80A.
AN ATTEMPT AT A THEORY OF β RAYS (*)

"Tentativo di un a teoria dei raggi β,"
Nuovo Cimento **11**, 1–19 (1934).

A quantitative theory of the emission of β rays is proposed in which the existence of the "neutrino" is assumed and the emission of electrons and neutrinos in β decay is treated in a way similar to the one followed in the theory of radiation for describing the emission of a quantum of light from an excited atom. We deduce the formulas for the lifetime and for the shape of the continuous spectrum of β rays and compare them with experimental data.

The fundamental hypotheses of the theory

§ 1. In the attempt to construct a theory of the nuclear electrons and the emission of β rays, one encounters, as is known, two principal difficulties. The first depends on the fact that the primary β rays are emitted from nuclei with a continuous velocity distribution. If we do not want to abandon the energy conservation principle, we are obliged to admit that a fraction of the energy which is released in the process of β decay escapes our present possibilities of observation. According to Pauli's proposal one can for instance assume the existence of a new particle, the so called "neutrino", having vanishing electric charge and mass on the order of magnitude of the electron mass or less. Thus we assume that in any β process are simultaneously emitted an electron, which is detected as a ray, and a neutrino which eludes the observation carrying a part of the energy away. In the present theory, we shall adopt the neutrino hypothesis.

A second difficulty for a theory of nuclear electrons depends on the fact that the present relativistic theories of the light particles (electrons or neutrinos) do not give a satisfactory explanation for the possibility that these particles are bound in orbits of nuclear size.

Consequently it seems more appropriate to agree with Heisenberg[1] and assume that all nuclei consist only of heavy particles, protons and neutrons. Then with the aim of understanding the possibility of emission of β rays, we will attempt to construct a theory of the emission of light particles from a nucleus in analogy with

*Cf. the preliminary note in *La Ricerca Scientifica*, **4** (2), 491 (1933)
[1]W. Heisenberg, *ZS. für Phys.* **77**, 1 (1932); E. Majorana, *ZS. für Phys.* **82**, 137 (1933).

the theory of the emission of a quantum of light from an excited atom in the usual process of radiation. In the theory of radiation, the total number of the light quanta is not constant; the quanta are created when being emitted from an excited atom and disappear when absorbed. In analogy with that we will try to establish the theory of β rays on these assumptions:

(a) The total number of electrons and neutrinos is not necessarily constant. Electrons (or neutrinos) can be created or destroyed. On the other hand this possibility has no analogy with the possibility of the creation or destruction of an electron-positron pair; in fact if we interpret a positron as a Dirac "hole", we can simply consider this latter process as a quantum jump of an electron from a state of negative energy to a state of positive energy, conserving the total number (infinitely large) of the electrons.

(b) The heavy particles, neutron and proton, can be considered, following Heisenberg, as two different internal states of the heavy particle. We shall formulate this fact by introducing an internal coordinate ρ of the heavy particle, which can assume only two values: $\rho = +\,1$, if the particle is a neutron; $\rho = -\,1$, if the particle is a proton.

(c) The Hamiltonian function of the overall system, consisting of heavy and light particles, must be chosen so that every transition from neutron to proton be accompanied by the creation of an electron and a neutrino; and the inverse process, transformation of a proton into a neutron, be accompanied by the disappearance of an electron and a neutrino. It must be remarked that in this way the conservation of the electric charge is assured.

The operators of the theory

§ 2. A mathematical formalism which allows us to construct a theory in agreement with the three points of the preceding section can be easily constructed by using the method of Dirac-Jordan-Klein[2] called "the method of second quantization." Then we shall consider the probability amplitudes ψ and φ of the electrons and neutrinos in ordinary space, and their complex conjugates ψ^* and φ^* as operators; while for describing the heavy particles we shall use the usual representation in configuration space, in which obviously also ψ will be considered as a coordinate.

We introduce first two operators Q and Q^* which operate on the functions of the two-valued variable ρ as the linear substitutions

$$Q = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \; ; \qquad Q^* = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \; . \tag{1}$$

One immediately realizes that Q determines the transitions from proton to neutron, and Q^* the inverse transitions from neutron to proton.

[2]Cf. e.g. P. Jordan and O. Klein, *ZS. für Phys.* **45**, 751 (1927); W. Heisenberg, *Ann. d. Phys.* **10**, 888 (1931).

The meaning of the probability amplitudes ψ and φ interpreted as operators is, as we know, the following. Let

$$\psi_1\psi_2\ldots\psi_s\ldots$$

be a system of individual quantum states of the electrons. Then put

$$\psi = \sum_s \psi_s a_s \ ; \qquad \psi^* = \sum_s \psi_s^* a_s^* \ . \tag{2}$$

The amplitudes a_s and the conjugate complex quantities a_s^* are operators which act on the functions of the occupation numbers N_1, N_2,. . .,N_s,. . . of the individual quantum states. If the Pauli principle holds, each of the N_s can assume only one of the values 0, 1; and the operators a_s and a_s^* are defined as follows:

$$\begin{aligned} a_s\Psi\ \ &(N_1, N_2, \ldots, N_s, \ldots) \\ &= (-1)^{N_1+N_2+\ldots+N_{s-1}}\,(1-N_s)\,\Psi\,(N_1, N_2, \ldots, 1-N_s, \ldots) \\ a_s^*\Psi\ \ &(N_1, N_2, \ldots, N_s, \ldots) \\ &= (-1)^{N_1+N_2+\ldots+N_{s-1}}\,(1-N_s)\,\Psi\,(N_1, N_2, \ldots, N_s, \ldots)\ . \end{aligned} \tag{3}$$

The operator a_s^* determines the creation, while the operator a_s determines the disappearance of an electron in the quantum state s.

Corresponding to (2), for the neutrinos we shall set:

$$\varphi = \sum \varphi_\sigma b_\sigma \qquad ; \qquad \varphi^* = \sum \varphi_\sigma^* b_\sigma^* \ . \tag{4}$$

The conjugate complex operators b_σ and b_σ^* operate on the functions of the occupation numbers M_1,M_2,. . .,M_σ,. . . of the individual quantum states φ_1, φ_2,. . .,φ_σ,. . . of the neutrinos. If we assume that the Pauli principle also holds for these particles, the numbers M_σ can only assume the two values 0, 1; and one has

$$\begin{aligned} b_\sigma\ \ &\Phi\,(M_1, M_2, \ldots, M_\sigma, \ldots) \\ &= (-1)^{M_1+M_2+\ldots+M_{\sigma-1}}\,(1-M_\sigma)\,\Phi\,(M_1, M_2, \ldots, 1-M_\sigma, \ldots) \\ b_\sigma^*\ \ &\Phi\,(M_1, M_2, \ldots, M_\sigma, \ldots) \\ &= (-1)^{M_1+M_2+\ldots+M_{\sigma-1}}\,(1-M_\sigma)\,\Phi\,(M_1, M_2, \ldots, M_\sigma, \ldots)\ . \end{aligned} \tag{5}$$

The operators b_σ and b_σ^* determine the disappearance and the creation of a neutrino in the state σ, respectively.

The Hamiltonian function

§ 3. The energy of the overall system constituted by the heavy and the light particles is the sum of the energy $H_{\rm hea}$ of the heavy particles + the energy $H_{\rm lig}$ of the light particles + the interaction energy $\mathcal{H}$ between the light and heavy particles.

Limiting ourselves for the sake of simplicity to consider only the heavy particle, we shall write the first term in the form

$$H_{\rm hea} = \frac{1+\rho}{2}\mathcal{N} + \frac{1-\rho}{2}\mathcal{P} \tag{6}$$

in which $\mathcal{N}$ and $\mathcal{P}$ are the operators which represent the energy of the neutron and the proton. We notice in fact that, for $\rho = +\,1$ (neutron), (6) reduces to $\mathcal{N}$; while for $\rho = -\,1$ (proton) it reduces to $\mathcal{P}$.

To write the energy H_{lig} in the simplest way, we shall consider the quantum states ψ_s and φ_σ of the electrons and neutrinos to be stationary states. For the electrons we shall take the eigenfunctions in the Coulomb field of the nucleus (conveniently shielded in order to take into account the action of the atomic electrons); for the neutrinos we simply shall take De Broglie plane waves, since possible forces acting on neutrinos are certainly very weak. Let $H_1, H_2, \ldots, H_s, \ldots$ and $K_1, K_2, \ldots, K_\sigma, \ldots$ be the energies of the stationary states of the electrons and the neutrinos; then we shall have

$$H_{\text{lig}} = \sum_s H_s N_s + \sum_\sigma K_\sigma M_\sigma \ . \qquad (7)$$

There still remains to write the interaction energy. It consists first of the Coulomb energy between proton and electrons; however, in the case of heavy nuclei the attraction exercised by only a proton has no importance[3] and in any case does not contribute in any way to the process of β decay. In order not to uselessly complicate the problem, we shall neglect this term. We must instead add a term to the Hamiltonian such that it satisfies the condition c) of § 1.

A term which necessarily joins the transformation of a neutron into a proton with the creation of an electron and a neutrino has, according with the results of § 2, the form

$$Q^* a_s^* b_\sigma^* \qquad (8)$$

while the conjugate complex operator

$$Q a_s b_\sigma \qquad (8)$$

joins together the inverse processes (transformation of a proton into a neutron and disappearance of an electron and a neutrino).

An interaction term satisfying the condition c) will then have the following form

$$\mathcal{H} = Q \sum_{s\sigma} c_{s\sigma} a_s b_\sigma + Q^* \sum_{s\sigma} c_{s\sigma}^* a_s^* b_\sigma^* \ , \qquad (9)$$

where $c_{s\sigma}$ and $c_{s\sigma}^*$ are quantities which may depend on the coordinates, the momenta, etc.... of the heavy particle.

A further determination of $\mathcal{H}$ must necessarily follow the principle of greatest simplicity; in any case the choices for $\mathcal{H}$ are restricted by the fact that $\mathcal{H}$ must be invariant with respect to a change of coordinates and moreover it must also satisfy momentum conservation.

[3]The Coulomb attraction due to the many other protons must obviously be taken into account as a static field.

If at first we neglect spin and relativistic effects, the simplest choice for (9) is the following

$$\mathcal{H} = g\left[Q\psi(x)\varphi(x) + Q^*\psi^*(x)\varphi^*(x)\right] \ , \tag{10}$$

where g is a constant with dimensions L^5MT^{-2}; x represents the coordinates of the heavy particle; ψ, φ, ψ^*, φ^* are given by (2) and (4) and must be evaluated at the position x, y, z of the heavy particle.

Obviously (10) is not the only possible choice for $\mathcal{H}$; any scalar expression as

$$L(p)\psi(x)M(p)\varphi(x)N(p) + \text{compl. conj.}$$

where $L(p)$, $M(p)$, $N(p)$, represent convenient functions of the momentum of the heavy particle, would have been admissible. On the other hand, since until now the consequences of (10) have been in agreement with experience, there is no need to resort to more complicated expressions.

On the contrary, it is essential to generalize (10) in such a way to be able to treat relativistically at least the light particles. Of course, also in this generalization, it does not seem possible to eliminate all arbitrariness. However, the most natural solution of the problem appears to be the following: Relativistically we have, in place of ψ and φ, two sets $\psi_1\psi_2\psi_3\psi_4$ and $\varphi_1\varphi_2\varphi_3\varphi_4$ of four Dirac functions. Let us consider the 16 independent bilinear combinations of $\psi_1\psi_2\psi_3\psi_4$ and $\varphi_1\varphi_2\varphi_3\varphi_4$. When the frame of reference undergoes a Lorentz transformation, the 16 bilinear combinations undergo a linear substitution which gives a representation of the Lorenz group. In particular the four bilinear combinations

$$\begin{aligned} A_0 &= -\psi_1\varphi_2 + \psi_2\varphi_1 + \psi_3\varphi_4 - \psi_4\varphi_3 \\ A_1 &= \psi_1\varphi_3 - \psi_2\varphi_4 - \psi_3\varphi_1 + \psi_4\varphi_2 \\ A_2 &= i\psi_1\varphi_3 + i\psi_2\varphi_4 - i\psi_3\varphi_1 - i\psi_4\varphi_2 \\ A_3 &= -\psi_1\varphi_4 - \psi_2\varphi_3 + \psi_3\varphi_2 + \psi_4\varphi_1 \end{aligned} \tag{11}$$

transform like the components of a four-vector, that is like the components of the electromagnetic four-potential. Then it is natural to introduce in the Hamiltonian of the heavy particle the four quantities

$$g\left(QA_i + Q^*A_i^*\right)$$

in a situation corresponding to that of the components of the four-potential. Here we run into a problem depending on the fact that we do not know a relativistic wave equation for the heavy particles. However, in the case in which the velocity of the heavy particle is small compared to c, one can limit oneself to the term corresponding to eV (V the scalar potential) and write

$$\mathcal{H} = g\left[Q\left(-\psi_1\varphi_2 + \psi_2\varphi_1 + \psi_3\varphi_4 - \psi_4\varphi_3\right) + Q^*\left(\psi_1^*\varphi_2^* + \psi_2^*\varphi_1^* + \psi_3^*\varphi_4^* - \psi_4^*\varphi_3^*\right)\right] \ . \tag{12}$$

To this term one must add other ones of the order of magnitude v/c. At the moment, however, we shall neglect these terms, since the velocities of the neutrons and protons inside the nuclei are in general small compared to c (Cf. § 9).

In matrix language, (12) can be written

$$\mathcal{H} = g\left[Q\tilde{\psi}^*\delta\varphi + Q^*\tilde{\psi}\delta\varphi^*\right] \ , \tag{13}$$

where ψ and φ are meant as matrices with one column and the symbol $\sim$ transforms a matrix into its transposed conjugate; and moreover

$$\delta = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix} \ . \tag{14}$$

With this notation, one finds by comparing (12) with (9)

$$c_{s\sigma} = g\tilde{\psi}_s^*\delta\varphi_\sigma \ ; \qquad c_{s\sigma}^* = g\tilde{\psi}_s\delta\varphi_\sigma^* \ , \tag{15}$$

where ψ and φ represent the four-component normalized eigenfunctions of the states s of the electron and σ of the neutrino, considered as functions of the position x, y, z occupied by the heavy particle.

The perturbation matrix

§ 4. With the Hamiltonian we have established one can develop a theory of β decay in complete analogy with the theory of radiation. In that theory, as is known, the Hamiltonian consists of the sum: Energy of the atom + Energy of the radiation field + Interaction between atom and radiation; the latter term is considered as a perturbation of the other two. Analogously we shall take

$$H_{\text{hea}} + H_{\text{lig}} \tag{16}$$

as the unperturbed Hamiltonian. The perturbation is represented by the interaction term (13).

The quantum states of the unperturbed system can be enumerated in the following way:

$$(\rho, n, N_1, N_2 \ldots N_s \ldots M_1, M_2 \ldots M_\sigma \ldots) \ , \tag{17}$$

where the first number ρ takes one of the values ± 1 and indicates if the heavy particle is a neutron or a proton. The second number n indicates the quantum state of the neutron or the proton. For $\rho = +\,1$ (neutron) let the corresponding eigenfunction be

$$u_n(x) \ , \tag{18}$$

while for $\rho = -\,1$ (proton) let the eigenfunction be

$$v_n(x) \ . \tag{19}$$

The other numbers $N_1, N_2 \ldots N_s \ldots M_1, M_2 \ldots M_\sigma \ldots$ can only take the values 0, 1 and indicate what states of the electrons and neutrinos are occupied.

By an examination of the general form (9) of the perturbation energy, one immediately realizes that it has nonvanishing matrix elements only for transitions in which either the heavy particle passes from neutron to proton, while in the meantime one electron and one neutrino are created, or viceversa.

Through (1), (3), (5), (9), (18), (19) one easily finds that the corresponding matrix element is

$$\mathcal{H}_{-1mN_1N_2\ldots 1_s M_1 M_2 \ldots 1_\sigma \ldots}^{\;1nN_1N_2\ldots 0_s M_1 M_2 \ldots 0_\sigma \ldots} = \pm \int v_m^* c_{s\sigma}^* u_n d\tau \;, \tag{20}$$

where the integration must be extended over the entire configuration space of the heavy particle (with the exception of the coordinate ρ); the $\pm$ sign means more precisely

$$(-1)^{N_1+N_2+\ldots+N_{s-1}+M_1+M_2+\ldots M_{\sigma-1}}$$

and in any case does not enter into the calculations that will follow. To the inverse transition corresponds a matrix element which is the conjugate complex of (20).

Taking (15) into account, (20) becomes

$$\mathcal{H}_{-1m1_s1_\sigma}^{\;1n0_s0_\sigma} = \pm \int v_m^* u_n \tilde{\psi}_s \delta \varphi_\sigma^* d\tau \;, \tag{21}$$

where for the sake of brevity in the left hand side we have omitted writing all the indexes which do not change.

Theory of β decay

§ 5. A β decay consists of a process in which a nuclear neutron transforms into a proton, while at the same time, in the way we have described, an electron, which is observed as a β particle, and a neutrino are emitted. To calculate the probability of this process, we shall assume that, at the time $t = 0$, a neutron is in a nuclear state of eigenfunction $u_n(x)$, and furthermore the electron state s and the neutrino state σ are free, that is $N_s = M_\sigma = 0$. Then for $t = 0$ we shall put the probability amplitude of the state $(1, n, 0_s, 0_\sigma)$ equal to 1, that is

$$a_{1,n,0_s,0_\sigma} = 1 \;, \tag{22}$$

whereas we shall put the probability amplitude of the state $(-1, m, 1_s, 1_\sigma)$, in which the neutron has been transformed into a proton with eigenfunction $v_m(x)$ emitting an electron and a neutrino in the states s and σ initially equal to zero.

By applying the usual formulas of perturbation theory, for a time short enough to still consider (22) approximately valid one finds

$$\dot{a}_{-1,m,1_s,1_\sigma} = -\frac{2\pi i}{h} \mathcal{H}_{-1m1_s1_\sigma}^{\;1n0_s0_\sigma} e^{\frac{2\pi i}{h}(-W+H_s+K_\sigma)t} \;, \tag{23}$$

where W stands for the difference in energy between the neutron state and the proton state.

By integrating (23) we obtain (since for $t = 0$, $a_{-1m1_s1_\sigma} = 0$)

$$a_{-1m1_s1_\sigma} = -\mathcal{H}\,{}^{1n0_s0_\sigma}_{-1m1_s1_\sigma}\,\frac{e^{\frac{2\pi i}{h}(-W+H_s+K_\sigma)t} - 1}{-W + H_s + K_\sigma}\ . \tag{24}$$

The probability of the transition we consider is then

$$\left|a_{-1m1_s1_\sigma}\right|^2 = 4\left|\mathcal{H}\,{}^{1n0_s0_\sigma}_{-1m1_s1_\sigma}\right|^2 \frac{\sin^2 \frac{\pi t}{h}\left(-W + H_s + K_\sigma\right)}{\left(-W + H_s + K_\sigma\right)^2}\ . \tag{25}$$

To calculate the lifetime of the neutron state u_n it is necessary to sum (25) with respect to all unoccupied states of the electrons and neutrinos. A strong reduction of this sum can be obtained by observing that the De Broglie wave length for electrons or neutrinos having energies of some millions of volts is much larger than the nuclear sizes. Thus one can, as a first approximation, consider the eigenfunctions ψ_s and φ_σ to be constants inside the nucleus. Thus (21) becomes

$$\mathcal{H}\,{}^{1n0_s0_\sigma}_{-1m1_s1_\sigma} = \pm g\tilde{\psi}_s\delta\varphi^*_\sigma \int v^*_m u_n d\tau\ , \tag{26}$$

where here and below ψ_s and φ_σ are meant to be taken in the nucleus (Cf. § 8). From (26) we draw:

$$\left|\mathcal{H}\,{}^{1n0_s0_\sigma}_{-1m1_s1_\sigma}\right|^2 = g^2\left|\int v^*_m u_n d\tau\right|^2 \tilde{\psi}_s\delta\varphi^*_\sigma\tilde{\varphi}^*_\sigma\tilde{\delta}\psi_\sigma\ . \tag{27}$$

States σ of the neutrino are characterized by their momentum p_σ and by the spin direction. If, for the convenience of normalization, we quantize inside a volume Ω, whose size later on will be made to tend to infinity, the normalized neutrino eigenfunctions are Dirac plane waves having density $1/\Omega$. Then simple algebraic considerations allow us to perform in (27) an average with respect to all the orientations of p_σ and of the spin. (And in this only the states of positive energy must be considered; the negative energy states must be eliminated through a device like the Dirac hole theory). One finds

$$\overline{\left|\mathcal{H}\,{}^{1n0_s0_\sigma}_{-1m1_s1_\sigma}\right|^2} = \frac{g^2}{4\Omega}\left|\int v^*_m u_n d\tau\right|^2\left(\tilde{\psi}_s\psi_s - \frac{\mu c^2}{K_\sigma}\tilde{\psi}_s\beta\psi_s\right)\ , \tag{28}$$

where μ is the rest mass of the neutrino and β the Dirac matrix

$$\beta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \tag{29}$$

By observing that the number of positive energy neutrino states with momentum between p_σ and $p_\sigma + dp_\sigma$ is $8\pi\Omega p^2_\sigma dp_\sigma/h^3$, that furthermore $\partial K_\sigma/\partial p_\sigma$ is the neutrino velocity for the state σ, and finally that (25) has a strong maximum for the value of p_σ for which there is no variation of the unperturbed energy, that is

$$-W + H_s + K_\sigma = 0 \;, \tag{30}$$

one can perform the sum of (25) with respect to σ in the usual way[4] and one finds

$$t\frac{8\pi^3 g^2}{h^4}\left|\int v_m^* u_n d\tau\right|^2 \frac{p_\sigma^2}{v_\sigma}\left(\tilde{\psi}_s\psi_s - \frac{\mu c^2}{K_\sigma}\tilde{\psi}_s\beta\psi_s\right) \;, \tag{31}$$

where p_σ is the value of the momentum of the neutrino for which (30) holds.

Determining elements of the transition probability

§ 6. (31) expresses the probability that in a time t a β decay takes place in which the electron is emitted in the state s. As must be the case, this probability turns out to be proportional to the time (t has been considered small with respect to the lifetime); the coefficient of t gives the transition probability for the process we consider; it turns out to be

$$P_s = \frac{8\pi^3 g^2}{h^4}\left|\int v_m^* u_n d\tau\right|^2 \frac{p_\sigma^2}{v_\sigma}\left(\tilde{\psi}_s\psi_s - \frac{\mu c^2}{K_\sigma}\tilde{\psi}_s\beta\psi_s\right) \;. \tag{32}$$

Note that:

(a) For the free states of the neutrinos one always has $K_\sigma \geq \mu c^2$. Then it is necessary, in order that (30) can be satisfied, that

$$H_s \leq W - \mu c^2 \tag{33}$$

The upper limit of the β ray spectrum corresponds to the = sign.

(b) Secondly, since for the unoccupied electron state one has $H_s \geq mc^2$, we obtain, in order that the decay be possible, the following condition:

$$W \geq (m+\mu)c^2 \tag{34}$$

Then, in order that the β decay be possible, one must have a rather high occupied neutron state over a free proton state.

(c) According to (32), P_s depends on the eigenfunctions u_n and v_m of the heavy particle in the nucleus, through the matrix element

$$Q_{mn}^* = \int v_m^* u_n d\tau \tag{35}$$

This matrix element plays a role, in the in the theory of β rays, which is analogous to that of the matrix element of the electric moment in the theory of radiation. The matrix element (35) has normally the order of magnitude

[4]For a description of the methods used for performing such sums, cf. any expository article on the theory of radiation. For instance, E. Fermi *Rev. of Mod. Phys.* **4**, 87, (1932).

1; nevertheless it often happens that, due to particular symmetries of the eigenfunctions u_n and v_m, Q^*_{mn} exactly vanishes. In that case we shall speak of "forbidden β transitions". On the other hand, one should not expect that the forbidden transitions are really impossible, since (32) is only an approximate formula. We shall come back to this matter in § 9.

The mass of the neutrino

§ 7. The transition probability (32) determines among other things the shape of the continuous spectrum of β rays. We will discuss here how the shape of this spectrum depends on the rest mass of the neutrino, in order to be able to determine this mass through a comparison with the experimental shape of the spectrum itself. The mass μ also enters into (32) through the factor p_σ^2/v_σ. The dependence of the shape of the curve of the energy distribution on μ is particularly pronounced in the proximity of the maximum energy E_0 of the β rays. It is easy to recognize that the distribution curve for energies E close to the maximum value E_0, behaves, apart from a factor independent of E, as

$$\frac{p_\sigma^2}{v_\sigma} = \frac{1}{c^3}\left(\mu c^2 + E_0 - E\right)\sqrt{\left(E_0 - E\right)^2 + 2\mu c^2\left(E_0 - E\right)}\ . \qquad (36)$$

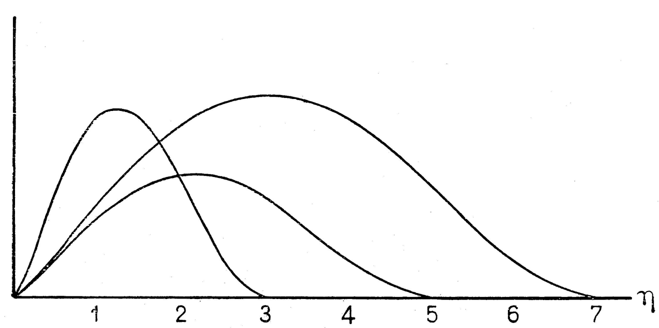

Figure 1.

In Figure 1 the end of the distribution curve is represented for $\mu = 0$, and for a small value and a large value of μ. The closest similarity of the theoretical curve to the experimental curves corresponds to $\mu = 0$. Thus we arrive at concluding that the mass of the neutrino is equal to zero or, in any case, much smaller than the mass of the electron[5]. In the calculations below, for the sake of simplicity, we always set $\mu = 0$.

[5]In a recent note F. Perrin, *C.R.*, **197**, 1625 (1933), by means of quantitative arguments arrives at a similar conclusion.

Then we have, also taking (32) into account

$$v_\sigma = c \; ; \qquad K_\sigma = cp_\sigma \; ; \qquad p_\sigma = \frac{K_\sigma}{c} = \frac{W - H_s}{c} \tag{37}$$

and the inequalities (33) and (34) become

$$H_s \leq W \; ; \qquad W \geq mc^2 \; . \tag{38}$$

Finally the transition probability takes the form

$$P_s = \frac{8\pi^3 g^2}{c^3 h^4} \left| \int v_m^* u_n d\tau \right|^2 \tilde{\psi}_s \psi_s \left(W - H_s\right)^2 \; . \tag{39}$$

Lifetime and shape of the energy distribution curve for allowed transitions

§ 8. From (39) one can derive a formula which expresses how many β transitions in which a β particle gets a momentum ranging from $mc\eta$ to $mc(\eta + d\eta)$ take place in unit time. For this it is necessary to calculate the sum of the values of $\tilde{\psi}_s \psi_s$ in the nucleus, extended to all the states (of the continuum) which belong to the indicated range of momentum. In this regard we point out that the relativistic eigenfunctions in the Coulomb field for the states with j=1/2 ($^2s_{1/2}$ e $^2p_{1/2}$) become infinite in the center. On the other hand the Coulomb law does not hold up to the center of the nucleus, but only up to a distance from it larger than R, where R is the nuclear radius. At this point, a tentative calculation shows that, if we make plausible assumptions on the behavior of the electric potential inside the nucleus, the value of $\tilde{\psi}_s \psi_s$ in the center of the nucleus turns out to be very close to the value which $\tilde{\psi}_s \psi_s$ should assume if the Coulomb law were valid at a distance R from the center. Applying the known formulas[6] for the relativistic eigenfunctions of the continuum spectrum in a Coulomb field, after a rather long but easy calculation, one finds

$$\sum_{d\eta} \tilde{\psi}_s \psi_s = d\eta \cdot \frac{32\pi m^3 c^3}{h^3 \left[\Gamma\left(3 + 2S\right)\right]^2} \left(\frac{4\pi mcR}{h}\right)^{2S} \eta^{2+2S} e^{\pi\gamma \frac{\sqrt{1+\eta^2}}{\eta}} \times$$
$$\times \left| \Gamma\left(1 + S + i\gamma \frac{\sqrt{1+\eta^2}}{\eta}\right) \right|^2 \; , \tag{40}$$

where we have set

$$\gamma = Z/137 \; ; \qquad S = \sqrt{1 - \gamma^2} - 1 \; . \tag{41}$$

[6] R.H. Hulme, *Proc. Roy. Soc.* **133**, 381 (1931).

The transition probability in an electric state in which the momentum has a value in the interval $mc\,d\eta$ then becomes (see (39))

$$P(\eta)d\eta = d\eta \cdot g^2 \frac{256\pi^4}{[\Gamma(3+2S)]^2} \frac{m^5c^4}{h^7} \left(\frac{4\pi mcR}{h}\right)^{2S} \left|\int v_m^* u_n d\tau\right|^2 \eta^{2+2S} \times$$
$$\times e^{\pi\gamma\frac{\sqrt{1+\eta^2}}{\eta}} \left|\Gamma\left(1+S+i\gamma\frac{\sqrt{1+\eta^2}}{\eta}\right)\right|^2 \left(\sqrt{1+\eta_0^2}-\sqrt{1+\eta^2}\right)^2 , \quad (42)$$

where η_0 is the maximum momentum of the emitted β rays, as measured in units of mc.

For a numerical evaluation of (42) we refer to the particular value $\gamma = 0.6$, which corresponds to $Z = 82.2$ since the atomic numbers of the radioactive substances are not far from this value. For $\gamma = 0.6$, we have from (41) $S = -0.2$. Moreover one finds that, for $\eta < 10$ it is possible to set, with a sufficient approximation

$$\eta^{1.6} e^{0.6\pi\frac{\sqrt{1+\eta^2}}{\eta}} \left|\Gamma\left(0.8+0.6i\frac{\sqrt{1+\eta^2}}{\eta}\right)\right|^2 \cong 4.5\eta + 1.6\eta^2 . \quad (43)$$

With this, (42) becomes, setting $R = 9 \cdot 10^{-13}$ in it,

$$P(\eta)d\eta = 1.75 \cdot 10^{95} g^2 \left|\int v_m^* u_n d\tau\right|^2 \left(\eta + 0.355\eta^2\right) \left(\sqrt{1+\eta_0^2}-\sqrt{1+\eta^2}\right)^2 . \quad (44)$$

The inverse of the lifetime is obtained by integrating (44) from $\eta = 0$ to $\eta = \eta_0$; one finds

$$\frac{1}{\tau} = 1.75 \cdot 10^{95} g^2 \left|\int v_m^* u_n d\tau\right|^2 F(\eta_0) , \quad (45)$$

where we have set

$$F(\eta_0) = \frac{2}{3}\sqrt{1+\eta_0^2} - \frac{2}{3} + \frac{\eta_0^4}{12} - \frac{\eta_0^2}{3} +$$
$$+0.355\left[-\frac{\eta_0}{4} - \frac{\eta_0^3}{12} + \frac{\eta_0^5}{30} + \frac{\sqrt{1+\eta_0^2}}{4}\log\left(\eta_0+\sqrt{1+\eta_0^2}\right)\right] . \quad (46)$$

For small values of the argument, $F(\eta_0)$ behaves like $\eta_0^6/24$; for larger values of the argument, the values of F are gathered together in the following table.

Table 1.

η_0	$F(\eta_0)$	η_0	$F(\eta_0)$	η_0	$F(\eta_0)$	η_0	$F(\eta_0)$
0	$\eta_0^6/24$	2	1.2	4	29	6	185
1	0.03	3	7.5	5	80	7	380

The forbidden transitions

§ 9. Before moving on to a comparison of the theory with experience, we still want to illustrate some properties of the forbidden transitions.

As we have already said, a transition is forbidden when the corresponding matrix element (35) vanishes. If the representation of the nucleus by means of individual quantum states of the protons and neutrons turns out to be a good approximation, the matrix element Q^*_{mn} vanishes, due to symmetry, when

$$i = i' \tag{47}$$

does not hold, where i and i' are the angular momentum, in units $h/2\pi$, of the neutron state u_n and the proton state v_m, respectively. When the individual quantum states do not turn out to be a good approximation, to the selection rule (47) coresponds the other one

$$I = I' \ , \tag{48}$$

where I and I' represent the angular momentum of the nucleus before and after the β decay.

The selection rules (47) and (48) are much less rigorous than the selection rules of optics. It is possible to find exceptions to them, particularly with the two following processes:

(a) Formula (26) has been obtained by neglecting the variations of ψ_s and φ_s inside the region of the nucleus. If on the contrary these variations are taken into account, one has the possibility of obtaining β transitions even when Q^*_{mn} vanishes. It is easy to recognize that the intensity of these transitions has a ratio, as an order of magnitude, with the intensity of the allowed processes given by $(R/\lambda)^2$, where λ is the De Broglie wave length of the light particles. It must be noted that, if the electron and the neutrino have the same energy, when the former is near the nucleus it has a higher kinetic energy, due to the electrostatic attraction and so the most important effect comes from the variations of ψ_s. An evaluation of the order of magnitude of the intensity of these forbidden processes show that, at the same energy of the emitted electrons, they must have an intensity of one hundredth of the intensity of the normal processes. Besides the relatively small intensity, a characteristics of the forbidden transitions of this type can be found in the different shape of the curve of the energy distribution of β rays, which, for the forbidden transitions, must give a number of particles with small energy lower than in the normal case.

(b) A second possibility to have β transitions forbidden by the rule (48) depends on the fact, already pointed out at the end of § 3, that when the velocity of neutrons and protons is not negligible in comparison with the velocity of light we must add to the interaction term (12) other terms of order v/c. If e.g. one would assume a relativistic wave equation of the Dirac type also for the heavy particles, one could add to (12) terms like

$$gQ\,(\alpha_x A_1 + \alpha_y A_2 + \alpha_z A_3) \;+\; \text{complex conjugate}\;, \tag{49}$$

where $\alpha_x\alpha_y\alpha_z$ are the usual Dirac matrices for the heavy particle and $A_1A_2A_3$ the spatial components of the four vector defined by (12). A term of the type (49) allows also β transitions which do not satisfy the selection rule (48), and their intensity is, with respect to that of normal processes, of the order of magnitude $(v/c)^2$, that is about 1/100. Thus we find a second possibility for forbidden transitions nearly 100 times less intense than the normal ones.

Comparison with experience

§ 10. (45) establishes a relation between the maximum momentum η_0 of the β rays emitted by a substance and its lifetime. In this relation, really, also an unknown element enters, the integral

$$\int v_m^* u_n d\tau \tag{50}$$

whose evaluation requires knowledge of the nuclear eigenfunctions u_n and v_m of the neutron and the proton. However, in the case of the allowed transitions, (50) is of the order of magnitude of unity. Then we expect the product

$$\tau F(\eta_0) \tag{51}$$

to have the same order of magnitude in all the allowed transitions. Instead, for the forbidden transitions, the lifetime will be, as an order of magnitude, one hundred times larger, and correspondingly also the product (51) will be larger. In the following table we collect the products $\tau F(\eta_0)$ for all the substances which disintegrate by emitting β rays and for which we have sufficiently exact data.

In this table the two groups we have expected are certainly recognizable; moreover such a division of the elements which emit primary β rays into two groups had been already observed experimentally by Sargent.[7] The values of η_0 have been taken from the quoted paper of Sargent (for a comparison, note that: $\eta_0 = (H\rho)_{max/1700}$). Besides the data in this table, Sargent gives the data for three other elements, warning that they are not as reliable as the other ones. They are UX_1 for which $\tau = 830$; $\eta_0 = 0.76$; $F(\eta_0) = 0.0065$; $\tau F(\eta_0) = 5.4$; then this element appears to be attributable to the first group. For AcB one has: $\tau = 0.87$; $\eta_0 = 1.24$; $F(\eta_0) = 0.102$; $\tau F(\eta_0) = 0.09$; then one finds a value of $\tau F(\eta_0)$ about ten times smaller than those of the first group. Finally for *RaD* one has: $\tau = 320000$; $\eta_0 = 0.38$ (largely uncertain); $F(\eta_0) = 0.00011$; $\tau F(\eta_0) = 35$. Then this element can be put roughly

[7] B.W. Sargent, *Proc. Roy. Soc.* **139**, 659, (1933).

Table 2.

Element	τ(hours)	η_0	$F(\eta_0)$	$\tau F(\eta_0)$
UX_2	0.026	5.4	115	3.0
RaB	0.64	2.04	1.34	0.9
ThB	15.3	1.37	0.176	2.7
ThC''	0.076	4.4	44	3.3
AcC''	0.115	3.6	17.6	2.0
RaC	0.47	7.07	398	190
RaE	173	3.23	10.5	1800
ThC	2.4	5.2	95	230
$MsTh_2$	8.8	6.13	73	640

half-way between the two groups. I have not succeeded in finding data for the other elements which emit primary β rays, that is Ms, Th_1, UY, Ac, AcC, UZ, RaC''.

On the whole one can conclude from this comparison between theory and experience that the agreement is certainly as good as one would have expected. The discrepancies observed for the elements with uncertain experimental data, RaD and AcB, can be explained well partly by the lack of precision of the measures, partly also by oscillations, quite plausible, in the value of the matrix element (50). Moreover one must notice that the fact that the majority of β decays are accompanied by emission of γ rays indicates that the larger part of the β processes can leave the proton in different excitation states and this gives a further mechanism which can determine oscillations in the value of $\tau F(\eta_0)$.

From the data of Table 2 one can infer an evaluation, even if rough, of constant g. If we admit, for instance, that when the matrix element (50) has the value 1, one has $\tau F(\eta_0) = 1\, hour = 3600\, s$; one finds from (45)

$$g = 4 \cdot 10^{-50} cm^3 \cdot erg$$

which gives nothing more than the order of magnitude.

Let us move on to discuss the shape of the curve of the velocity distribution of β rays. In the case of allowed processes, the distribution curve, as a function of η (that is, apart from a factor 1700, of $H\rho$) is represented in Fig. 2, for values of the maximum momentum η_0.

The curves are satisfactorily similar to experimental ones collected by Sargent.[8] Only in the range of small energy Sargent's curves are a little lower than the theoretical ones, and this is more easily evident in the curves of Fig. 3 where the abscissas are the energies instead of the momenta. But we must remark that the part of the curves of small energy is not perfectly known experimentally.[9] Moreover, for

[8]B.W. Sargent, *Proc. Camb. Phil. Soc.* **28**, 538 (1932).
[9]Cf. e.g., Rutherford, Ellis and Chadwick, *Radiation from Radio-active Substances*, Cambridge, 1930. See, in particular p. 407.

the forbidden transitions, also theoretically, in the range of small energies the curve must be lower than the curves of the allowed transitions, represented in Figs. 2 and 3. Of this fact one must particularly take into account for the case of RaE, which is the best known from an experimental point of view. The emission of β rays from this element, as results from the abnormally large value of $\tau F(\eta_0)$ (Cf. Table 2), is certainly forbidden, or better it is possible that it is allowed only in the second approximation. I hope, in a future article, to be able to better specify the behavior of distribution curves for the forbidden transitions.

To summarize, it seems justified to assert that the theory in the form described here does agree with the experimental data, which in any case are not always sufficiently accurate. On the other hand, even if in a further comparison of the theory with experience, one should arrive at some discrepancy, it would be always be possible to modify the theory without changing its conceptual foundations in an essential way. It would be possible precisely to keep equation (9) but choose the $c_{s\sigma}$ in a different way. This will carry us, in particular, to a different form of the selection rule (48) and to a different form of the curve of the energy distribution.

Only a further development of the theory, as also an increase in the precision of the experimental data, will be able to indicate if such a change will be necessary.

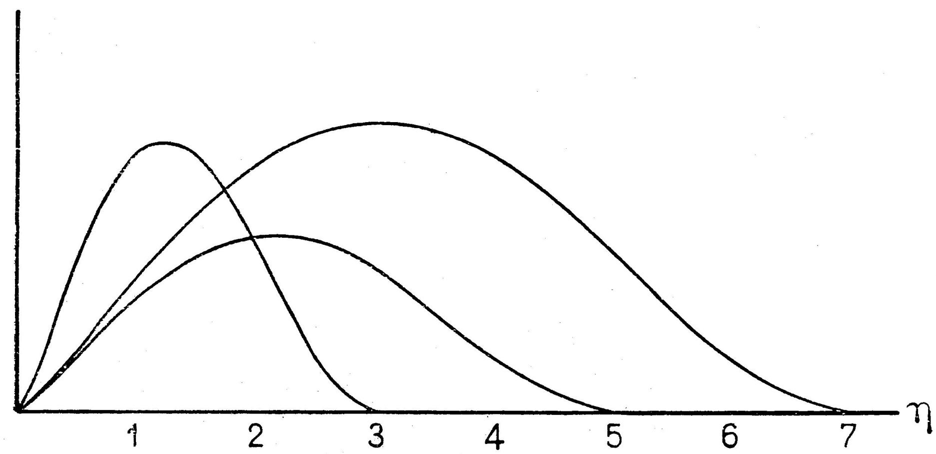

Figure 2.

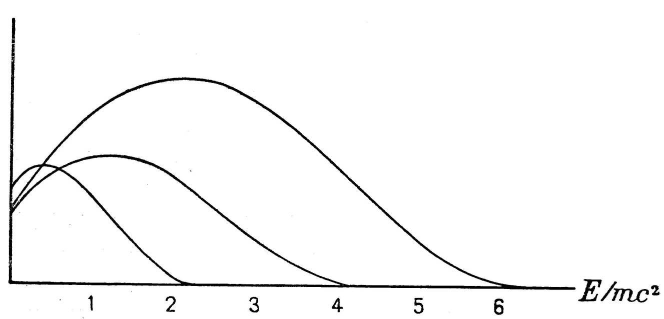

Figure 3.