac{e^2}{2r_0}\,, \qquad p = 0\,. \tag{8}$$

These are time-independent because of the time-independence of the electric field in this frame. The energy W is assigned to be the self-energy $m_{\rm em}$ of the electron due to the surrounding Coulomb field

$$W = m_{\rm em}\,. \tag{9}$$

Note that the tracefree condition $T^{00} = T^{11} + T^{22} + T^{33}$ in the Cartesian inertial coordinates when integrated over the same region yields the condition

$$\int_\Sigma T^{00}_{\rm em} dV = \int_\Sigma \left(T^{11}_{\rm em} + T^{22}_{\rm em} + T^{33}_{\rm em}\right) dV\,, \tag{10}$$

but by spherical symmetry each of the terms on the right hand side has the same value

$$\int_\Sigma T^{11}_{\rm em} dV = \int_\Sigma T^{22}_{\rm em} dV = \int_\Sigma T^{33}_{\rm em} dV = \frac{1}{3}\int_\Sigma T^{00}_{\rm em} dV\,. \tag{11}$$

Consider a second inertial system K' with inertial Cartesian coordinates (t', x'^1, x'^2, x'^3) moving with uniform velocity $-v$ along the x^1-axis with respect to the rest system K, so that the charged sphere modeling the electron moves with velocity v relative to K', and let $U' = \partial/\partial t'$ the 4-velocity of the new time lines. The corresponding energy and momentum are obtained from those at rest by the Lorentz transformation $x'^\mu = L^\mu{}_\nu x^\nu$, namely

$$t' = \gamma(t + vx^1)\,,\ x' = \gamma(x^1 + vt)\,,\ \gamma = (1 - v^2)^{-1/2}\,. \tag{12}$$

The rest frame 4-vector components $(W, 0, 0, 0)$ are transformed by the same transformation to (W', p'^1, p'^2, p'^3), whose nonzero values are

$$W' = \gamma W = \gamma m_{\rm em}\,, \qquad p'^1 = \gamma W v = \gamma m_{\rm em} v\,. \tag{13}$$

On the other hand, by Lorentz-transforming the electromagnetic energy-momentum tensor but maintaining the definitions (17) with the integration performed over the new time coordinate hyperplanes, Abraham and Lorentz found

$$W' = \int_{\Sigma'} T'^{00}_{\rm em} dV' = \gamma\left(1 + \frac{v^2}{3}\right) m_{\rm em}\,, \qquad p'^1 = \int_{\Sigma'} T'^{0k}_{\rm em} dV' = \frac{4}{3}\gamma m_{\rm em} v\,, \tag{14}$$

which do not agree with the previous expressions even in the nonrelativistic limit $\nu^2 \ll 1$, $\gamma \to 1$. The famous unwanted 4/3 factor multiplies the inertial mass factor in constructing the momentum from the velocity and gamma factor.

To rederive their result, for motion along the x^1-axis as assumed, the stress-energy tensor transforms as follows

$$\begin{aligned} T'^{00}_{\rm em} &= \gamma^2 [T^{00}_{\rm em} + 2vT^{01}_{\rm em} + v^2 T^{11}_{\rm em}]\,, \\ T'^{01}_{\rm em} &= \gamma^2 [T^{01}_{\rm em} + v(T^{00}_{\rm em} + T^{11}_{\rm em}) + v^2 T^{01}_{\rm em}]\,, \end{aligned} \tag{15}$$

which simplifies since $T^{01}_{\rm em} = 0$. Since the 3-volume element transforms according to $dV' = dV/\gamma$ due to the Lorentz contraction of the differential dx^1, taking the symmetry property (11) into account, the definitions (14) then give

$$\begin{aligned} W' &= \int_{\Sigma'} T'^{00}_{\rm em} dV' = \gamma \left(1 + \frac{1}{3}v^2\right) \int_{\Sigma} T^{00}_{\rm em} dV = \gamma \left(1 + \frac{v^2}{3}\right) W\,, \\ p'^1 &= \int_{\Sigma'} T'^{01}_{\rm em} dV' = \gamma v \left(1 + \frac{1}{3}\right) \int_{\Sigma} T^{00}_{\rm em} dV = \frac{4}{3}\gamma v W\,. \end{aligned} \tag{16}$$

Here the integral over Σ' of the scalar integrand equals the integral over Σ (once the volume element is transformed) because its value at x'^i, re-expressed in terms of the old coordinates x^i of the same point, is independent of t, and so has the same value at the corresponding point of Σ.

In the nonrelativistic limit $|v| \ll 1$, the energy is unchanged, but the momentum has an unwanted extra factor of $4/3$. This is the famous $4/3$ problem.

These integrals define the components of a 4-vector for each inertial coordinate system, i.e.,

$$P^\mu_{\rm AL}(\Sigma) = \int_\Sigma T^{\mu 0}_{\rm em} dV = \int_\Sigma T^{\mu\nu}_{\rm em} d\Sigma_\nu\,, \tag{17}$$

where $d\Sigma_\nu = -U_\nu dV = \delta^0{}_\nu dV$ is the spacelike hyperplane volume element 4-vector, and the subscript AL stands for Abraham-Lorentz. The problem is that although this is independent of the choice of time hypersurface for a given inertial coordinate system, this leads to different 4-vectors on the time hyperplanes associated with inertial coordinate systems in relative motion, since the components of this functional in the primed and unprimed coordinate systems are not related by a Lorentz transformation as shown above due to the unwanted correction terms. The only way to guarantee that such an integral give the same 4-vector when evaluated in different inertial coordinate systems is to reformulate it so that it will indeed do so. The obvious problem is that the stress-energy tensor has two indices each of which transforms like a 4-vector under Lorentz transformations, so the only way to get a single 4-vector index is to freeze out one of them, which in tensor analysis terms means contracting it with a fixed 4-vector. Contraction with the 4-velocity of the rest frame of the electron will do the job, as shown in detail in the final section.

There were in the literature many attempts to overcome this difficulty of the Abraham-Lorentz theory after Poincaré, of whom Fermi, Kwal and Rohrlich made independent contributions which we briefly comment on below.

Fermi's contribution

The problem of restoring Lorentz invariance without introducing Poincaré stresses was solved by Fermi [4], who showed that the lack of invariance was due to an incorrect assumption about the rigid motion of the individual charge elements in the spherical model of the electron which is incompatible with special relativity. His first paper in 1921 (On the dynamics of a rigid system of electric charges in translational motion) studied a special relativistic system of electrons in rigid motion and found the 4/3 factor in its inertial mass formula, while this factor was not present in the gravitational mass he calculated using general relativity in his second paper (On the electrostatics of a homogeneous gravitational field and on the weight of electromagnetic masses) [29], referring to Levi-Civita's uniformly accelerated metric for the calculations [30]. These were all took place within five years of the birth of Einstein's theory in 1916, during which Fermi was first a high school stduent and then a university student writing his first scientific papers. During the next year 1922 in preparation for his revisit to the problem, Fermi wrote his third paper on his famous Fermi coordinate system adapted to the local rest spaces along the world line of a particle in motion (On phenomena occurring close to a world line), and then used it to resolve this 4/3 puzzle in his fourth paper (two versions published in Italian and one in German: Correction of a contradiction between electrodynamic and relativistic electromagnetic mass theories) without explicitly referring to the third paper.

His approach was to use a variational principle based on an infinitesimal thin sandwich slice of spacetime, within which one has to make certain assumptions on how the relative motion of the individual charge elements in a distribution move. The only way an electron can move rigidly so that its shape in its rest frame does not change (in order to be compatible with special relativity as discussed by Born) is if the individual world lines of the charge distribution all cut the local rest frame time slices orthogonally. This is important for establishing the boundary conditions on the otherwise arbitrary variations of the world lines at the starting and ending time slices, for the variational principle to yield equations of motion. If one takes two parallel inertial time slices and lets the variations of the world lines depend only on that time variable so the shape is rigid with respect to those time slices (but therefore not in any other family of time slices in relative motion), one finds the equations of motion with the incorrect inertial mass factor. As explained in the introduction, an accelerated electron distribution must change shape with respect to an inertial frame due to the time varying Lorentz contraction, so this cannot be right. By instead using two successive time slices of the local rest spaces of the electron charge distribution, separated by a fixed proper time of that local rest system, and only constant translations of the entire family of world lines along those time slices (i.e., a translation only depending on the proper time of the local rest system and not on position), one finds the expected result for the inertial mass appearing in the equations of motion. This latter situation corresponds to fixing the time slices in a Fermi coordinate system adapted to a world line fixed in the electron charge distribution, preferably the center of the assumed spherical symmetry.

Fermi considers a laboratory frame with inertial coordinates (t, x^1, x^2, x^3) in which the accelerated electron is momentarily at rest near the spatial origin at the initial coordinate time which we will assume for simplicity to be $t = 0$. Assuming that the Fermi coordinate system (T, X^1, X^2, X^3) is adapted to a world line in the electron charge distribution passing through the origin of these spatial coordinates at $t = 0$, its time hypersurface $T = 0$ can be chosen to coincide with $t = 0$, but after a small interval dt of laboratory time along the central world line, the Fermi time slice is instead tilted slightly to remain orthogonal to that world line as shown in Fig. 4 in the appendix. The metric in the Fermi coordinate system is

$$ds^2|_{T=0=t} = -N^2 dT^2 + \delta_{ij} dX^i\, dX^j\,, \quad N = 1 + \Gamma_i X^i/c^2\,, \tag{18}$$

where $\Gamma_i = \dot{v}^i$ are the Cartesian components of the acceleration of the central world line at $t = 0$ where $v^i = 0$. The proper time along the central Fermi coordinate time line is approximately $dT = dt$, but away from the origin there is a linear correction factor due to the lapse function in the Fermi coordinate system. The proper time interval along the normal to the initial hypersurface (measured by the increment in t) to the nearby Fermi time slice is the increment $N\, dT = (1 + \Gamma_i x^i/c^2)dt$. Thus in the integral along the time direction in the variational principle over a thin sandwich region of spacetime from $t = 0$ to $t = 0 + dt$ (his variation A), he simply had to multiply the remaining spatial integral by the arbitrary coefficient dt (ignoring the dependence of the integrand on t in the limit $dt \to 0$, but for the corresponding integral starting at $t = 0$ and ending on the nearby Fermi time slice (his variation B), he needed to include the lapse correction factor $(1 + \Gamma_i x^i/c^2)$ in the remaining spatial integral. In both cases, factoring out the constant factor dt which is nonzero, the remaining spatial integral has to have zero variation with respect to the allowed variations of the spatial coordinates of the world lines. The extra term in the integral with coefficient $\Gamma_i x^i$ provides exactly the necessary correction to produce the desired result in the inertial mass coefficient in the equations of motion for the spherical shell model of the electron.

Consider an accelerated system of electric charges in special relativity held at rest relative to each other by external forces (i.e., in rigid motion) and let the speed of light c not be unity in this section in order to compare with the original Fermi discussion. The corresponding action is given by (see e.g., Pauli [31])

$$S = S(A_\mu, x^\alpha) = \int (\mathcal{L}_{\rm em} - 2A_\mu J^\mu) d^4x\,, \qquad \mathcal{L}_{\rm em} = \frac{1}{2} F^{\alpha\beta} F_{\alpha\beta}\,, \tag{19}$$

where the integration is extended to the whole spacetime, and the 4-current $J^\mu = \rho U^\mu$ depends on the arclength parametrized world lines of the charged particles, whose unit 4-velocity is $U^\mu = dx^\mu/d\tau$.

Varying S with respect to the vector potential A_μ, fixing the world lines of the charge distribution, leads to inhomogeneous Maxwell's equations. In fact

$$\begin{aligned}
\delta S|_{x^\alpha=const.} &= \int d^4x \left(F^{\alpha\beta} \frac{\delta F_{\alpha\beta}}{\delta A_\mu} - 2J^\mu \right) \delta A_\mu \\
&= \int d^4x \left(2F^{\alpha\beta} \frac{\delta}{\delta A_\mu} (\partial_\alpha A_\beta) - 2J^\mu \right) \delta A_\mu \\
&= -2 \int d^4x \left(\partial_\alpha F^{\alpha\beta} + J^\mu \right) \delta A_\mu \,, \qquad (20)
\end{aligned}$$

that is

$$\partial_\alpha F^{\beta\alpha} = J^\mu \,. \qquad (21)$$

The variation of S with respect to the coordinates of the charge world lines where the above variations A and B are relevant can be evaluated as follows. Let de be the invariant charge element at the time slice $t = 0$ where the laboratory and electron rest frames coincide, related to the charge density by $de = \rho\, dV$, while $J = \rho U$ is the associated 4-current and $U^\alpha = dx^\alpha/dt$ is the charge element 4-velocity there (since the charge element proper time coincides with t at $t = 0$). Since variations at $A_\alpha = const.$ of the $\mathcal{L}_{\mathrm{em}}$ integral vanish, i.e.,

$$\delta \int \left(\frac{1}{2} F^{\alpha\beta} F_{\alpha\beta} \right) = 0 \,, \qquad (22)$$

we find

$$\begin{aligned}
\delta S|_{A_\alpha=const.} &= -2\delta \left(\int d^4x\, J^\mu A_\mu \right) \\
&= -2\delta \left(\int dt\, dV\, \rho U^\mu A_\mu \right) = -2\delta \left(\int de\, dt\, U^\mu A_\mu \right) \\
&= -2 \int de\, dt\, \left[(\partial_\sigma A_\mu) U^\mu + A_\mu (\partial_\sigma U^\mu) \right] \delta x^\sigma \\
&= -2 \int de\, dt\, \left[(\partial_\sigma A_\mu) U^\mu - \frac{dA_\mu}{d\tau} \delta^\mu_\sigma \right] \delta x^\sigma \\
&= -2 \int de\, dt\, \left[U^\mu (\partial_\sigma A_\mu - \partial_\mu A_\sigma) \right] \delta x^\sigma \\
&= -2 \int de\, dt\, F_{\sigma\mu} U^\mu \delta x^\sigma \,, \qquad (23)
\end{aligned}$$

that is

$$-\frac{1}{2} \delta S = \int de\, d\tau E(U)_\mu \delta x^\mu \,, \qquad (24)$$

since

$$E(U)_\mu = F_{\mu\nu} U^\nu \,, \qquad (25)$$

is the electric field as measured in the rest frame of the charge element. Therefore $\delta S = 0$ implies

$$\int de\, d\tau E(U)_\mu \delta x^\mu = 0 \,. \qquad (26)$$

Fermi pointed out that the results of such a variation depend on the way in which the variation itself is performed. The correct method is the one that agrees with the relativity principle and is compatible with the the rigidity conditions of the system.

Fermi considered then two different kinds of variations on two distinct thin sandwich regions of integration over spacetime as described above. In each case factoring out the constant factor dt arising from the time integration step, the coefficients of the arbitrary variations δx^a must vanish.

A) Laboratory frame boundary time slices, initial time hypersurface $t = 0$, final time hypersurface $t = dt$, spatial displacements functions of time only: $\delta t = 0$, $\delta x^a = \delta x^a(t)$. In this case Eq. (26) gives

$$\int de\, dt\, E_a \delta x^a = 0 \quad \rightarrow \int de\, E_a = 0\,; \tag{27}$$

B) Rest frame of the system boundary time slices, initial time hypersurface $t = 0$, final time hypersurface $t = dt\,(1 + \Gamma_i x^i/c^2)$, spatial displacements which are arbitrary constants: $\delta t = 0$, $\delta x^a = \delta x^a_0$. In this case Eq. (26) gives

$$\int de\, dt E_a \delta x^a = 0 \quad \rightarrow \int de\,(1 + \Gamma_i x^i/c^2) E_a = 0\,. \tag{28}$$

These final integral conditions at the initial time $t = 0$ may then be used at any laboratory time t at which the charge distribution is momentarily at rest. Clearly when the acceleration is identically zero $\Gamma_i = 0$, the final conditions are the same for both cases A dnd B, so one must have nonzero acceleration to see a difference in these two cases.

- Variations of type A

Consider first the system of variations A.

$$0 = \int E_a\, de\,. \tag{29}$$

Let $E_{\rm self}$ and $E_{\rm ext}$ the contributions to the total field due to the self-interaction of the system and to the external field respectively, that is $E = E_{\rm self} + E_{\rm ext}$. Eq. (29) thus becomes

$$F^a_{\rm ext} \equiv \int E^a_{\rm ext}\, de = -\int E^a_{\rm self}\, de \equiv -F^a_{\rm self}\,. \tag{30}$$

The self-force is the result of the interaction of each small element of charge of the sphere with every other element. The explicit details of the calculation involving the retarded times can be found, e.g., in Jackson's textbook [26], although the Third Edition omits the final explicit evaluation of the famous 4/3 term. The self-field can be expressed in terms of the self-potentials A and ϕ by

$$E_{\rm self} = -\nabla\phi - \frac{1}{c}\frac{\partial A}{\partial t}\,, \tag{31}$$

so that

$$F_{\text{ext}} = \int \rho \left[\nabla\phi + \frac{1}{c}\frac{\partial A}{\partial t} \right] d^3\mathbf{x}\,, \tag{32}$$

since the charge element is $de = \rho\, d^3\mathbf{x}$. We now adopt the Jackson notation that $\mathbf{x}$ is the spatial position vector in the Cartesian coordinate system and $dV = d^3\mathbf{x}$ is the spatial volume element, and let $\mathbf{v}$ and $\mathbf{a} = \dot{\mathbf{v}} = \Gamma$ be the velocity and acceleration of the charge distribution, which at the initial time t of our calculation satisfies $\mathbf{v}(t) = 0$ (all elements of the charge distribution are simultaneously at rest) and $\mathbf{a} = \mathbf{a}(t)$ (the acceleration is the same for all elements of the charge distribution at that moment), expressing the rigidity of the charge distribution.

By evaluating the potentials at the retarded time $t' = t - |\mathbf{x} - \mathbf{x}'|/c$, i.e.,

$$A = \frac{1}{c}\int \frac{[J(t',\mathbf{x}')]_{\text{ret}}}{|\mathbf{x}-\mathbf{x}'|}\, d^3\mathbf{x}'\,, \qquad \phi = \int \frac{[\rho(t',\mathbf{x}')]_{\text{ret}}}{|\mathbf{x}-\mathbf{x}'|}\, d^3\mathbf{x}'\,, \tag{33}$$

and using the rule (Taylor series expansion about the time $t' = t$)

$$[\ldots]_{\text{ret}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{|\mathbf{x}-\mathbf{x}'|}{c} \right)^n \frac{\partial^n}{\partial t^n}[\ldots]|_{t'=t}\,, \tag{34}$$

Eq. (32) becomes

$$\begin{aligned} F_{\text{ext}} \;&=\; \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,c^n} \int d^3\mathbf{x} \int d^3\mathbf{x}'\, \rho(t,\mathbf{x}) \frac{\partial^n}{\partial t^n} \Big[\rho(t,\mathbf{x}')\nabla(|\mathbf{x}-\mathbf{x}'|^{n-1}) \\ &\quad + \frac{|\mathbf{x}-\mathbf{x}'|^{n-1}}{c^2} \frac{\partial J(t,\mathbf{x}')}{\partial t} \Big]\,. \end{aligned} \tag{35}$$

Consider the first term in the brackets. The $n = 0$ term

$$\int d^3\mathbf{x} \int d^3\mathbf{x}'\, \rho(t,\mathbf{x})\rho(t,\mathbf{x}')\nabla|\mathbf{x}-\mathbf{x}'|^{-1} \tag{36}$$

vanishes in the case of a spherically symmetric charge distribution, whereas the $n = 1$ term is identically zero (gradient of a constant), implying that the first nonvanishing contribution comes from $n = 2$. Changing the summation indices thus leads to

$$\begin{aligned} F_{\text{ext}} \;&=\; \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,c^{n+2}} \int d^3\mathbf{x} \int d^3\mathbf{x}'\, \rho(t,\mathbf{x})|\mathbf{x}-\mathbf{x}'|^{n-1} \frac{\partial^{n+1}}{\partial t^{n+1}} \Big[J(t,\mathbf{x}') \\ &\quad + \frac{\partial\rho(t,\mathbf{x}')}{\partial t} \frac{\nabla(|\mathbf{x}-\mathbf{x}'|^{n+1})}{(n+1)(n+2)|\mathbf{x}-\mathbf{x}'|^{n-1}} \Big]\,. \end{aligned} \tag{37}$$

The continuity equation, spherical symmetry and angular averaging can be used to simplify this expression, taking into account also that for a rigid charge distribution the current is $J(t,\mathbf{x}') = \rho(t,\mathbf{x}')\mathbf{v}(t)$, where $\mathbf{v}(t) = 0$ holds at the time

t at which this calculation is carried out, so only its time derivatives contribute to the series expansion. The term in this expansion containing the first time derivative of the acceleration $\dot{\Gamma} = \ddot{\mathbf{v}}$ is associated with the radiation reaction.

The final result, obtained by neglecting all nonlinear powers of the acceleration and its derivatives (which appear for $n \geq 4$), can be written as

$$F_{\rm ext} = -F_{\rm self} = \frac{2}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{I_n}{c^{n+2}}\frac{\partial^n}{\partial t^n}\dot{\mathbf{v}}\,, \tag{38}$$

where

$$I_n = \int\int d^3\mathbf{x}d^3\mathbf{x}'\,\rho(t,\mathbf{x})|\mathbf{x}-\mathbf{x}'|^{n-1}\rho(t,\mathbf{x}')\,. \tag{39}$$

In the point particle limit, I_0 diverges corresponding to the infinite self-energy of a point particle, $I_1 = e^2$, and $I_n = 0$ for $n > 1$. When the charge is uniformly distributed over the surface of the sphere one has $I_n = 2e^2(2r_0)^{n-1}/(n+1)$, and in the non-relativistic limit (i.e. considering only the $n = 0$ term of the series) Eq. (38) becomes

$$F_{\rm self}^{\rm NR} = -\frac{4}{3}\frac{U_{\rm em}}{c^2}\dot{\mathbf{v}}\,, \tag{40}$$

so that the Newton's equation of motion for the system takes the form

$$F_{\rm ext}^{\rm NR} = \frac{4}{3}m_{\rm em}\dot{\mathbf{v}}\,, \qquad m_{\rm em} = \frac{U_{\rm em}}{c^2}\,, \tag{41}$$

provided higher terms in expansion (38) are neglected. This is 4/3 times the electromagnetic mass. However, Fermi pointed out that this result has been obtained through improper use of the variational method, which is not Lorentz-invariant since it is carried out in a particular reference frame, where the result is equivalent to assuming that the constant laboratory time cross-sections of the world tube of the charge distribution are unchanging.

- Variations of type B

The correct result in which the unwanted factor of 4/3 is removed is achieved if instead the variation B is performed, so that in the previous calculation of Jackson we must replace the factor of $\rho(t,\mathbf{x})$ in the double spatial integral by $\rho(t,\mathbf{x})(1+\dot{\mathbf{v}}(t)\cdot\mathbf{x})$. Thus the vanishing integral $n = 0$ term, namely Eq. (36), of the original expansion now becomes

$$\begin{aligned}&\int d^3x\int d^3x'\,\rho(t,\mathbf{x})[1+\dot{\mathbf{v}}(t)\cdot\mathbf{x}/c^2]\rho(t,\mathbf{x}')\nabla|\mathbf{x}-\mathbf{x}'|^{-1}\\ =&\int d^3x\int d^3x'\,\rho(t,\mathbf{x})[\dot{\mathbf{v}}(t)\cdot\mathbf{x}/c^2]\rho(t,\mathbf{x}')\nabla|\mathbf{x}-\mathbf{x}'|^{-1}\,.\end{aligned} \tag{42}$$

Fermi noted that this double spatial integral will give the same value if the two dummy vector integration variables are switched, and hence can also be replaced by the average of these two ways of writing the same integral. Letting $\nabla|\mathbf{x}-\mathbf{x}'|^{-1} = -(\mathbf{x}-\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|^3$

$$\begin{aligned} & c^{-2}\int d^3x \int d^3x'\, \rho(t,\mathbf{x})\rho(t,\mathbf{x}')[\dot{\mathbf{v}}(t)\cdot\mathbf{x}](\mathbf{x}'-\mathbf{x})/|\mathbf{x}-\mathbf{x}'|^3 \\ = \; & c^{-2}\int d^3x \int d^3x'\, \rho(t,\mathbf{x}')\rho(t,\mathbf{x})[\dot{\mathbf{v}}(t)\cdot\mathbf{x}'](\mathbf{x}-\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|^3 \\ = \; & -c^{-2}\frac{1}{2}\int d^3x \int d^3x'\, \rho(t,\mathbf{x})\rho(t,\mathbf{x}')[\dot{\mathbf{v}}(t)\cdot(\mathbf{x}'-\mathbf{x})](\mathbf{x}'-\mathbf{x})/|\mathbf{x}-\mathbf{x}'|^3 \end{aligned} \tag{43}$$

Now imposing spherical symmetry about the origin, the components of this vector integral are nonzero only along the acceleration vector, with a coefficient which can be replaced by the average value of the vector component integral

$$-[\dot{\mathbf{v}}(t)\cdot(\mathbf{x}'-\mathbf{x})](\mathbf{x}'-\mathbf{x}) \to -\dot{\mathbf{v}}(t)\frac{1}{3}(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{x}'-\mathbf{x}) = -\dot{\mathbf{v}}(t)\frac{1}{3}|\mathbf{x}'-\mathbf{x}|^2 \tag{44}$$

so it reduces to

$$-\frac{1}{3}\frac{\dot{\mathbf{v}(t)}}{c^2}\left[\frac{1}{2}\int d^3x \int d^3x'\, \rho(t,\mathbf{x})\rho(t,\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|\right] = -\frac{1}{3}\frac{U_{\rm em}}{c^2}\dot{\mathbf{v}}(t)\,, \tag{45}$$

since the expression in square brackets is the self-energy of the charge distribution at the time t. This is the only additional term linear in the acceleration which contributes to the lowest terms of the previous calculation (so that the lowest order radiation reaction term is unchanged)

$$F^{\rm NR}_{\rm ext} = \frac{4}{3}\frac{U_{\rm em}}{c^2}\dot{\mathbf{v}} - \frac{1}{3}\frac{U_{\rm em}}{c^2}\dot{\mathbf{v}} = \frac{U_{\rm em}}{c^2}\dot{\mathbf{v}}\,, \tag{46}$$

which leads to the desired result

$$F^{\rm NR}_{\rm ext} = m_{\rm em}\dot{\mathbf{v}}\,, \qquad m_{\rm em} = \frac{U_{\rm em}}{c^2}\,. \tag{47}$$

in the non-relativistic limit, according to Newton's law with the electromagnetic mass $m_{\rm em} = U_{\rm em}/c^2$.

It needs to be emphasized that Fermi started with Hamiilton's principle of least action to derive the correct equations of motion subject to the symmetry constraints of the rigid charge configuration, while the Abraham and Lorentz calculation assumes that the total self-force at a given instant of laboratory time should vanish in the absence of external forces. Why indeed should the total force on the acceleration electron configuration as calculated in an inertial frame in which the electron is momentarily at rest be zero? Fermi shows this to be false yet the same argument is repeated in Jackson [26] and Rohrlich [22]. The rigidity of the charge distribution is a strong constraint on the action principle, and imposing symmetry on an action does not in general commute with deriving the equations of motion, a spectacular example being the failure of the class B spatially homogeneous cosmological models to have a Lagrangian principle [32]. One must carefully examine the boundary conditions as Fermi indeed does. For an outsider to this field, it seems puzzling that such contradictions in reasoning can remain in place for over a century.

Kwal's and Rohrlich's contributions

The Fermi idea that one must consider time slices orthogonal to the world lines of the electron charge distribution was rediscovered independently many years later by Kwal [7] and Rohrlich [8] in the context of the integral definition of the 4-momentum of the unaccelerated classical electron exterior field. Recognizing that the definition (17) of the electromagnetic energy and momentum does not lead to an electromagnetic 4-momentum endowed with the correct transformation properties, they pointed out that when considering these integrals for a moving electron, it is also necessary to select a hypersurface which is linked to the motion of the electron in a relativistically invariant way. According to Fermi's original idea, in order to evaluate the energy and momentum of a moving charge, one should integrate over a hypersurface whose unit normal n is everywhere parallel to the velocity of the charge, i.e., $n = U$, implying that the surface element has the form

$$d\Sigma^{\mu} = -U^{\mu} dV \tag{48}$$

Eq. (48) gives the invariant link between the 4-velocity of the electron and the spacelike hyperplane which must be chosen to correctly define the electromagnetic 4-momentum associated with it. This condition physically expresses the rigidity of the charged particle in the relativistic sense: the world lines associated with each point of the moving system are always orthogonal to a family of spacelike hypersurfaces, so that the shape of the system is the same on any of them. They argued that one simply had to restrict the region of integration to a time slice belonging to the rest frame of the electron, but of course one is still free to express the components of the tensor involved in any inertial coordinate system. By contracting one index of the stress-energy tensor with the fixed 4-velocity of the rest frame, only one index remains free to change under Lorentz transformations of those coordinates. Thus one defines a single 4-vector whose components transform correctly under changes of inertial coordinates, an obvious observation noted by Jackson [26].

To implement this idea so that one can define an integral functional of the hyperplane region exterior to the electron world tube that works for any Cartesian inertial coordinate system, one must slip in between the contraction of the hypersurface element and stress-energy tensor the temporal projection $-U_{\beta} U^{\alpha}$ along the 4-velocity of the electron frame. Hypersurface integrals are about integrating 3-forms, which are then translated into metric language using the normal and hypersurface volume element geometry. Thus alter the previous integral definition in this way

$$P^{\mu}_{\rm AL}(\Sigma) = \int_{\Sigma} T^{\mu\nu}_{\rm em} d\Sigma_{\nu} \qquad \to \qquad P^{\mu}(\Sigma) = \int_{\Sigma} T^{\mu\alpha}_{\rm em} (-U_{\alpha} U^{\nu}) d\Sigma_{\nu} \,, \tag{49}$$

where now U refers to the rest frame 4-velocity, and below U^{α} and $U'^{\alpha} = L^{\alpha}{}_{\beta} U^{\beta}$ are its components with respect to the rest frame and the moving frame. Note that from Eq. (48) one finds $U^{\nu} d\Sigma_{\nu} = -U_{\nu} U^{\nu} dV = dV$, so that the newly defined momentum can also be written as

$$P^\mu(\Sigma) = -\int_\Sigma T^{\mu\alpha}_{\rm em} U_\alpha dV \,, \tag{50}$$

as in the Kwal original calculation [7], where it is understood that Σ can only be a time slice in the electron rest frame where dV is the volume element, but the components of the integrand may be referred to any inertial coordinate system. Since there is one free index, only that index transforms under a Lorentz transformation, giving the appropriate 4-vector character to the evaluated integral.

We can be a little bit more sophisticated over half a century later. The 3-form $dV = U^\nu d\Sigma_\nu$ is really the dual ${}^*U = \frac{1}{3!} U^\nu \eta_{\nu\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma$, which is an integrable 3-form which admits a family of integral submanifolds which are the time slices in the rest frame of the electron, and on one of those slices it reduces to $dV = dx^1 \wedge dx^2 \wedge dx^3$ which we then represent as $dx^1 dx^2 dx^3$ in an ordinary triple integral.

We can integrate this same 3-form over any time slice Σ' (corresponding to a moving frame K') exterior to the electron world tube and it will restrict to the value $dV = \gamma dV'$ on that time slice, where γ is the gamma factor of the new time slice relative to the electron rest frame. Since the integrand in our case is independent of the time in the electron rest frame, its value on the new time slice is the same as on one of the rest frame slices at the corresponding point connected by a world line at rest in the rest frame against any function which depends only on the rest frame spatial coordinates (like inertial Cartesian components of $T^{\mu\alpha}_{\rm em} U_\alpha$) is the same as its integral over a hyperplane region Σ (rest frame time slice of the exterior region), solving the problem. In the differential volume notation, for an integral at constant time t' with unit 4-velocity n', this 3-form looks like $U'^\nu d\Sigma'_\nu = U'^\nu(-n'_\nu dV') = \gamma dV' = dV$, due to the Lorentz contraction $dV' = dV/\gamma$ of the volume element.

Thus if we integrate the new components of the momentum integral over a new time hypersurface, we get

$$\begin{aligned} P'^\mu(\Sigma') &= \int_{\Sigma'} T'_{\rm em}{}^{\mu\alpha}(-U'_\alpha U'^\nu) d\Sigma'_\nu = \int_{\Sigma'} L^\mu{}_\beta T^{\beta\alpha}_{\rm em}(-U_\alpha U'^\nu) d\Sigma'_\nu \\ &= L^\mu{}_\beta \int_\Sigma -T^{\beta\alpha}_{\rm em} U_\alpha dV = L^\mu{}_\beta P^\beta(\Sigma) \,, \end{aligned} \tag{51}$$

using the fact that $T^{\beta\alpha}_{\rm em}(x') = T^{\beta\alpha}_{\rm em}(x)$ for spacetime points x and x' which lie on the same static world line of the electron rest frame. In other words this altered definition defines a unique 4-vector whose components transform as they should between different inertial coordinate systems. However, unless there exists a simultaneous rest frame for the entire charge distribution, one cannot introduce this modified definition of the energy-momentum 4-vector and one is stuck with the noncovariant Abraham-Lorentz definition.

Concluding Remarks

It is unfortunate that the first four papers by one of the leading physicists of the twentieth century were never translated from their original Italian. The fourth paper, which appeared in two preliminary versions in Italian and German, was the culmination of Fermi's early work in relativity only a few years after the birth of general relativity and written while he was a young university student. Its actual contents seem to have remained a mystery to all those who have cited it in discussions of the classical theory of the electron which still interests people even today, while the leading textbook on classical electrodynamics still repeats the Abraham-Lorentz derivations of the equations of motion without Fermi's correction. Ironically Fermi's third paper (see [33] for a historical discussion), which he considered only a tool for obtaining his result in that fourth paper, and which Fermi never even explicitly cited, did make an indelible mark on relativity with the terms Fermi coordinates and Fermi-Walker transport, although even the much later paper by Walker that coupled together their names forever ignores Fermi's original paper in Italian. Even the text by Rohrlich updated only recently four decades after its original publication fails to connect his adjustment of the definition of the 4-momentum of the electromagnetic field of the classical electron to Fermi's argument about the equations of motion, which is described in our appendix. We hope the present work restores Fermi's message to its rightful place.

Appendix. Gauss's theorem and "conservation laws"

For a divergence-free stress-energy tensor in all of Minkowski spacetime which falls off sufficiently fast at spatial infinity, its integral over any two parallel inertial time hyperplanes would be the same by Gauss's law, as explained in most standard text in relativity, see Chapter 5 of Misner, Thorne and Wheeler [28], for example, or Appendix A1–5 of Rohrlich's Third Edition [22], or Anderson [27]. However, in the present case the boundary term on the timelike world tube of the surface of the electron sphere interferes with this more familiar picture, explaining the unwanted correction terms to the evaluation of the above stress-energy tensor.

Fig. 1 generalizes Fig. 5.3.b of Misner, Thorne and Wheeler: it represents a constant x^2, x^3 slice of the unaccelerated electron world tube centered at the origin of the unprimed spatial coordinates in spacetime. As in section 2, the unprimed coordinates are associated with the rest frame K of the electron, while the primed coordinates are associated with a frame K' in relative motion with respect to the unprimed frame is in the x^1 direction with velocity $-v < 0$ as shown in the figure. Consider the spacetime region devoid of electromagnetic sources between two spacelike hyperplanes Σ'_1 and Σ'_2 of constant inertial times t'_1 and $t'_2 > t'_1$ and outside of an internal lateral boundary σ between them which is a subset of the cylindrical timeline surface representing the world tube

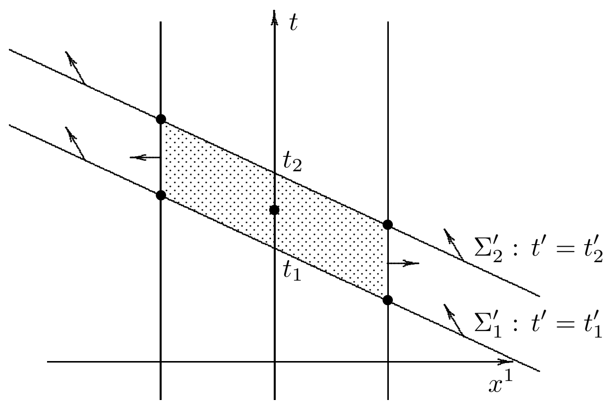

Figure 1: Figure 5.3.b from Misner, Thorne and Wheeler redrawn with an inner cylindrical boundary which is the world tube of the electron sphere boundary. The arrows show the unit normal direction for the orientation of each hypersurface, but in the single Gauss law relation for the region of spacetime between Σ_1 and Σ_2 excluding the shaded region inside the cylinder, the sum of the outward normal integral contributions is zero for a divergence-free vector field. Here the boundary term due to the portion σ of the cylinder between the two parallel hyperplanes vanishes by spherical symmetry.

of the electron spherical surface. Let $\overline{\Sigma}'_1$ and $\overline{\Sigma}'_2$ be the portions of those planes exterior to this cylinder. Suppose Σ'_1 and Σ'_2 are oriented by their future-pointing unit normal vector fields and σ by its inward unit normal $\partial/\partial r$ relative to the region of spacetime in question. Let Q be any constant 4-vector so that $q^\mu = Q_\nu T^{\nu\mu}_{\rm em}$ is a divergence-free vector field in the spacetime region bounded by the three hypersurfaces $\overline{\Sigma}$, $\overline{\Sigma}'$ and σ, as well as by the lateral boundary at spacelike infinity, a region to which Gauss's law with zero volume integral and outward pointing normals applies. Taking the orientations into account relative to the outward normal on each boundary hypersurface, one then has

$$\int_{\overline{\Sigma}'_2} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu - \int_{\overline{\Sigma}'_1} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu = \int_\sigma Q_\mu T^{\mu\nu}_{\rm em} d\sigma_\nu \,. \tag{52}$$

If the lateral boundary term vanishes, then the integral is the same over each of the two time hypersurfaces outside the world tube of the electron sphere. Indeed for time slices in the rest frame of the electron, or in the moving frame, these integrals are time-independent, which corresponds exactly to the vanishing of the integral over the electron surface tube between the two slices. This follows for all possible projections Q_α in the explicit evaluation of the lateral integral from the vanishing of $T^{0r}_{\rm em}$ itself and of the surface integral of the spatial stress

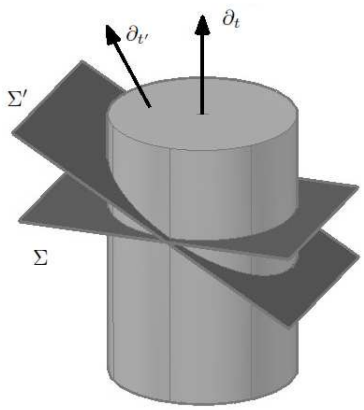

Figure 2: The world tube of the electron sphere is a cylinder in spacetime about the t axis, shown here with one spatial dimension suppressed. The time slices $t = 0$ (Σ) and $t' = 0$ (Σ') cut this cylinder, intersecting in the spacelike 2-plane $x^1 = 0, t = 0$, which separates the spacetime region between these time slices into two disjoint subregions $x^1 > 0$ and $x^1 < 0$. Gauss's law applies separately to each of these two simply connected regions outside the electron sphere cylinder, but the signs of the outward normals of the time slices switch between these two regions, while remaining the same for the cylindrical portion of their boundaries.

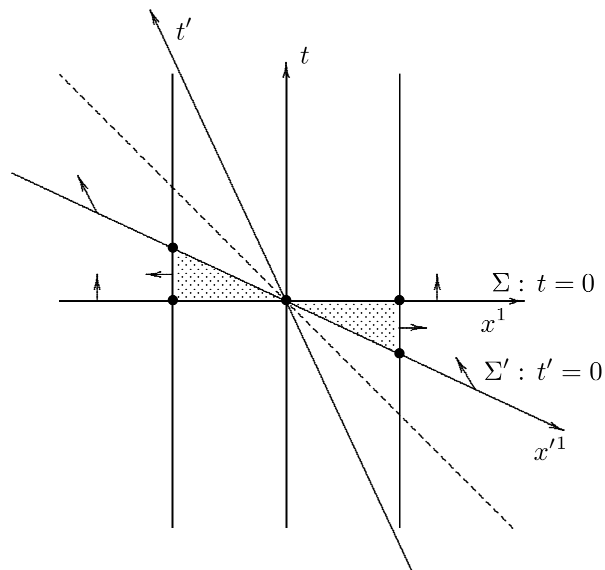

Figure 3: Fig. 5.3.c from Misner, Thorne and Wheeler (or Fig. A1–3 from Rohrlich's Third Edition) redrawn with an inner cylindrical boundary which is the world tube of the electron sphere boundary, showing a constant x^2, x^3 slice of the previous figure. The arrows show the unit normal direction for the orientation of each hypersurface, which changes sign relative to the unit outward normal of the exterior region outside the cylinder going from $x^1 > 0$ to $x^1 < 0$. Here the boundary term due to the portion σ of the cylinder between the two parallel hyperplanes is now nonvanishing. The two halves σ_+ ($x^1 > 0$) and σ_- ($x^1 < 0$) contribute terms with opposite signs to the two separate Gaussian integral relations because of the change in sign of the outward normals on Σ and Σ', and hence in the difference relation needed to reassemble the two halves of those time hypersurface integrals, they contribute a nonzero correction term.

components

$$T_{\rm em}^{x^i r} = T_{\rm em}^{r x^i} = T_{\rm em}^{rr}\frac{\partial x^i}{\partial r} = -T_{\rm em}^{00}\frac{x^i}{r} \tag{53}$$

over the 2-sphere $r = r_0$, which follows from the spherical symmetry and the fact that the integral along the time direction on the cylinder is the constant rest frame time difference $t_2 - t_1 = \gamma(t'_2 - t'_1)$. However, even though for each such inertial coordinate system, the integral at constant time is time-independent, we must do a second calculation to relate the results of the integration with respect to inertial coordinate systems in relative motion.

The situation between the time hyperplanes of two different inertial frames is more complicated since the hyperplanes necessarily intersect, as shown in Fig. 2 with one spatial dimension suppressed, assuming that the relative velocity v

along the direction x^1 of the electron rest frame relative to the moving primed frame is positive, as in the previous figure. Fig. 3 shows a constant x^2, x^3 slice of Fig. 2 generalizing Fig. 5.3.c of Misner, Thorne and Wheeler [28] (or Fig. A1–3 from Rohrlich's Third Edition), but with an additional internal lateral boundary, here the portion σ of the cylinder representing the electron sphere centered around the t axis and extending between the two time slices. Consider the region of spacetime exterior to the electron sphere bounded by the time hypersurfaces $t = 0$ and $t' = 0$, with unit future-pointing normals $U = \partial/\partial t$ and $U' = \partial/\partial t'$. Let $\sigma = \sigma_- \cup \sigma_+$ be the portion of the cylindrical world tube of the electron sphere between these two time hyperplanes, divided into two disjoint parts σ_+ for $x^1 > 0$ and σ_- for $x^1 < 0$, each with the orientation induced by the outward radial normal $\partial/\partial r$ relative to the sphere. For each point on the electron sphere, σ consists of the region between $t = 0$ and $t = -vx^1$, so the integral on σ along t leads to a factor $\Delta t = 0 - (-vx^1) = vx^1 > 0$ for $x^1 > 0$ and a factor $\Delta t = -vx^1 - 0 = -vx^1 > 0$ for $x^1 < 0$ since the integrand is independent of t along the cylinder.

Similarly let $\Sigma = \Sigma_- \cup \Sigma_+$ and $\Sigma' = \Sigma'_- \cup \Sigma'_+$, each with the future-pointing normal orientation, and let $\overline{\Sigma} = \overline{\Sigma}_- \cup \overline{\Sigma}_+$ and $\overline{\Sigma}' = \overline{\Sigma}'_- \cup \overline{\Sigma}'_+$ be the portions of those regions outside the world tube of the electron sphere. One can separately apply Gauss's law to the two disjoint regions with these boundaries and reassemble the pieces to get a relation between the integrals over $\overline{\Sigma}$, $\overline{\Sigma}'$ and σ. Since the outer normal directions switch directions for $\overline{\Sigma}$ and $\overline{\Sigma}'$ but not σ going from $x^1 > 0$ to $x^1 < 0$, one must take the difference of the two separate Gauss law relations to reassemble the total integrals over $\overline{\Sigma}$ and $\overline{\Sigma}'$, which leads to a net nonvanishing contribution from σ in spite of the spherical symmetry. One has

$$\begin{aligned}
\int_{\overline{\Sigma}_+} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu - \int_{\overline{\Sigma}'_+} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu &= \int_{\sigma_+} Q_\mu T^{\mu\nu}_{\rm em} d\sigma_\nu \,, \\
\int_{\overline{\Sigma}'_-} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu - \int_{\overline{\Sigma}_-} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu &= \int_{\sigma_-} Q_\mu T^{\mu\nu}_{\rm em} d\sigma_\nu \,,
\end{aligned} \tag{54}$$

and therefore taking the difference

$$\begin{aligned}
&\int_{\overline{\Sigma}'} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu - \int_{\overline{\Sigma}} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu \\
&= \int_{\overline{\Sigma}'_+} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu - \int_{\overline{\Sigma}_+} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu \\
&\qquad - \left(\int_{\overline{\Sigma}_-} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu - \int_{\overline{\Sigma}'_-} Q_\mu T^{\mu\nu}_{\rm em} d\sigma_\nu \right) \\
&= -\int_{\sigma_+} Q_\mu T^{\mu\nu}_{\rm em} d\sigma_\nu + \int_{\sigma_-} Q_\mu T^{\mu\nu}_{\rm em} d\sigma_\nu \,.
\end{aligned} \tag{55}$$

Consider applying the above relation in this setting for $Q = -U'$, so that

$q^\alpha = -U'_\nu T^{\nu\alpha}_{\rm em} = T^{t'\alpha}_{\rm em} = \gamma(T^{t\alpha}_{\rm em} + vT^{x^1\alpha}_{\rm em})$. Then

$$\int_{\overline{\Sigma}'} q^\alpha d\Sigma'_\alpha = \int_{\overline{\Sigma}'} T^{t't'}_{\rm em}\, dV' = W' \tag{56}$$

while

$$\int_{\overline{\Sigma}} q^\alpha d\Sigma_\alpha = \int_{\overline{\Sigma}} T^{t't}_{\rm em}\, dV = \gamma W\,. \tag{57}$$

The cylindrical world tube integrals, since the integrand is independent of t, are

$$\begin{aligned}
\int_{\sigma_+} q^\alpha d\sigma_\alpha &= \int_{\sigma_+} T^{t'r}_{\rm em}\, dt\, dS = \int_{\sigma_+} \gamma(T^{tr}_{\rm em} + vT^{x^1 r}_{\rm em})\, dt\, dS \\
&= \int_{-\pi/2}^{\pi/2}\int_0^\pi (0-(-vx^1))\gamma v \left(\frac{x^1}{r_0} T^{rr}_{\rm em}\right) r_0^2 \sin\theta\, d\theta d\phi \\
&= \gamma v^2 r_0^3 T^{rr}_{\rm em} \int_{-\pi/2}^{\pi/2}\int_0^\pi (\sin\theta\cos\phi)^2 \sin\theta\, d\theta d\phi \\
&= \frac{1}{6}\gamma v^2 (4\pi r_0^3 T^{rr}_{\rm em}) = -\frac{1}{6}\gamma v^2 W\,.
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
\int_{\sigma_-} q^\alpha d\sigma_\alpha &= \int_{\pi/2}^{3\pi/2}\int_0^\pi ((-vx^1)-0)\gamma v \left(\frac{x^1}{r_0} T^{rr}_{\rm em}\right) r_0^2 \sin\theta\, d\theta d\phi \\
&= -\gamma v^2 r_0^3 T^{rr}_{\rm em} \int_{\pi/2}^{3\pi/2}\int_0^\pi (\sin\theta\cos\phi)^2 \sin\theta\, d\theta d\phi \\
&= -\gamma v^2 r_0^3 (4\pi T^{rr}_{\rm em})\frac{1}{6} = \frac{1}{6}\gamma v^2 W\,.
\end{aligned} \tag{59}$$

Since the outward normals on Σ and Σ' reverse direction on the second set of integrals, but the outward normal on σ does not, the separate Gauss's law relations are

$$\begin{aligned}
\int_{\overline{\Sigma}'_+} q^\alpha d\Sigma'_\alpha - \int_{\overline{\Sigma}_+} q^\alpha d\Sigma_\alpha &= -\int_{\sigma_+} q^\alpha d\sigma_\alpha \\
\int_{\overline{\Sigma}'_-} q^\alpha d\Sigma'_\alpha - \int_{\overline{\Sigma}_-} q^\alpha d\Sigma_\alpha &= \int_{\sigma_-} q^\alpha d\sigma_\alpha
\end{aligned} \tag{60}$$

and their sum is

$$\begin{aligned}
W' - \gamma W &= \int_{\overline{\Sigma}'} q^\alpha d\Sigma'_\alpha - \int_{\overline{\Sigma}} q^\alpha d\Sigma_\alpha \\
&= -\int_{\sigma_+} q^\alpha d\sigma_\alpha + \int_{\sigma_-} q^\alpha d\sigma_\alpha = \frac{1}{3}v^2\gamma W\,.
\end{aligned} \tag{61}$$

Thus the unwanted correction factor is exactly the integral over the cylindrical boundary over the electron sphere of the moving frame 4-velocity component of the stress-energy tensor, with the factor of 1/3 equal to

$$\int_0^{2\pi}\int_0^{\pi}(\sin\theta\cos\phi)^2\sin\theta\,d\theta d\phi$$
$$= \frac{1}{4\pi}\int_{S_2}\frac{(x^1)^2}{r_0^4}r_0^2 d\Omega = \frac{1}{3}\frac{1}{4\pi}\int_{S_2}\frac{r_0^2}{r_0^2}r_0^2\,d\Omega = \frac{1}{3}\,, \tag{62}$$

whose value follows from spherical symmetry as expressed in Eq. (11).

One can repeat this calculation for $Q = \partial/\partial x'^1$ in order to express the momentum correction factor as an integral over this boundary, with one less factor of v in the correction term since

$$T_{\rm em}^{x'^1 r} = \gamma(T_{\rm em}^{x^1 r} + vT_{\rm em}^{tr}) = \gamma T_{\rm em}^{x^1 r} \tag{63}$$

compared to the previous calculation where

$$T_{\rm em}^{t'^1 r} = \gamma(T_{\rm em}^{tr} + vT_{\rm em}^{x^1 r}) = \gamma v T_{\rm em}^{x^1 r}\,. \tag{64}$$

With this corresponding correction term the integral relationship now becomes

$$p'^1 - \gamma vW = \frac{1}{3}\gamma vW \to p'^1 = \frac{4}{3}\gamma vW\,. \tag{65}$$

explaining the famous factor of 4/3.

If we shrink the radius $r_0 \to 0$ for a divergence-free stress-energy tensor which is not singular at the origin, the contributions due to the hypersurface integral over σ also go to zero and show that the 4-momentum integral is the same over both bounding inertial time hypersurfaces associated with the two frames in relative motion.

$$\int_{\Sigma'} q^\alpha d\Sigma'_\alpha = \int_\Sigma q^\alpha d\Sigma_\alpha = \int_\Sigma Q_\mu T^{\mu\nu} d\Sigma_\nu\,. \tag{66}$$

In this case the 4-vector $\int_\Sigma T^{\mu\nu} d\Sigma_\nu$ has the same value on any inertial time hypersurface, i.e., its components transform correctly under Lorentz transformations. This applies to any (time-dependent) source-free electromagnetic field defined over all space. Of course this cannot be done for the Coulomb field of the electron which goes infinite in the limit of shrinking its radius to zero.

Finally suppose we accept that the total 4-momentum of the electromagnetic field at a given instant of time in the instantaneous rest frame of an accelerated electron is the Kwal-Rohrlich integral and express the limit definition of its time derivative through a limiting Gauss's law volume integral over the region between the two successive rest frame time slices. Let $\Sigma, \Sigma_\pm$ and $\Sigma', \Sigma_\pm{}'$ now refer to the entire hyperplanes or half hyperplanes corresponding to the starting time and ending time τ and $\tau' = \tau + \Delta\tau$ in the rest frame of the electron distribution, where the intersection of the two hyperplanes away from the electron world tube divides them into halves, one Σ_- containing a portion of the electron world tube and the other not. Let $R_\pm$ be the two wedges of spacetime between

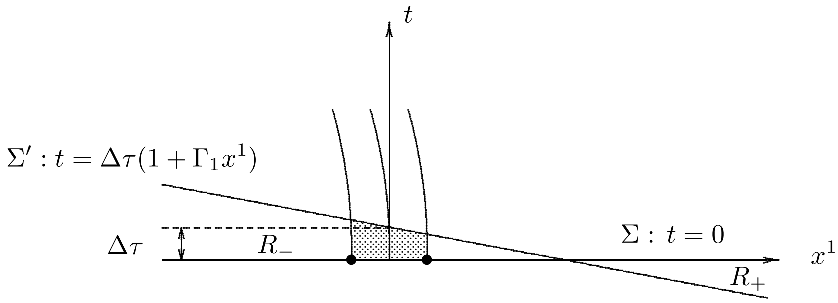

Figure 4: Figure 5.3.c from Misner, Thorne and Wheeler redrawn in a constant x^2, x^3 slice showing the world tube of an electron sphere instantaneously at rest at $t = 0$ but accelerated in the negative x^1 direction ($\Gamma_1 < 0$) and two successive rest frame time slices separated by proper time $\Delta\tau$ at the center of the sphere, with the time slices intersecting to the right of the world tube (equivalent to the assumption $|\Gamma_1| r_0 < 1$). The spacetime region within the electron world tube between the two slices (shaded in this plane cross-section) occurs in the Gauss's law application to the wedge between the two time slices, namely $R_- \cap R_+$, two regions which are separated from each other by a plane of constant x^1 within the hypersurface $t = 0$ shown as the intersection point in this diagram.

these two planes, and let x^α be the inertial coordinates instantaneously at rest with the electron distribution initially at $t = 0$. The successive time slice in the rest frame of the electron after a small proper time interval $\Delta\tau$ is then described by $t = \Delta\tau(1 + \Gamma_i x^i)$, where Γ_i are the components of the acceleration of the electron frame at $t = 0$, and $N = 1 + \Gamma_i x^i$ is the lapse function in the Fermi coordinate system. Fig. 4 describes this situation for the special case in which the acceleration is aligned with the x^1 direction at $t = 0$.

Then the hypersurface integral of the stress-energy tensor on each hyperplane of constant proper time τ, using the notation $P(\tau)$ now instead of the previous $P(\Sigma)$, is the Kwal-Rohrlich integral

$$\begin{aligned} Q_\mu(P^\mu(\tau') - P^\mu(\tau)) &= \int_{\Sigma'} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma'_\nu - \int_\Sigma Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu \\ &= \int_{R_-} Q_\mu T^{\mu\nu}_{{\rm em}\,,\nu}\, d^4x - \int_{R_+} Q_\mu T^{\mu\nu}_{{\rm em}\,,\nu}\, d^4x \\ &= -\int_{R_-} Q_\mu F^\mu{}_\nu J^\nu\, d^4x \,. \end{aligned} \tag{67}$$

The intermediate equalities state Gauss's law for the wedge regions $R_- \cap R_+$, which is implied by Rohrlich's Fig. A1–3 and his Eq. (A1–61), equivalent to assigning a negative orientation to the half region R_+, if both Σ_- and Σ_+ are oriented by their future-pointing normals.

The proper time derivative (letting Γ_i be the spatial components of the acceleration at $t = 0$) is then

$$
\begin{aligned}
Q_\mu \frac{dP^\mu(\tau)}{d\tau} &= Q_\mu \lim_{\Delta\tau\to 0} \frac{P^\mu(\tau+\Delta\tau) - P^\mu(\tau)}{\Delta\tau} \\
&= -Q_\mu \lim_{\Delta\tau\to 0} \frac{\int_{R_-} F^\mu{}_\nu J^\nu \, d^4x}{\Delta\tau} \\
&= -Q_\mu \lim_{\Delta\tau\to 0} \frac{\int \int_{t=0}^{t=\Delta\tau(1+\Gamma_j x^j)} F^\mu{}_\nu J^\nu \, dt\, d^3x}{\Delta\tau} \\
&= -Q_\mu \int (1+\Gamma_j x^j) F^\mu{}_\nu J^\nu \, d^3x \,, \qquad (68)
\end{aligned}
$$

where the integral is evaluated at $t = 0$ over the entire region where this integrand is nonzero. With $J^\mu = \rho U^\mu = \rho\, \delta^\mu{}_0$ since the electron distribution is instantaneously at rest, this reduces to

$$
\begin{aligned}
Q_\mu \frac{dP^\mu(\tau)}{d\tau} &= -Q_\mu \int (1+\Gamma_j x^j) F^\mu{}_\nu \rho U^\nu \, d^3x \\
&= -Q_\mu \int (1+\Gamma_j x^j) E^\mu \rho \, d^3x \,, \qquad (69)
\end{aligned}
$$

where $E^\mu = \delta^\mu{}_i E^i$ at $t = 0$ when this integral is evaluated, so that $dP^0/d\tau = 0$. Thus taking Q_μ to pick out the spatial directions, we get

$$
\frac{dP^i(\tau)}{d\tau} = -\int (1+\Gamma_j x^j) E^i \rho \, d^3x \,. \qquad (70)
$$

The vanishing of the right hand side is the starting point of the second half of Fermi's variation B discussion, which is therefore equivalent to the conservation of this definition of the total spatial momentum in the electromagnetic field, as redefined by Kwal and Rohrlich.

The Abraham-Lorenz argument proceeds without the lapse factor in parentheses, as reproduced in Jackson [26] and as reviewed in Rohrlich [22], but Fermi explicitly argues that this is wrong because the condition that the integral vanish is equivalent to the variation A of the action which violates the relativity principle. If the charge distribution is rigid in the relativistic sense as well as accelerated, then the total force on the charge distribution cannot be zero simultaneously in the laboratory frame. By including the lapse factor, one relies on the conservation of total momentum in the rest frame of the charge distribution to determine the equations of motion which are compatible, and the extra factor of the acceleration leads to the correction of the unwanted 4/3 factor in the mass coefficient.

Note that by applying Gauss's law to the situation of Fig. 1 (dropping the primes from the time coordinates) between two hyperplanes of infinitesimally separated laboratory times $t_2 = t_1 + dt$, one finds instead an expression for the time rate of change of the Abraham-Lorentz momentum

$$\frac{dP^i_{\rm AL}(t)}{dt} \quad = \quad -\int E^i \rho \, d^3x \,, \tag{71}$$

which is Rohrlich's Eqs. (4–141,142) [22] in a laboratory frame in which the charge distribution is momentarily at rest. For an accelerated charge distribution, this will not be zero so the total Abraham-Lorentz momentum will not be conserved due to momentum losses in the field caused by the sources. Similarly although the energy will have zero time derivative at this moment, if the charge distribution is accelerated, the time derivative of the energy will become nonzero away from this time slice when it is no longer at rest.

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