11.
THE PRINCIPLE OF ADIABATICS AND SYSTEMS WHICH DO NOT ADMIT ANGLE COORDINATES

"Il principio delle adiabatiche ed i sistemi
che non ammettono coordinate angolari,"
Nuovo Cimento **25**, 171–175, (1923).

§ 1. – The importance of Ehrenfest's principle of adiabatics for the determination of the selection rules for the stationary orbits of a system, in the Bohr theory, is well-known.[1] This principle, as we know, can be stated in the following way: Suppose that in a mechanical system the forces or the constraints are continuously changed with time but very slowly compared with the system's own periods, or using Ehrenfest's expression, adiabatically; the principle of adiabatics states that if the system is initially in a preferred quantum orbit, it will still be found there at the end of the transformation.

Let us consider a pendulum, for instance, and imagine shortening its string at a very slow rate in comparison with the period of the pendulum itself. The frequency ν of the pendulum will then grow slowly, but it is easy to realize that energy u also will grow and grow precisely so that the ratio u/ν remains constant. In this way if this ratio was initially an integer multiple of the Planck constant h, it always remain so and therefore the state of the system will remain quantum mechanically preferred during the entire transformation. For further examples we refer to Ehrenfest's memoir.

The formal basis for the principle of adiabatics is provided by Burgers' theorem.[2] Let us consider a system that in certain general coordinates $q_1, q_2, ..., q_f$ allows separation of variables.[3] Then set

$$I_K = \oint p_K dq_K \,, \qquad (K = 1, 2, \ldots, f) \tag{1}$$

[1]Ehrenfest, *Ann. d. Phys.* **51**, 327 (1916).
[2]Burgers, Versl. Akad. van. Wetensch. – Amsterdam 1916, 1917; Ann. d. Phys. 52, 195 (1917).
[3]For the validity of Burgers' conclusions it is sufficient, more generally, that the system admits angle coordinates, i.e., it is possible to introduce in place of q_K, p_K new variables w_K, j_k such that the q_K's, expressed in terms of the (w_K, j_K), are periodic with period ℓ in the variables w_K, and the energy, in the new coordinates, turns out to be a function only of the j's. Then, because of the Hamilton equations, the j's turn out to be constant and the w's linear functions of the time, and the q's as functions of the time can be expanded in Fourier series with f indices.

where the p_K are the momenta canonically conjugate to q_K and the integral is extended, according to the rules of quantum theory, over a complete oscillation of the coordinate q_K; in this way the conditions that an orbit of the system under consideration be quantum preferred are:

$$I_1 = n_1 h \ ; \ I_2 = n_2 h \ ; \ \ldots ; \ I_f = n_f h \tag{2}$$

where $n_1, n_2, ..., n_f$ are integers. Let us suppose now that we modify our system adiabatically, but in a way which allows separation of variables at any instant. Burgers' theorem tells us that in this case the integrals $I_1, I_2, ..., I_f$ do not change during the transformation, i.e., that they are adiabatic invariants. Therefore, if conditions (2) are satisfied at the onset of the transformation, they will also be satisfied at the end, and so Burger's theorem on the principle of adiabatics is satisfied.

In this Note I intend to show by means of a simple example that if a system adiabatically transforms into another system and the initial and final states both admit separation of variables, but the intermediate states do not, then the I_K are no longer adiabatic invariants. In this case the principle of adiabatics loses its basis.

§ 2.– Let us consider a mass point moving on a plane inside a rectangle; we shall assume that no force acts on the point while it is inside the rectangle, but it bounces off the walls when it hits them. Consider sides AB and AC of the rectangle as coordinate axes x, y. Now, it is clear that our system admits separation of variables in these coordinates. Calling a, b the lengths of sides AB, AC, the coordinate x in fact oscillates between values $0, a$; the coordinate y between values $0, b$.

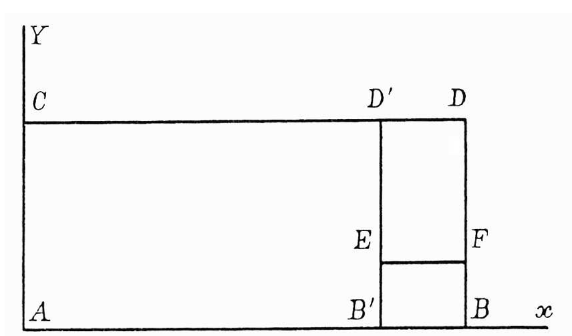

Figure 1.

Moreover, if at a certain instant the components of the velocity are u, v, at any instant whatever they will be $\pm u$, $\pm v$, where one must choose the sign + or − according to whether the relative coordinate is increasing or decreasing at the instant under consideration.

The momenta conjugate to x and y will be $\pm mu$, $\pm mv$, where m is the mass of the point; then one will have

$$I_x = \oint (\pm mu)\, dx = \int_0^a mu\, dx + \int_a^0 (-mu)\, dx = 2\, mua \tag{3}$$

and analogously

$$I_y = 2\, mvb\,. \tag{3$'$}$$

Now we want to study how I_x and I_y change if we transform our system adiabatically. We just intend to transform the rectangle ABCD into the other AB$'$CD$'$; we remark that such a transformation can be carried out in three ways:

(1) one shifts the line segment BD parallel to itself until it arrives at B$'$D$'$;
(2) one shifts the line segment BB$'$ parallel to itself until it arrives at DD$'$, so that at an intermediate instant, the mass point can move inside the concave polygon AB$'$EFDC;
(3) one deforms in any way the broken line B$'$BDD until made to coincide with the line segment B$'$D$'$.

Excluding the last case, which is clearly somewhat complicated, from our considerations, we will limit ourselves to discussing the first two.

As to the first one, we remark that in this case at any instant the point can always move inside a rectangle, and therefore also in the intermediate instants separation of variables is always possible; according to Burgers' theorem, in this case we must expect that I_x and I_y remain invariant. This is obviously evident for I_y, since neither b nor v change during the transformation and thus, due to (3$'$), nor does I_y. As to I_x, instead, a decreases during the transformation, being reduced from $a =$ AB to $a' =$ AB$'$; but at the same time u increases as a consequence of the bounces against the moving wall and an immediate consideration shows that things go just so that the product au, and so also I_x, remains constant,[4] obviously under the condition that the transformation is realized slowly enough.

Passing on to consider case (2), it is easy to recognize that now things are different. As to I_x, in fact one immediately sees that the x component of the velocity remains unchanged (except for the sign), since it could change its absolute value only hitting a moving wall parallel to the x axis, but the only moving wall EF moves parallel to y; instead a decreases from AB to AB$'$. In all therefore I_x reduces by the ratio a'/a and thus does not remain constant. Likewise I_y also does not remain

[4]In fact the number of collisions with the moving wall BD in the time interval dt is obviously $\dfrac{u}{2a}dt$; on the other hand, if V is the velocity of the wall BD, the velocity of the point will experience an increase of $2V$ with every collision; then the increase of u in the time dt will be:

$$du = 2V\frac{u}{2a}dt = \frac{u}{a}V dt = -\frac{u}{a}da$$

since obviously $-da = V dt$. By integrating the preceding equation, we find exactly $ua = const.$, as claimed above.

constant; in fact b remains unchanged whereas v increases due to the collisions against the moving wall EF. An immediate evaluation shows that v, and then also I_y, increases by the ratio a/a'. From the above considerations we can conclude that the integrals I_K are adiabatic invariants only if in the intermediate states the system always admits separation of variables or at least, according to Burgers' theorems, always admits a system of angular coordinates. On the contrary, at least in general, this is not true if the system does not always have a multiperiodic motion.

On the other hand, this fact is easily understandable also from the point of view of quantum theory. In fact one knows, following Bohr, that a well-defined quantization is possible only if the motion of the system is multiperiodic. Thus one can recognizes that, if in the intermediate states the system cannot be quantized rigorously, this inexactitude is also transmitted to the final state.

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