the square wheel calculus problem
Italy 2017

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The following problem appeared on the Italian national written exam taken by all scientific high school students graduating in 2017 after a year of single variable calculus. There was a choice between this interesting problem and a straightforward problem, and since test pressure and stress is an important consideration for students everywhere, almost no one chose to do this interesting problem. However, freed from the time constraints and test pressure, this is a reasonable problem that any serious calculus student should be able to do. The solution can of course easily be found on the internet, so this is only intended for students who are looking for a fun math challenge.

PROBLEM 1

Can one easily ride a bicycle with square wheels? At the MoMath Museum of Mathematics in New York, one can, in one of the exhibits dedicated to mathematical fun (Figure 1). However, it is necessary that the profile of the platform on which the wheel turns satisfy certain requirements.

In Figure 2 is a representation of the situation in the Cartesian x-y plane: the square of side length $DE = 2$ (in appropriate length units) and with center C represents the bicycle wheel, and the graph of the function $f(x)$ represents the platform profile.

Figure 1

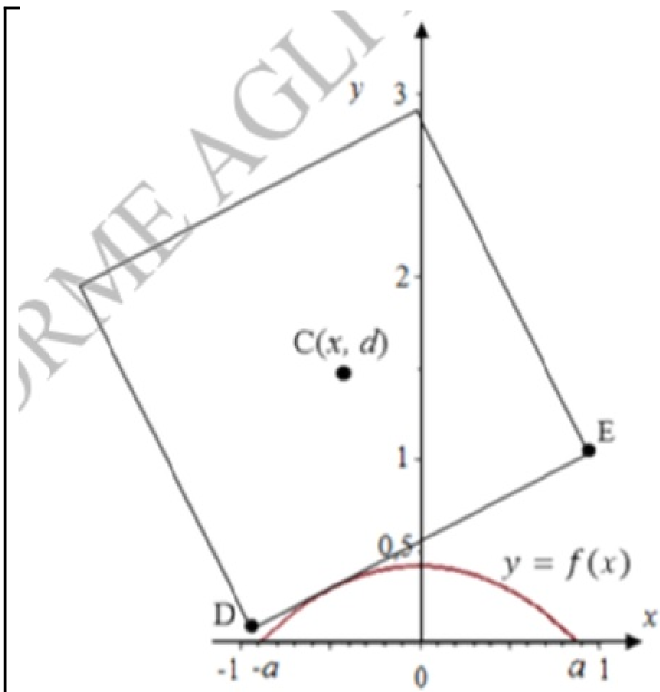

Figure 2

1) On the basis of the information obtainable from the graphic in Figure 2, show with appropriate explanation that the function

$$f(x) = \sqrt{2} - \frac{e^{x} + e^{-x}}{2}, \quad x \in \mathbb{R}$$

adequately represents the platform profile for $x \in [-a, a]$; in addition determine the values of the extremes a and $-a$ of the interval.

To visualize the complete profile of the platform on which the bicycle should move, one places a number of copies side by side of the graph of the function $f(x)$ relative to the interval $[-a, a]$ as shown in Figure 3.

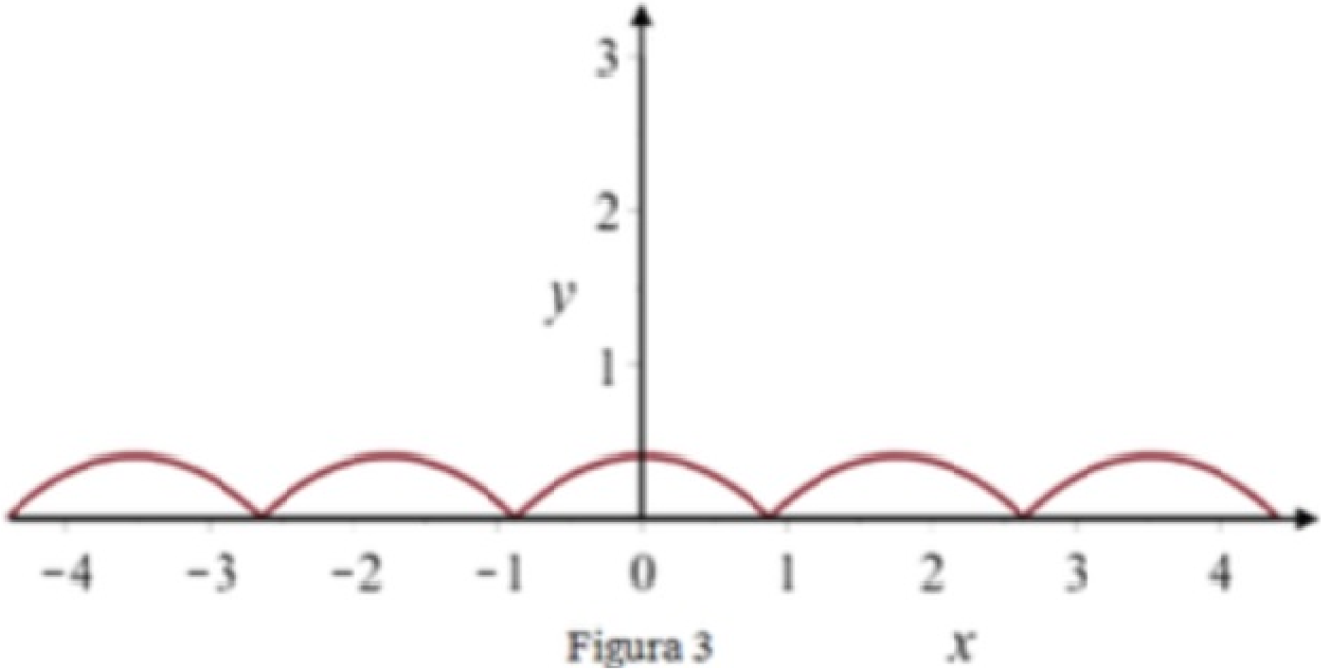

Figure 3

2) In order that the bicycle can proceed easily on the platform it is necessary that:
- ♦ to the left and right of the nondifferentiable points the pieces of the graph should be orthogonal;
- ♦ the length of the side of the square wheel should turn out to be equal to the length of a "hump", namely of the arc of the curve with equation $y = f(x)$ for $x \in [-a, a]$.

Establish that these conditions are satisfied.

Hint: In general the length of an arc of a curve having the equation $y = \varphi(x)$ between the values x_1 and x_2 is given by

$$\int_{x_1}^{x_2} \sqrt{1 + (\varphi'(x))^2}\, dx .$$

3) Considering the similarity of the triangles ACL and ALM in Figure 4, and remembering the geometric significance of the derivative, verify that the value d of the y coordinate of the center of the wheel remains constant during the motion. Thus the cyclist seems to move on a plane surface.

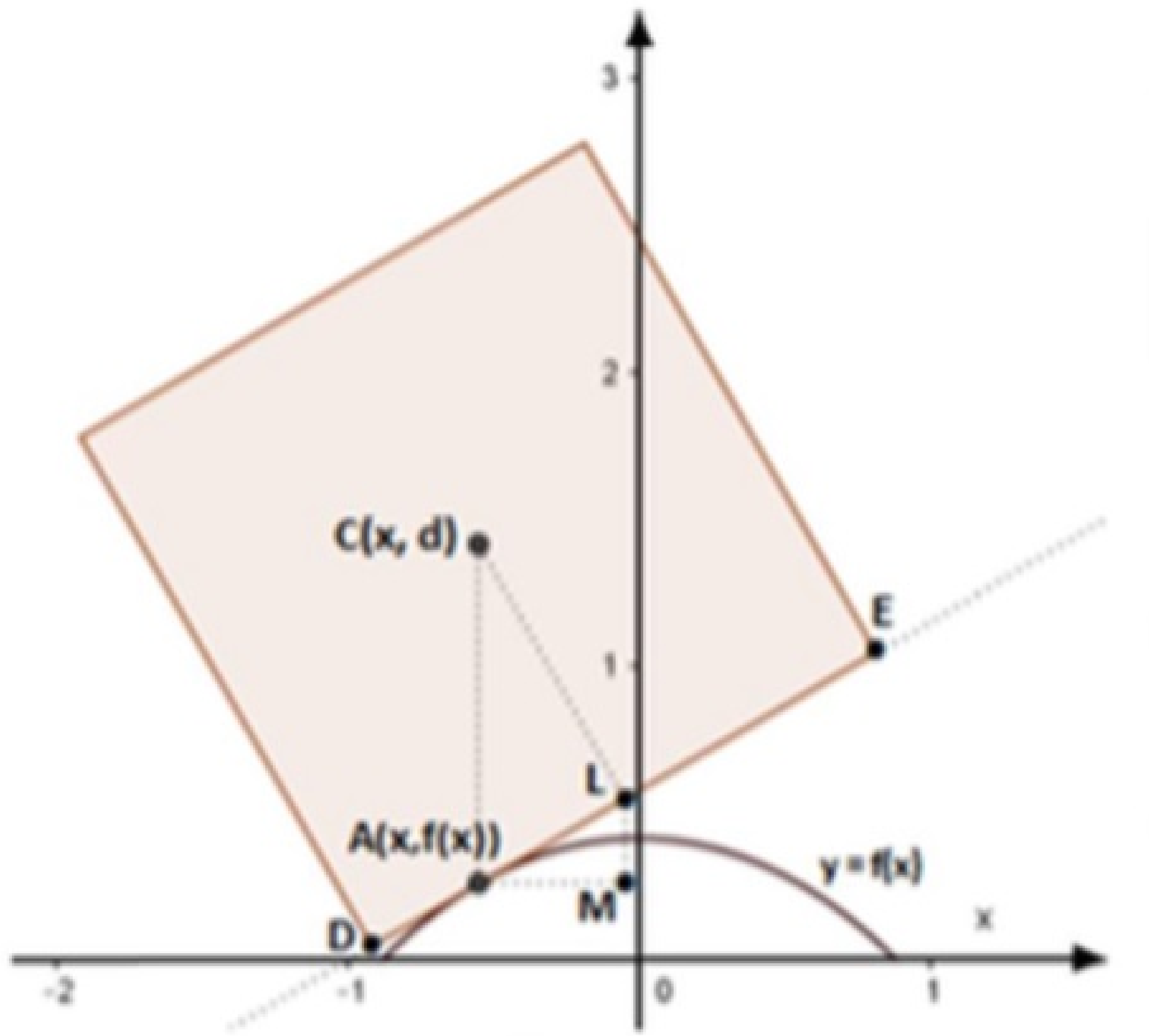

Figura 4

Figure 4

Also the graph of the function

$$f(x) = \frac{2}{\sqrt{3}} - \frac{e^{x} + e^{-x}}{2} \quad \text{for } x \in \left[-\frac{\ln(x)}{2}, \frac{\ln(x)}{2} \right]$$

if replicated several times can represent the profile of a platform adapted to be traveled by a bicycle with very particular wheels, having the form of a regular polygon.

4) Determine this regular polygon, explaining your result.