1.

ON THE DYNAMICS OF A RIGID SYSTEM OF OF ELECTRIC CHARGES ON TRANSLATIONAL MOTION

"Sulla dinamica di un sistema rigido di cariche
elettriche in moto traslatorio,"
Nuovo Cimento **22**, 199–207 (1921).

§ 1. – When a rigid system of electric charges moves arbitrarily, the electric field it generates is different from that which Coulomb's law would predict. Now, the electric field produced by the entire system exerts some forces on each element of charge of the system. The resultant of these forces, namely the resultant of the internal electric forces, would of course be identically zero if Coulomb's law were valid, but it no longer is, however, at least in general, when the system moves, since in such a case that law is no longer valid.

This resultant gives the electromagnetic inertial reaction, and the aim of the present work is precisely its evaluation in the case of an arbitrary system in translational motion. In the case in which the system is a spherical distribution of surface electricity, as it is assumed in most electronic models, it is known that one finds [1] that such a resultant, at least in the first approximation, is given by

$$-\frac{2e^2}{3\mathrm{R}c^3}\,\mathbf{\Gamma} + \frac{2e^2}{3c^2}\,\dot{\mathbf{\Gamma}}\ , \tag{1}$$

where e, R denote the total charge and the radius of the system, c is the speed of light, $\mathbf{\Gamma}$ and $\dot{\mathbf{\Gamma}}$ are the acceleration and its derivative with respect to time. For quasi-stationary motions the second term of (1) becomes negligible, so that (1) reduces to

$$-m\mathbf{\Gamma}\ , \tag{2}$$

where m is the elecromagnetic mass.

In § 2 one finds the generalization of (1) to the case of any system, referring for example to molecular models, always assuming that the velocity is negligible with respect to the speed of light. If F_i $(i = 1, 2, 3)$ are the components of the resultant in question, one finds

$$\mathrm{F}_i = -\sum_k m_{ik}\Gamma_k + \sum_k \sigma_{ik}\dot{\Gamma}_k\ , \tag{3}$$

[1] Richardson, *Electron Theory of Matter*, Chapter XIII. The difference between my formulas and those of Richardson is due to the fact that he adopts Heaviside units.

where m_{ik}, σ_{ik} are some quantities depending on the properties of the system. Therefore one can no longer refer to a scalar electromagnetic mass, but instead in its place one introduces the tensor m_{ik}.

§ 3 is devoted to the dynamical study of the law for quasi-stationary motions:

$$\mathrm{K}_i = \sum_k m_{ik}\Gamma_k \ , \tag{4}$$

where K_i are the components of the external force. One shows that with such a law the fundamental principles of the living force and of Hamilton continue to hold.

Finally in § 4 the law (4) for quasi-stationary motions, which holds only for small velocities, is generalized to the case of arbitrary velocity using special relativity.

With this the study of electromagnetic masses as inertial masses will be complete. In a forthcoming paper I will consider electromagnetic masses as masses endowed with weight from the point of view of the general theory of relativity.

§ 2.– It is known [2] that the electric force due to a point charge 1 in motion is the sum of two forces, which by assuming the ratio between the velocity v of the particle and the speed c of light to be negligible, are: the first one, $\mathbf{E}_1$, the force given by Coulomb's law; the second one $\mathbf{E}_2$ has the form

$$\mathbf{E}_2 = \frac{\mathbf{\Gamma}^* \cdot \mathbf{a}}{c^2 r}\, a - \frac{1}{c^2 r}\,\mathbf{\Gamma}^* \ . \tag{5}$$

In this formula r represents the distance between the particle M and the point P at which the force is calculated and $\mathbf{a}$ is a vector of magnitude 1 and orientation MP. Finally $\mathbf{\Gamma}^*$ is the acceleration of the particle at the time $t-(r/c)$. If instead of the charge 1 at M there is the charge $\rho\, d\tau$ (ρ is the electric density, $d\tau$ the volume element), the force at P will be $\rho\, d\tau(\mathbf{E}_1+\mathbf{E}_2)$, so that the force exerted at P by all charges will be $\int_\tau \rho(\mathbf{E}_1+\mathbf{E}_2)\, d\tau$, where the integration must be extended over the whole space τ occupied by charges. Now if at the point P there is the charge $\rho'\, d\tau'$, the force $\rho'\, d\tau' \int_\tau \rho(\mathbf{E}_1+\mathbf{E}_2)\, d\tau$ is acting on it.

The force acting on the entire system is therefore

$$\mathbf{F} = \iint \rho\rho'(\mathbf{E}_1+\mathbf{E}_2)\, d\tau\, d\tau' \ ,$$

where the two integrations must be extended over the same domain. On the other hand, one clearly has

$$\iint \rho\rho'\mathbf{E}_1\, d\tau\, d\tau' = 0 \ ,$$

so that

$$\mathbf{F} = \iint \rho\rho'\mathbf{E}_2\, d\tau\, d\tau' \ .$$

[2]See, e.g., Richardson, op. cit.

If we now denote by $\mathbf{\Gamma}$ and $\dot{\mathbf{\Gamma}}$ the acceleration and its derivative with respect to time, at the time t, if r is small enough, we can set

$$\mathbf{\Gamma}^* = \mathbf{\Gamma} - \frac{r}{c}\,\dot{\mathbf{\Gamma}} \ ,$$

obtaining finally

$$\mathbf{F} = \iint \left(\frac{\mathbf{\Gamma}\cdot\mathbf{a}}{c^2 r}\, a - \frac{\mathbf{\Gamma}}{c^2 r} \right) \rho\rho'\, d\tau\, d\tau' + \iint \left(\frac{\dot{\mathbf{\Gamma}}\cdot\mathbf{a}}{c^3}\, a - \frac{\dot{\mathbf{\Gamma}}}{c^3} \right) \rho\rho'\, d\tau\, d\tau' \ . \qquad (6)$$

Ww denote orthogonal cartesian coordinates by x_1, x_2, x_3, and let (x_i) be the coordinates of M, (x'_i) that of P. The components of $\mathbf{a}$ are $a_i = \dfrac{x'_i - x_i}{r}$. Writing the (6) in scalar form, one thus obtains

$$\mathrm{F}_i = -\sum_k m_{ik}\Gamma_k + \sum_k \sigma_{ik}\dot{\Gamma}_k \ , \qquad (7)$$

noting that, under the assumption of translational motion, Γ_i and $\dot{\Gamma}_i$ are constant when the integration is performed.

Here one has set:

$$\begin{cases} m_{ii} = \dfrac{2\mathrm{U}}{c^2} - \displaystyle\iint \frac{\rho\rho'(x'_i - x_i)^2}{c^2 r^3}\, d\tau\, d\tau' \ , \\[2ex] m_{ik} = m_{ki} = -\displaystyle\iint \frac{\rho\rho'(x'_i - x_i)(x'_k - x_k)}{c^2 r^3}\, d\tau\, d\tau' \ , \qquad i \neq k \ , \end{cases} \qquad (8)$$

$$\begin{cases} \sigma_{ii} = \dfrac{e^2}{c^3} - \displaystyle\iint \frac{\rho\rho'(x'_i - x_i)^2}{c^3 r^2}\, d\tau\, d\tau' \ , \\[2ex] \sigma_{ik} = \sigma_{ki} = -\displaystyle\iint \frac{\rho\rho'(x'_i - x_i)(x'_k - x_k)}{c^3 r^2}\, d\tau\, d\tau' \ , \qquad i \neq k \ . \end{cases} \qquad (9)$$

In these formulae U represents the electrostatic energy of the system $= \dfrac{1}{2}\displaystyle\iint \frac{\rho\rho'}{r}\, d\tau\, d\tau'$, and e the total electric charge $= \displaystyle\int \rho\, d\tau = \int \rho'\, d\tau'$.

From the expressions (8), (9) it immediately follows that if the axes (x_i) are substitued by others (y_i) using the orthogonal substitution

$$y_i = \sum_k \alpha_{ik} x_k \ ,$$

the m_{ik} and σ_{ik} corresponding to the new axes are given by:

$$m'_{ik} = \sum_{rs} \alpha_{ir}\alpha_{ks} m_{ik} \ ,$$

$$\sigma'_{ik} = \sum_{rs} \alpha_{ir}\alpha_{ks} \sigma_{ik} \ .$$

Hence both m_{ik} and σ_{ik} are symmetric covariant tensors. Each of them will have three orthogonal principal directions such that, taking the axes to be parallel to them, one has either $m_{ik} = 0$ or $\sigma_{ik} = 0$ when $i \neq k$.

The principal axes of tensors m, σ, however, will be different in general. If the case that the system has spherical symmetry one can do the integrations (8) and (9), since instead of $\frac{(x'_i - x_i)(x'_k - x_k)}{r^2}$ one can put the mean value of this expression over all possible directions MP, since in this case to the two points MP correspond an infinite number of pairs which differ only by orientation. Now, this mean value if $i = k$ is given by $\frac{2\pi}{4\pi}\int_0^\pi \cos^2\theta \sin\theta \, d\theta$; if instead $i \neq k$, it is zero.

So one then has

$$m_{11} = m_{22} = m_{33} = \frac{4\mathrm{U}}{3c^2} \ ; \qquad m_{23} = m_{31} = m_{12} = 0 \ ;$$

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \frac{2}{3}\frac{e^2}{c^3} \ ; \qquad \sigma_{23} = \sigma_{31} = \sigma_{12} = 0 \ .$$

By substituting these values into (7), one obtains well known formulas if the system consists of a homogeneous spherical layer.

§ 3.− Returning to the general case, we note that for quasi-stationary motions (5) can be replaced by:

$$\mathrm{F}_i = -\sum_k m_{ik}\Gamma_k \ .$$

If one thinks of an external force (X_i) acting on the system, the total force will be $(\mathrm{X}_i + \mathrm{F}_i)$. If one now supposes that the system has no material mass one must have $\mathrm{X}_i + \mathrm{F}_i = 0$, and so

$$\mathrm{X}_i = \sum_k m_{ik}\Gamma_k \ . \tag{10}$$

It is easy to show how with the law (10) the principle of the living forces and of Hamilton's force are preserved. In fact, denoting the velocity by $\mathrm{V} \equiv (\mathrm{V}_1, \mathrm{V}_2, \mathrm{V}_3)$ and multiplying (10) by V_i, then summing with respect to i one obtains

$$\sum_i \mathrm{X}_i\mathrm{V}_i = \sum_{ik} m_{ik}\mathrm{V}_k\frac{d\mathrm{V}_i}{dt} \ .$$

Interchanging i and k in the second sum, and noting that $m_{ik} = m_{ki}$

$$\sum_i \mathrm{X}_i\mathrm{V}_i = \sum_{ik} m_{ik}\mathrm{V}_i\frac{d\mathrm{V}_k}{dt} \ ,$$

and summing

$$2\sum_i \mathrm{X}_i\mathrm{V}_i = \sum_{ik} m_{ik}\left(\mathrm{V}_i\frac{d\mathrm{V}_k}{dt} + \mathrm{V}_k\frac{d\mathrm{V}_i}{dt}\right) = \frac{d}{dt}\ \sum_{ik} m_{ik}\mathrm{V}_i\mathrm{V}_k \ .$$

The first left hand side is twice the potential P of the external forces. Thus one has

$$P = \frac{dT}{dt} \ , \qquad \text{where} \qquad T = \frac{1}{2}\sum_{ik} m_{ik}\mathrm{V}_i\mathrm{V}_k \ . \tag{11}$$

Multiplying, instead, the two sides of (10) by δx, and then summing, one similarly gets

$$\sum_i \mathrm{X}_i \delta x_i = \frac{1}{2}\sum_{ik} m_{ik}\left(\frac{d^2x_k}{dt^2}\,\delta x_i + \frac{d^2x_i}{dt^2}\,\delta x_k\right)$$
$$= \frac{d}{dt}\left\{\frac{1}{2}\sum_{ik} m_{ik}(\dot{x}_k\delta x_i + \dot{x}_i\delta x_k)\right\} - \frac{1}{2}\sum_{ik} m_{ik}(\dot{x}_k\delta\dot{x}_i + \dot{x}_i\delta\dot{x}_k)$$
$$= \frac{d}{dt}\left\{\frac{1}{2}\sum_{ik} m_{ik}(\dot{x}_k\delta x_i + \dot{x}_i\delta x_k)\right\} - \delta T \ .$$

Multiplying by dt and integrating between two limits t', t'' at which the variations δx_i are assumed to be zero, one obtains

$$\int_{t'}^{t''}\left(\delta T + \sum_i \mathrm{X}_i\delta x_i\right) = 0 \ , \tag{12}$$

expressing Hamilton's principle.

If one refers to the principal axes of the tensor m_{ik} instead of arbitrary ones, (10) takes the simple form:

$$\mathrm{X}_i = m_{ii}\Gamma_i \ . \tag{13}$$

§ 4. – This formula holds only if V/c is negligible. To generalize it to an arbitrary velocity let us denote by $\mathrm{S} \equiv (x_1, x_2, x_3, t)$ the indicated reference frame and by $\mathrm{S}^* \equiv (x, y, z, t)$ a frame fixed with respect to S with the $x-$axis orientated along the velocity of the system at a certain fixed but generic time $\bar{t}$, and finally let $\mathrm{S}' \equiv (x', y', z', t')$ be a system with spatial axes parallel to xyz which moves uniformly with respect to S^* with velocity equal to that of the moving one at time $\bar{t}$, whose magnitude is v. One will have

$$t' = \beta\left(t - \frac{v}{c^2}\,x\right) \ ; \quad x' = \beta\,(x - vt) \ ; \quad y' = y \ ; \quad z' = z \ ; \quad \beta = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} \ , \tag{14}$$

where, once $\bar{t}$ is fixed, v and hence β are constant.

Let us asumme that the forces acting on our system are due to an external electromagnetic field ($\mathbf{E}$, $\mathbf{H}$); since at the instant t the system has velocity zero with respect to S', (10) will hold for it, and so one will therefore have, with an obvious meaning for the symbols:

$$\begin{aligned} e\,\mathrm{E}'_x &= m_{xx}\Gamma'_x + m_{xy}\Gamma'_y + m_{xz}\Gamma'_z \\ e\,\mathrm{E}'_y &= m_{yx}\Gamma'_x + m_{yy}\Gamma'_y + m_{yz}\Gamma'_z \\ e\,\mathrm{E}'_z &= m_{zx}\Gamma'_x + m_{zy}\Gamma'_y + m_{zz}\Gamma'_z \ . \end{aligned}$$

But one has

$$e\,\mathrm{E}'_x = e\,\mathrm{E}_x \ , \qquad e\,\mathrm{E}'_y = e\beta\left(\mathrm{E}_y - \frac{v}{c}\,\mathrm{H}_z\right) \ , \qquad e\,\mathrm{E}'_z = e\beta\left(\mathrm{E}_z + \frac{v}{c}\,\mathrm{H}_y\right) \ .$$

So therefore setting

$$\mathbf{k} = e\left(\mathrm{E} + \frac{1}{c}\,\mathrm{V}\times\mathrm{H}\right) \ , \tag{15}$$

one finds

$$e\,\mathrm{E}'_x = e\,\mathrm{E}_x \ , \qquad e\,\mathrm{E}'_y = e\beta k_y \ , \qquad e\,\mathrm{E}'_z = e\beta k_z \ .$$

On the other hand:

$$\Gamma'_x = \frac{d^2x'\,dt' - d^2t'\,dx'}{dt'^3} \ ,$$

but at time $\bar{t}$, $\dfrac{dx'}{dt'} = 0$, hence $\Gamma'_x = \dfrac{d^2x'}{dt'^2}$. Taking t as the independent variable, and noting that $\dfrac{dx}{dt} = v$,

$$\Gamma'_x = \beta^3\Gamma_x \ . \qquad \text{Analogously} \qquad \Gamma'_y = \beta^2\Gamma_y \ , \qquad \Gamma'_z = \beta^2\Gamma_z \ .$$

Substituting

$$\begin{cases} k_x = m_{xx}\beta^3\ddot{x} + m_{xy}\beta^2\ddot{y} + m_{xz}\beta^2\ddot{z} \\[2ex] k_y = m_{yx}\beta^3\ddot{x} + m_{yy}\beta\ddot{y} + m_{yz}\beta\ddot{z} \\[2ex] k_z = m_{zx}\beta^3\ddot{x} + m_{zy}\beta\ddot{y} + m_{zz}\beta\ddot{z} \ . \end{cases} \tag{16}$$

Denoting by α_{xi} the cosine of the angle between the $x-$axis and the x_i-axis, one has

$$k_i = \alpha_{xi}k_x + \alpha_{yi}k_y + \alpha_{zi}k_z \ .$$

On the other hand, being m_{i0} covariant, one has for instance

$$m_{xy} = \sum_r m_{rr}\alpha_{xr}\alpha_{yr} \ .$$

Analogously

$$\ddot{x} = \sum_j \ddot{x}_j\alpha_{xj} \ .$$

Multiplying then (16) by $\alpha_{xi}, \alpha_{yi}, \alpha_{zi}$ and summing, one finds

$$k_i = \sum_{rj} m_{rr} \ddot{x}_j \begin{bmatrix} \beta^3 \alpha_{xr}^2 \alpha_{xj} \alpha_{xi} + \beta^2 \alpha_{xr} \alpha_{yr} \alpha_{yj} \alpha_{xi} + \beta^2 \alpha_{xr} \alpha_{zr} \alpha_{zj} \alpha_{xi} \\ + \beta^2 \alpha_{yr} \alpha_{xr} \alpha_{xj} \alpha_{yi} + \beta \alpha_{yr}^2 \alpha_{yj} \alpha_{yi} + \beta \alpha_{yr} \alpha_{zr} \alpha_{zj} \alpha_{yi} \\ + \beta^2 \alpha_{zr} \alpha_{xr} \alpha_{xj} \alpha_{zi} + \beta \alpha_{zr} \alpha_{yr} \alpha_{yj} \alpha_{zi} + \beta \alpha_{zr}^2 \alpha_{zj} \alpha_{zi} \end{bmatrix} .$$

But one has $\alpha_{xi} = \frac{\dot{x}_i}{v}$. Taking into account the relations between the α's, one finally finds the sought after generalization of (13)

$$k_i = \beta \sum_{rj} \ddot{x}_j m_{rr} \left\{ (\beta - 1)^2 \frac{\dot{x}_i \dot{x}_j \dot{x}_r^2}{v^4} \right. \\ \left. + (\beta - 1) \left[(jr) \frac{\dot{x}_i \dot{x}_r}{v^2} + (ir) \frac{\dot{x}_j \dot{x}_r}{v^2} \right] + (ir)\,(jr) \right\} , \qquad (17)$$

where

$$(jr) = 1 , \quad \text{if} \quad j = r ; \qquad (jr) = 0 , \quad \text{if} \quad j \neq r .$$

In the case of spherical symmetry, setting $m_{11} = m_{22} = m_{33} = m$, one can evaluate the sum in (17), finding:

$$k_i = \beta m \ddot{x}_i + m\beta(\beta^2 - 1) \frac{\dot{x}_i}{v^2} \sum_j \dot{x}_j \ddot{x}_j ,$$

from which, recalling that

$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} ,$$

one recovers the well known formula of electronic dynamics

$$k_i = \frac{d}{dt} \frac{m \dot{x}_i}{\sqrt{1 - \frac{v^2}{c^2}}} .$$

Pisa, January 1921.