# 38B.
# A THEOREM OF CALCULATION OF PROBABILITY AND SOME OF ITS APPLICATIONS

Teacher's Diploma Thesis of the Scuola Normale di Pisa.
Presented on June 20, 1922.

"Un teorema di calcolo delle probabilità ed alcune sue applicazioni"
Tesi di Abilitazione della Scuola Normale Superiore di Pisa

§ 1. The theorem we want to deal with concerns the properties of the sums of many incoherent addenda having a known statistical distribution. The fundamental theorem on these sums is due to Laplace[1]. We state the theorem together with a short account of its demonstration from which we shall move to establish a new theorem.

Let $n$ be a very great number and let $y_1, y_2 \dots y_n$ represent $n$ unknowns, for each of which we know the statistical distribution; that is, we know that the probability that $y_i$ has a value ranging between $y_i$ and $y_i + dy_i$ is $\varphi_i(y_i)dy_i$, where $\varphi_i$ is a known function for which, obviously

$$\int_{-\infty}^{\infty} \varphi_i(y)dy = 1 \ , \tag{1}$$

which means that $y_i$ must certainly have a value between $-\infty$ and $+\infty$.

In addition we will assume that the statistical distribution of $y_i$ is not affected by the values that the other $y$'s can assume, that is, we assume the $y_i$'s are completely incoherent among themselves. Then we take the average of the $y_i$ to be zero, namely:

$$\bar{y}_i = \int_{-\infty}^{\infty} y\varphi_i(y)dy = 0 \ . \tag{2}$$

Finally we set the average of the squared $y_i$ to be

$$\bar{y}_i^2 = \int_{-\infty}^{\infty} y^2\varphi_i(y)dy = k_i^2 \tag{3}$$

and assume that, for any $i$, $k_i^2$ is negligible with respect to $\sum_1^n k_i^2$.

Under these assumptions, Laplace's theorem holds which says that:

[1] Théorie analytique des probabilités, Oeuvres, VII, p. 309.

The probability that inequalities

$$x \leq \sum_{1}^{n} y_i \leq x + dx \tag{4}$$

hold at the same time is given by

$$F(x)dx = \frac{1}{\sqrt{2\pi \sum_1^n k_i^2}} e^{-\frac{x^2}{2\sum_1^n k_i^2}} dx \ . \tag{5}$$

To demonstrate this, we indicate by $r$ a number $\leq n$ and let $F(r,x)dx$ be the probability that the inequalities

$$x \leq \sum_{1}^{r} y_i \leq x + dx \tag{6}$$

hold.

Now if $p$ is any value, let us look for the probability that inequalities

$$\sum_{1}^{r-1} y_i < p < \sum_{1}^{r} y_i \tag{7}$$

hold simultaneously, that is, that the addition of $y_r$ to $\sum_1^{r-1} y_i$ does not exceed $p$. This probability is obviously given by

$$\int_0^\infty d\xi \ F(r-1, p-\xi) \int_\xi^\infty \varphi_r(y) dy \ .$$

Analogously, the probability that inequalities

$$\sum_{1}^{r-1} y_i > p > \sum_{1}^{r} y_i \tag{8}$$

hold simultaneously is

$$\int_0^\infty d\xi \ F(r-1, p+\xi) \int_\xi^\infty \varphi_r(y) dy \ .$$

The difference of these two probabilities is obviously given by the difference between the probability that $\sum_1^r y_i > p$ and the probability that $\sum_1^{r-1} y_i > p$, that is by

$$\int_p^\infty F(r,x)dx - \int_p^\infty F(r-1,x)dx \ \ .$$

Then we have

$$\int_p^\infty F(r,x)dx - \int_p^\infty F(r-1,x)dx = \int_0^\infty d\xi \ F(r-1,p-\xi) \int_\xi^\infty \varphi_r(y)dy$$

$$- \int_0^\infty d\xi F(r-1,p+\xi) \int_\xi^\infty \varphi_r(y)dy \ .$$

On the right hand side we can reverse the integrations using the formulas

$$\int_0^\infty d\xi \int_\xi^\infty dy = \int_0^\infty dy \int_0^y d\xi \quad ; \quad \int_0^\infty d\xi \int_{-\infty}^{-\xi} dy = \int_{-\infty}^0 dy \int_0^{-y} d\xi$$

and it becomes, also changing $\xi$ into $-\xi$ in the second term

$$\int_{-\infty}^{\infty} \varphi_r(y) dy \int_0^y F(r-1, p-\xi) d\xi \ .$$

We put, as an approximation

$$F(r-1, p-\xi) = F(r-1, p) - \xi \frac{\partial F(r-1,p)}{\partial p} \ .$$

Thus the above expression becomes

$$F(r-1,p) \int_{-\infty}^{\infty} \varphi_r(y) dy \int_0^y d\xi - \frac{\partial F(r-1,p)}{\partial p} \int_{-\infty}^{\infty} \varphi_r(y) dy \int_0^y \xi d\xi$$

$$= F(r-1,p) \int_{-\infty}^{\infty} y \varphi_r(y) dy - \frac{1}{2} \frac{\partial F(r-1,p)}{\partial p} \int_{-\infty}^{\infty} y^2 \varphi_r(y) dy$$

namely, remembering (2) and (3):

$$-\frac{k_r^2}{2} \frac{\partial F(r-1,p)}{\partial p}.$$

In this way we obtain the equality

$$\int_p^\infty F(r,x) dx - \int_p^\infty F(r-1,x) dx = -\frac{k_r^2}{2} \frac{\partial F(r-1,p)}{\partial p}. \tag{9}$$

Differentiating it with respect to $p$ we obtain

$$-F(r,p) + F(r-1,p) = -\frac{k_r^2}{2} \frac{\partial^2 F(r-1,p)}{\partial p^2} \ . \tag{10}$$

Let us change in it $r-1$ into $r$, $p$ into $x$, and in our approximation, set

$$F(r+1,x) - F(r,x) = \frac{\partial}{\partial r} F(r,x) \ .$$

Then (10) gives for $F(r,x)$ the differential equation

$$\frac{\partial}{\partial r} F(r,x) = -\frac{k_{r+1}^2}{2} \frac{\partial^2}{\partial x} F(r,x) \ . \tag{11}$$

Changing $r$ into another variable

$$t = \int_0^{r+1} k_i^2 di \tag{12}$$

(11) becomes

$$\frac{\partial F}{\partial t} = \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \ . \tag{13}$$

Then one has obviously the condition that for any $t$

$$\int_{-\infty}^{\infty} F dx = 1 \tag{14}$$

and that for $t = 0$, $F$ has a nonvanishing value only when $|x|$ is infinitesimal. It is known that these conditions are more than sufficient to determine $F$. They are satisfied by setting

$$F = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \ .$$

By giving to $t$ its value, which to our degree of approximation is $\sum_1^r k_i^2$, we find

$$F(r, x) = \frac{1}{\sqrt{2\pi \sum_1^r k_i^2}} e^{-\frac{x^2}{2\sum_1^r k_i^2}} \ . \tag{15}$$

Then one obviously has $F(x) = F(n, x)$, and therefore

$$F(x) = \frac{1}{\sqrt{2\pi \sum_1^n k_i^2}} e^{-\frac{x^2}{2\sum_1^n k_i^2}} \qquad \text{Q.E.D.}$$

§ 2. Let us mantain the notations and the assumptions made at the beginning of the previous section and in addition assume that all $\varphi_i(y)$ are equal (as a consequence we will supress their index). Then let us indicate by $a$ any positive value. Thus we can state the following

**Theorem 0.1.** *(REMOVE THEOREM NUMBER!!) The probability that at least one of the quantities*

$$y_1, y_1 + y_2, y_1 + y_2 + y_3, \ldots, \sum_1^n y_n$$

*exceeds $a$ is given by*

$$\frac{2}{\sqrt{\pi}} \int_{\frac{a}{\sqrt{2nk^2}}}^{\infty} e^{-x^2} dx$$

*provided that $a$ is sufficiently large compared to $k$.*

In particular, if $n$ tends to infinity, such a probability tends to 1, i.e., to certainty.

To demonstrate this, let us indicate by $F(r, x)dx(x < a)$ the probability that the inequalities (6) are fulfilled and in addition all $r$ quantities

$$y_1, y_1 + y_2, \ldots, \sum_1^r y_i \tag{16}$$

are smaller than $a$. The same arguments of the previous section now show us that $F(r, x)$ still will satisfy the differential equation (11), which in this case can be written as

$$\frac{\partial F}{\partial r} = \frac{k^2}{2} \frac{\partial^2 F}{\partial x^2} \ . \tag{17}$$

The boundary conditions will instead be changed.

In fact, we observe that

$$\int_{-\infty}^{a} F(r,x)dx$$

gives the probability that none of quantities (16) exceeds $a$ and then

$$-\int_{-\infty}^{a} F(r+1,x)dx + \int_{-\infty}^{a} F(r,x)dx$$

gives the proability that, because of the addition of $y_{r+1}$, $\sum_1^r y_i$ is made to exceed $a$. A calculation analogous to that performed in the previous section shows us that this probability is

$$\int_0^{\infty} F(r,a-\xi)d\xi \int_{\xi}^{\infty} \varphi(y)dy$$

i.e., to our degree of approximation, neglecting $\xi$ with respect to $a$

$$F(r,a)\int_0^{\infty} d\xi \int_{\xi}^{\infty} \varphi(y)dy$$

or by reversing the quadratures

$$F(r,a)\int_0^{\infty} \varphi(y)dy \int_0^{y} d\xi = F(r,a)\int_0^{\infty} y\varphi(y)dy.$$

By now setting

$$h = \int_0^{\infty} y\varphi(y)dy \tag{18}$$

we find

$$\int_{-\infty}^{a} \{F(r+1,x) - F(r,x)\}\, dx = -hF(r,a)\ .$$

But, to our usual degree of approximation, we can set

$$F(r+1,x) - F(r,x) = \frac{\partial F(r,x)}{\partial r}$$

and the previous equation becomes

$$\frac{\partial}{\partial r}\int_{-\infty}^{a} F(r,x)dx = -hF(r,a)\ . \tag{19}$$

After all, our unknown function $F$ must satisfy the differential equation (17) in the interval $-\infty, a$ and must satisfy equation (19) at the extremity $a$; then it must vanish together with its derivatives at the extremity $-\infty$ and, for $r = 0$, have a nonvanishing value only for very small $|x|$, but with the condition that the area coontained between it and the $x$-axis is $= 1$.

It is easy to prove that at least when $h$ is positive, as in our case, these conditions are sufficient to determine $F$. Therefore, we observe that, by multiplying (17) by $dx$ and integrating it between $-\infty$ and $a$, one finds

$$\frac{k^2}{2}\left(\frac{\partial F}{\partial x}\right)_a = \frac{\partial}{\partial r}\int_{-\infty}^{a} F(r,x)dx$$

with which (19) becomes

**[This equation should be numbered (19')!!]**

$$\frac{k^2}{2h}\left(\frac{\partial F(r,x)}{\partial x}\right)_a + F(r,a) = 0\ . \tag{19}$$

Then, for our purposes, it is evidently sufficient to prove that, if a function $\Phi(r,x)$ is $=0$ for $r=0$ and satisfies the equations

$$\frac{\partial \Phi}{\partial r} = \frac{k^2}{2}\frac{\partial^2 \Phi}{\partial x^2} \qquad ; \qquad \frac{k^2}{2h}\left(\frac{\partial \Phi}{\partial x}\right)_{x=a} + \phi(r,a) = 0 \tag{20}$$

and, for $x=-\infty$, it is always zero, it is certainly identically zero. In fact one has

$$\int_{-\infty}^{a}\left(\frac{\partial \Phi}{\partial x}\right)^2 dx = \int_{-\infty}^{a}\frac{\partial}{\partial x}\left(\Phi\frac{\partial \Phi}{\partial x}\right)dx - \int_{-\infty}^{a}\Phi\frac{\partial^2 \Phi}{\partial x^2}dx$$

that is, due to (20)

$$\int_{-\infty}^{a}\left(\frac{\partial \Phi}{\partial x}\right)^2 dx = \left(\Phi\frac{\partial \Phi}{\partial x}\right)_{-\infty}^{a} - \frac{2}{k^2}\int_{-\infty}^{a}\Phi\frac{\partial \Phi}{\partial r}dx =$$

$$= \Phi(r,a)\left(\frac{\partial \Phi}{\partial x}\right)_{x=a} - \frac{1}{k^2}\frac{\partial}{\partial r}\int_{-\infty}^{a}\Phi^2 dx = -\frac{2h}{k^2}\Phi^2(r,a) - \frac{1}{k^2}\frac{\partial}{\partial r}\int_{-\infty}^{a}\Phi^2 dx$$

i.e.,

$$\int_{-\infty}^{a}\left(\frac{\partial \Phi}{\partial x}\right)^2 dx + -\frac{2h}{k^2}\Phi^2(r,a) + \frac{1}{k^2}\frac{\partial}{\partial r}\int_{-\infty}^{a}\Phi^2(r,x)dx = 0\ . \tag{21}$$

Let us now suppose that, for some value of $r$ and $x$, $\Phi$ could be different from zero; then for some value $\overline{r}$ of $r$ $\int_{-\infty}^{a}\Phi^2 dx$ would be certainly positive; in addition, since for $r=0$ is $\phi=0$, and then $\int_{-\infty}^{a}\Phi^2(0,x)dx = 0$, there will be certainly between zero and $\overline{r}$ some value of $r$ for which $\frac{d}{dr}\int_{-\infty}^{a}\Phi^2(r,x)dx$ is positive. Now, the first two terms in (21) cannot be negative; the first one is, at least in some cases, positive and this is absurd. Then we will certainly always have $\phi(r,x)=0$. Q.E.D.

Granted this, it will be enough for us to find any solution whatsoever satisfying the imposed conditions cecause we are sure it is the solution we were looking for.

Let us try to see if our conditions can be satisfied by setting

$$F(r,x) = \frac{1}{k\sqrt{2\pi r}}e^{-\frac{x^2}{2rk^2}} - \frac{1}{k\sqrt{2\pi}}\int_0^r \frac{u(\rho)e^{-\frac{(a-x)^2}{2(r-\rho)k^2}}}{\sqrt{r-\rho}}d\rho \tag{22}$$

where $u(\rho)$ is a function to be determined.

With this position, the differential equation (17) and the limit conditions for $x = -\infty$ and $r = 0$ are certainly satisfied. Then it remains to determine $u(\rho)$ so that (19) is satisfied as well.

Now from (22) we have

$$F(r,a) = \frac{1}{k\sqrt{2\pi r}} e^{-\frac{a^2}{2rk^2}} - \frac{1}{k\sqrt{2\pi}} \int_0^r \frac{u(\rho)d\rho}{\sqrt{r-\rho}}$$

$$\int_{-\infty}^a F(r,x)dx = \frac{1}{k\sqrt{2\pi r}} \int_{-\infty}^a e^{-\frac{x^2}{2rk^2}} dx - \frac{1}{k\sqrt{2\pi}} \int_0^r \frac{u(\rho)d\rho}{\sqrt{r-\rho}} \int_{-\infty}^a e^{-\frac{(a-x)^2}{2(r-\rho)k^2}} dx =$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{a}{k\sqrt{2r}}} e^{-x^2} dx - \frac{1}{2} \int_0^r u(\rho) d\rho \qquad (23)$$

and then

$$\frac{\partial}{\partial r} \int_{-\infty}^a F(r,x)dx = -\frac{a e^{-\frac{a^2}{2rk^2}}}{2k\sqrt{2\pi r^3}} - \frac{1}{2} u(r)$$

in this way (19) becomes

$$\frac{e^{-\frac{a^2}{2rk^2}}}{k\sqrt{2\pi r}} \left( h - \frac{a}{2r} \right) = \frac{h}{k\sqrt{2\pi}} \int_0^r \frac{u(\rho)d\rho}{\sqrt{r-\rho}} + \frac{u(r)}{2} \qquad (24)$$

namely an integral equation of second kind for the unknown function $u(\rho)$. In spite of all our efforts, we have not suceeded in solving it exactly; we only have an approximate solution. We shall deal with this in a little while. We want to prove first, without approximations, that one has

$$\int_0^\infty u(r)dr = 1 \; .$$

Therefore, let $\vartheta$ be an arbitrary positive quantity and let us multiply both sides of (24) by $\sqrt{\theta} e^{-\theta r} dr$ and integrate then from $r = 0$ to $r = \infty$. One finds

$$\frac{\sqrt{\theta} h}{k\sqrt{2\pi}} \int_0^\infty \frac{e^{-\theta r - \frac{a^2}{2rk^2}}}{\sqrt{r}} dr - \frac{a\sqrt{\theta}}{2k\sqrt{2\pi}} \int_0^\infty \frac{e^{-\theta r - \frac{a^2}{2rk^2}}}{r^{3/2}} dr =$$

$$= \frac{h\sqrt{\theta}}{k\sqrt{2\pi}} \int_0^\infty e^{-\theta r} dr \int_0^r \frac{u(\rho)d\rho}{\sqrt{r-\rho}} + \frac{\sqrt{\theta}}{2} \int_0^\infty e^{-\theta r} u(r) dr =$$

$$= \frac{h\sqrt{\theta}}{k\sqrt{2\pi}} \int_0^\infty u(\rho)d\rho \int_\rho^\infty \frac{e^{-\theta r} dr}{\sqrt{r-\rho}} + \frac{\sqrt{\theta}}{2} \int_0^\infty e^{-\theta r} u(r) dr =$$

$$= \frac{h}{k\sqrt{2}} \int_0^\infty e^{-\theta \rho} u(\rho) d\rho + \frac{\sqrt{\theta}}{2} \int_0^\infty e^{-\theta r} u(r) dr \; .$$

In addition one has

$$\sqrt{\theta} \int_0^\infty \frac{e^{-\theta r - \frac{a^2}{2rk^2}}}{\sqrt{r}} dr = 2 \int_0^\infty e^{-x^2 - \frac{a^2 \theta}{2k^2 x^2}} dx = \sqrt{\pi} e^{-\frac{a\sqrt{2\theta}}{k}} \; .$$

Passing to the limit for $\theta = 0$ the above equation then becomes

$$\frac{h}{k\sqrt{2}} = \frac{h}{k\sqrt{2}} \int_0^\infty u(\rho) d\rho .$$

From which

$$\int_0^\infty u(\rho) d\rho = 1 \qquad \text{Q.E.D.} \tag{25}$$

At this point we can already get an interesting result. In fact from (23) we have

$$\lim_{r=\infty} \int_{-\infty}^{a} F(r,x) dx = \lim_{r=\infty} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{a}{k\sqrt{2r}}} e^{-x^2} dx - \frac{1}{2} \int_0^\infty u(r) dr = 0 \ . \tag{26}$$

If we remember the meaning of $F(r,x)$ this result can be read:

The probability that at least one of values (16) exceeds $a$ becomes certainty when $r$ tends to infinity. We remark that this result holds true independently of the approximation we are going to make to solve (24).

Let us pass now to the approximate solution of (24).

For this we observe that, as one can immediately verify

$$w(r) = \frac{a e^{-\frac{a^2}{2rk^2}}}{k\sqrt{2\pi r^3}} \tag{27}$$

is a solution of the integral equation of second kind

$$\frac{e^{-\frac{a^2}{2rk^2}}}{k\sqrt{2\pi r}} \left( h + \frac{a}{2r} \right) = \frac{h}{k\sqrt{2\pi}} \int_0^r \frac{w(\rho) d\rho}{\sqrt{r-\rho}} + \frac{1}{2} w(r) \tag{28}$$

which differs from (24) only in the sign inside the bracket of the left-hand side. Now, owing to the assumptions we have made, whenever $r$ is great enough so that $e^{-\frac{a^2}{2rk^2}}$ is not too small $a/2r$ is negligible with respect to $h$ and then we shall be allowed to assume $w(r)$ as an approximate solution of (24), by putting

$$u(r) = \frac{a e^{-\frac{a^2}{2rk^2}}}{k\sqrt{2\pi r^3}} \tag{29}$$

It is easy to check that from (29) it is $\int_0^\infty u(r) dr = 1$.

Now from (23) we get

$$\int_{-\infty}^{a} F(r,x) dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{a}{k\sqrt{2r}}} e^{-x^2} dx - \frac{1}{2} \frac{a e^{-\frac{a^2}{2\rho k^2}}}{k\sqrt{2\pi\rho^3}} d\rho =$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{a}{k\sqrt{2r}}} e^{-x^2} dx - \frac{1}{\sqrt{\pi}} \int_{\frac{a}{k\sqrt{2r}}}^{\infty} e^{-x^2} dx = 1 - \frac{2}{\sqrt{\pi}} \int_{\frac{a}{k\sqrt{2r}}}^{\infty} e^{-x^2} dx \ .$$

And then

$$1 - \int_{-\infty}^{a} F(r,x) dx = \frac{2}{\sqrt{\pi}} \int_{\frac{a}{k\sqrt{2r}}}^{\infty} e^{-x^2} dx \ . \tag{30}$$

Remembering now the meaning of $F(r, x)$ one immediately realizes that

$$1 - \int_{-\infty}^{a} F(r, x) dx$$

represents the probability that at least one of expressions (16) is greater than $a$. Therefore (30) completely demonstrates the theorem we have enunciated.

§ 3. The theorem just proved is susceptible of an immediate application to a famous theorem of calculus of probability:

Peter and Paul play a game of chance. In each game each one has probability 1/2 to win; the stake is always $k$ lire. Now Peter is infinitely rich, on the contrary Paul owns only $a$ lire. If at a certain moment Peter is able to win all the substance of Paul, the latter is ruined and is obliged to stop the game. So we are in the case considered in the above theorem and we can conclude that, after a sufficient number of games Peter will certainly ruin Paul; moreover, if $a$ is much greater than $k$ the probability that this fact happens in $n$ games is

$$\frac{2}{\sqrt{\pi}} \int_{\frac{a}{k\sqrt{2n}}}^{\infty} e^{-x^2} dx$$

§ 4. We want now to apply the above theorem to an astronomical problem.

Let us consider an elliptic comet which intersects Jupiter's orbit. The cometary orbit will obviously be perturbed by the action of Jupiter, and this particularly when Jupiter and the comet pass very close. Now it may happen that in these continuous transformations the comet's orbit ends by changing into a parabolic or hyperbolic orbit; then the comet will go away forever escaping from the attraction of Jupiter and the Sun. I want to study what is the probability that this happens in a certain time.

As far as I know the theory of the influence of Jupiter on the cometary orbits has never been studied from this point of view; people only dealt with this matter[2] looking for an explanation of the capture of comets with parabolic orbits when passing by chance close to Jupiter.

We will make the following simplifying assumptions, the same of the restricted 3-body problem: The comet has a negligible mass, so that it does not perturb either Jupiter or the Sun. The mass of Jupiter ($m$) is negligible with respect to the mass of the Sun (M). In this way we are allowed to assume the Sun is fixed and to consider the orbit of the comet being appreciably perturbed only when passing in the close vicinity of Jupiter. Jupiter's orbit is circular. Comet's orbit is coplanar

[2]TISSERAND, ≪*Traité de mécanique céleste*≫, vol. IV, pp. 198-216; CALLANDREAU, ≪Ann. de l'observatoire≫ vol. 22; A. NEWTON, ≪Mem. of the Nat. Acad. of Sci.≫, vol. 6.

with Jupiter's orbit. We let $u$ be the velocity of Jupiter and V the velocity of the comet when it crosses Jupiter's orbit with respect to a reference frame moving along this orbit with velocity $u$; we indicate by $\theta$ the angle between the direction of V and Jupiter's orbit. If $v$ is the absolute velocity of the comet, when it is crossing Jupiter's orbit one will have

$$v^2 = u^2 + \mathrm{V}^2 + 2u\mathrm{V}\cos\theta \; . \tag{31}$$

Let us suppose that once, while the comet is crossing Jupiter's orbit, it passes very close to this planet. Then it will be affected by a strong perturbation. Let $b$ be the smallest distance between the two bodies if they were not attracted to one another. According to our assumptions, in order that the perturbation be considerable $b$ must be very small compared to the curvature radii of the two unperturbed orbits so that, during this "collision", the comet will essentially describe a Keplerian hyperbolic orbit during its motion around Jupiter.

§ 5. Thus, let us consider this relative motion, referring to polar coordinates $(r, \varphi)$ having Jupiter as a pole and the polar axis parallel to the direction of the incoming comet.

Since the motion is Keplerian, we have

$$\frac{1}{r} = \mathrm{A} - \mathrm{B}\cos(\varphi - \varphi_0) \tag{32}$$

where A, B, $\varphi_0$ are constants.

Moreover, for $\varphi = 0$, $r$ must be infinite, that is

$$\mathrm{A} - \mathrm{B}\cos\varphi_0 = 0 \; . \tag{33}$$

Then it must be that

$$b = \lim_{r=\infty} r\sin\varphi = \lim_{\varphi=0} \frac{\sin\varphi}{\mathrm{A} - \mathrm{B}\cos(\varphi - \varphi_0)} = -\frac{1}{\mathrm{B}\sin\varphi_0} \; . \tag{34}$$

The area constant is then evidently V$b$ and owing to the well known formulas of Keplerian motion one has

$$\mathrm{A} = \frac{m}{\mathrm{V}^2 b^2} \; . \tag{35}$$

From (33) and (34) we can now obtain the other two constants. One finds exactly

$$\tan\varphi_0 = -\frac{\mathrm{V}^2 b}{m} \quad , \quad \mathrm{B} = \frac{1}{b}\sqrt{1 + \frac{m^2}{b^2 \mathrm{V}^4}} \; . \tag{36}$$

Now, let $\psi$ be the angle between the direction of the comet when approaching and its direction when going away. Obviously one will have:

$$\psi = 2\varphi_0 - \pi$$

and then

$$\tan\frac{\psi}{2} = -\cot\varphi_0 = \frac{m}{\mathrm{V}^2 b} \; . \tag{37}$$

We can conclude that the perturbation consists of keeping V unchanged and altering $\theta$ by the angle $\psi$ given by (37).

Now it is convenient to calculate the averages of the squares of $\psi$. Therefore we observe that one has:

$$\psi = 2 \arctan \frac{m}{\mathrm{V}^2 b}$$

and then

$$\int_{-\infty}^{\infty} \psi^2 db = 4 \int_{-\infty}^{\infty} \left( \arctan \frac{m}{\mathrm{V}^2 b} \right)^2 db =$$

$$= \frac{4m}{\mathrm{V}^2} \int_{-\infty}^{\infty} \left( \arctan \frac{1}{x} \right)^2 dx = \frac{8m}{\mathrm{V}^2} \int_0^{\infty} \left( \arctan \frac{1}{x} \right)^2 dx$$

by setting

$$h = \int_0^{\infty} \left( \arctan \frac{1}{x} \right)^2 dx \approx 2.5$$

then one has

$$\int_{-\infty}^{\infty} \psi^2 db = \frac{8mh}{\mathrm{V}^2} \ . \tag{38}$$

Now, $b$ being very small, the probability that its value lies between $b$ and $b + db$ is obviously

$$\frac{db}{2\pi \mathrm{R} \sin \theta}$$

where R is the radius of Jupiter's orbit.

The average of the squares of $\psi$ therefore is

$$\bar{\psi}^2 = \int_{-\infty}^{\infty} \psi^2 \frac{db}{2\pi \mathrm{R} \sin \theta} = \frac{4mh}{\pi \mathrm{R}\ \mathrm{V}^2 \sin \theta} \ . \tag{39}$$

§ 6. In its motion around the Sun the energy constant of our comet is given by

$$\frac{v^2}{2} - \frac{\mathrm{M}}{\mathrm{R}} = \mathrm{W} \ .$$

As is well known, a Keplerian orbit is elliptic, parabolic or hyperbolic accordingly as the energy constant is negative, null or positive; now remembering (31) we find for our comet:

$$\mathrm{W} = \frac{1}{2} \left( u^2 + V^2 + 2uV \cos \theta - 2 \frac{\mathrm{M}}{\mathrm{R}} \right)$$

but since for Jupiter we have the relation:

$$\frac{u^2}{\mathrm{R}} = \frac{\mathrm{M}}{\mathrm{R}^2}$$

we can write

$$2\mathrm{W} = \mathrm{V}^2 + 2u\mathrm{V}\cos\theta - \frac{\mathrm{M}}{\mathrm{R}} \; .$$

Since in the subsequent perturbations V is not changed and only $\theta$ changes, in order that the comet can become hyperbolic it is necessary that W, negative at present, can become positive corresponding to suitable values of $\theta$. Then one must have

$$\mathrm{V}^2 + 2u\mathrm{V} > \frac{\mathrm{M}}{\mathrm{R}}$$

but we remark that

$$u = \sqrt{\frac{\mathrm{M}}{\mathrm{R}}}$$

therefore the above inequality can be written:

$$\left(\mathrm{V} + \sqrt{\frac{\mathrm{M}}{\mathrm{R}}}\right)^2 > \frac{2\mathrm{M}}{\mathrm{R}}$$

from which * and reduces at the end to

$$\mathrm{V} > \left(\sqrt{2} - 1\right)\sqrt{\frac{\mathrm{M}}{\mathrm{R}}} = \left(\sqrt{2} - 1\right) u \; . \tag{40}$$

Therefore we will assume that this inequality is certainly satisfied. Moreover, for some values of $\theta$, W must certainly be negative, otherwise the cometary orbit could not be elliptic; so it must be that:

$$\mathrm{V}^2 + 2u\mathrm{V} < \frac{\mathrm{M}}{\mathrm{R}}$$

from which as above

$$\mathrm{V} > \left(\sqrt{2} + 1\right)\sqrt{\frac{\mathrm{M}}{\mathrm{R}}} = \left(\sqrt{2} + 1\right) u \; . \tag{41}$$

Therefore let us assume that V satisfies (40) and (41) and indicate by $\theta_0$ that particular value of $\theta$ for which the comet's orbit is hyperbolic, i.e., one has $\mathrm{W} = 0$, that is

$$\mathrm{V}^2 + 2u\mathrm{V}\cos\theta_0 = \frac{\mathrm{M}}{\mathrm{R}}$$

and then

$$\cos\theta_0 = \frac{\frac{\mathrm{M}}{\mathrm{R}} - \mathrm{V}^2}{2u\mathrm{V}} = \frac{u^2 - \mathrm{V}^2}{2u\mathrm{V}} \; . \tag{42}$$

When $\theta$ is greater than $\theta_0$, one has $\mathrm{W} < 0$ and then the comet describes an elliptic orbit; on the contrary, when $\theta$ is less then $\theta_0$ the orbit is hyperbolic.

*Editor's Note: At this point, in the Fermi's manuscript there is a blank line which, obviously, would have contained the expansion of the square of the last formula (N.d.R.).

Now we will suppose that initially the orbit is elliptic and very stretched, so that $\theta$ is very close to $\theta_0$, and precisely slightly greater. We call $\theta^*$ this initial value. Whenever the comet goes beyond Jupiter's orbit $\theta$ is changed of an amount $\psi$; the average of the squares of $\psi$ depends indeed on $\theta$, as (39) shows, but since we have supposed that $\theta$ remains always very close to $\theta_0$ we can set

$$\bar{\psi}^2 = \frac{4mh}{\pi \mathrm{R}\ \mathrm{V}^2 \sin\theta_0} \tag{43}$$

if after a certain time $\theta$ became ¡ $\theta_0$ the comet would become hyperbolic and would go away forever. Therefore we are in condition of being able to apply the theorem of §2. Then we must set $a = \theta^* - \theta_0$; $k^2 = \frac{4mh}{\pi \mathrm{R}\ \mathrm{V}^2 \sin\theta_0}$. And the theorem we proved tells us that:

The probability that the comet will be changed into a hyperbolic one after having crossed Jupiter's orbit $n$ times is:

$$\frac{2}{\sqrt{\pi}} \int_{\frac{\theta^* - \theta_0}{\sqrt{\frac{8mhn}{\pi \mathrm{R}\ \mathrm{V}^2 \sin\theta_0}}}}^{\infty} e^{-x^2} dx \tag{44}$$

and therefore tends to 1 when $n$ tends to infinity.

In the strict sense one could object that the above calculations would fail if the value of V were such that, when the orbit is parabolic, the comet took the same time as Jupiter to go from A to B, where A is the point where the comet enters Jupiter's orbit, and B the point where it goes out.

In Figure 1, S is the Sun, AJB Jupiter's orbit, AKB the orbit of the comet. But it is easy to realize that this case certainly cannot happen if the comet describes its trajectory with direct motion. In fact, if $v$ is the absolute velocity in A of the comet in its parabolic orbit, one has

$$v^2 = u^2 + \mathrm{V}^2 + 2u\mathrm{V}\cos\theta_0$$

and then from (42)

$$v^2 = 2u^2$$

that is:

$$v > u\ . \tag{45}$$

Now the velocity of the comet is not constant, but in whole tract AKB it is always greater than in the extremes A and B, thus inequality (45) holds true with all the more reason in the whole tract AKB. On the other hand, if the motion is direct one has that arc AKB is shorter than arc AJB, and since it is covered with even higher velocity it is certain that the comet will arrive at B before Jupiter.

If on the contrary the motion of the comet were retrograde, and it described for instance the orbit AK'B' in the sense indicated by the arrow one would have

$$\text{arc AK'B'} > \text{arc AJB'}$$

and then, though (45) still holds, it is evident that for a particular value of the parameter of the cometary orbit it can happen that the two heavenly bodies take the same time to go from A to B'; of course this can only happen for a particular value of V.

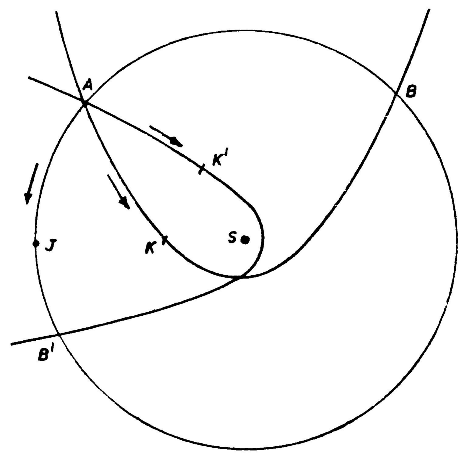


Figure 1.

Now if this happened it could occur that the comet, elliptic at first, crossed Jupiter when passing through A and got changed into a parabolic one; but in this case it would meet Jupiter again when passing through B and could possibly have a new perturbation which would change it into an elliptic comet again.

For this reason we consider this particular value of V ruled out from our calculations.

§ 7. Finally we want to consider the possibility that before being changed into a hyperbolic one the comet can crash into Jupiter and then be destroyed. What is the probability of this event?

For this let us look first for the probability that the comet, crossing once Jupiter's orbit, collides with the planet. If we indicate by $\rho$ the sum of the radii of Jupiter and the comet, to have the collision it is necessary that the periheliac distance of Jupiter from the comet, as calculated though the formulas of the Keplerian motion is smaller than $\rho$.

Let $\delta$ be this periheliac distance; from the formulas of §5 one finds that

$$\frac{1}{\delta} = \mathrm{A} + \mathrm{B}$$

and then from (35) and (36)

$$\frac{1}{\delta} = \frac{m}{\mathrm{V}^2 b^2} + \frac{1}{b}\sqrt{1 + \frac{m}{\mathrm{V}^4 b^2}}$$

If we want the collision to occur it must be that $\delta < \rho$ and therefore

$$\frac{m}{\mathrm{V}^2 b^2} + \frac{1}{b}\sqrt{1 + \frac{m}{\mathrm{V}^4 b^2}} > \frac{1}{\rho}$$

by multiplying this inequality by the quantity, certainly positive

$$\rho\left(\frac{1}{b}\sqrt{1 + \frac{m}{\mathrm{V}^4 b^2}} > \frac{1}{\rho} - \frac{m}{\mathrm{V}^2 b^2}\right)$$

we find

$$\frac{\rho}{b^2}\frac{1}{b}\sqrt{1 + \frac{m}{\mathrm{V}^4 b^2}} > \frac{1}{\rho} - \frac{m}{\mathrm{V}^2 b^2}$$

and summing the last two inequalities

$$\left(\frac{2m}{\mathrm{V}^2} + \rho\right)\frac{1}{b^2} > \frac{1}{\rho}$$

from which finally

$$|b| < \sqrt{\rho^2 + \frac{2m\rho}{\mathrm{V}^2}} \ . \tag{46}$$

We recall now that the probability that the value of $b$ lies between $b$ and $b + db$ is $\frac{db}{2\pi \mathrm{R} \sin\theta_0}$ and that the probability $p$ that the collision occurs in only one crossing of Jupiter's orbit is given by

$$p = \frac{1}{\pi \mathrm{R} \sin\theta_0}\sqrt{\rho^2 + \frac{2m\rho}{\mathrm{V}^2}} \ . \tag{47}$$

We will assume $p$ to be very small, and this obviously is equivalent to considering Jupiter's radius negligible if compared with the radius of its orbit.

Let us now look for the probability that the collision occurs at the $n$-th time the comet crosses Jupiter's orbit. Therefore it is evidently necessary that the collision has not occurred before and the probability of this is obviously $(1-p)^{n-1}$, that is in our approximation

$$e^{-pn} \ .$$

That the comet has not yet been changed into a hyperbolic one; and having supposed $p$to be extremely small, remembering (44) and setting for the sake of brevity:

$$\frac{\theta^* - \theta_0}{\sqrt{\frac{8mh}{\pi \mathrm{R}\mathrm{V}^2 \sin\theta_0}}} = \mathrm{H}$$

we can claim that the probability of this event is given by

$$1 - \frac{2}{\sqrt{\pi}}\int_{\frac{\mathrm{H}}{\sqrt{n}}}^{\infty} e^{-x^2} dx = \frac{2}{\sqrt{\pi}}\int_0^{\frac{\mathrm{H}}{\sqrt{n}}} e^{-x^2} dx \ .$$

And finally that the collision really occurs, for which we have the probability $p$.

After all the probability that the collision occurs the $n$-th time is

$$\frac{2e^{-pn}p}{\sqrt{\pi}}\int_0^{\frac{\mathrm{H}}{\sqrt{n}}} e^{-x^2}dx$$

and therefore the probability that the collision occurs at any time whatsoever will be the sum of the above expression from $n = 1$ to $n = \infty$, or replacing the sum by an integral

$$\frac{2p}{\sqrt{\pi}}\int_0^{\infty} e^{-pn}dn\int_0^{\frac{\mathrm{H}}{\sqrt{n}}} e^{-x^2}dx \ .$$

In this expression it is convenient to reverse the integration by the formula

$$\int_0^{\infty} dn\int_0^{\frac{\mathrm{H}}{\sqrt{n}}} dx = \int_0^{\infty} dx\int_0^{\frac{\mathrm{H}}{x^2}} dn$$

and in this way one finds for the desired probability the expression:

$$\frac{2p}{\sqrt{\pi}}\int_0^{\infty} e^{-x^2}dx\int_0^{\frac{\mathrm{H}}{x^2}} e^{-pn}dn = \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-x^2}\left(1-e^{-\frac{p\mathrm{H}}{x^2}}\right)dx$$

$$= 1-\frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-x^2-\frac{p\mathrm{H}}{x^2}}dx = 1-e^{-2\sqrt{p\mathrm{H}}} \ .$$

The probability that the collision never occurs is therefore:

$$e^{-2\sqrt{p\mathrm{H}}} \ .$$