Appendix A

Formulas from differential geometry

This summary of formulas from differential geometry continues the remarks of the introduction. It is assumed that the reader has some familiarity with many of these concepts, at least in coordinate notation, or at the level of the text *Gravitation* by Misner, Thorne and Wheeler [1973]. The purpose of this appendix is to have a reference guide for the particular choice of conventions used here and the specific formulas for the frame components of many geometric objects whose coordinate frame representation is more widely known. More details may be found by consulting Spivak [1965], Hicks [1971], Choquet-Bruhat, DeWitt-Morette and Dillard-Bleick [1982], Schouten [1954] or other well known texts on differential geometry.

A.1 Manifold

Let M be an n-dimensional manifold, with local coordinates $\{x^\alpha\}_{\alpha=1,\ldots,n}$ when needed. These local coordinates define a local coordinate frame $\{\partial/\partial x^\alpha\}$ with dual frame $\{dx^\alpha\}$.

A.2 Frame and dual frame

It is often more convenient to work with a (local) noncoordinate frame $\{e_\alpha\}$ with dual frame $\{\omega^\alpha\}$ (sometimes called co-frame) satisfying the duality condition

$$\omega^\alpha(e_\beta) = \delta^\alpha{}_\beta \ . \tag{A.1}$$

If one expresses these fields in local coordinates

$$e_\alpha = e^\beta{}_\alpha \partial/\partial x^\beta \ , \qquad \omega^\alpha = \omega^\alpha{}_\beta dx^\beta \ , \tag{A.2}$$

then the matrices of components $(e^\alpha{}_\beta)$ and $(\omega^\alpha{}_\beta)$ are inverse matrices.

The differential properties of the frame are characterized by the structure functions of the frame, which define the Lie brackets of the frame vector fields and the exterior derivatives of the dual frame 1-forms

$$[e_\alpha, e_\beta] = C^\gamma{}_{\alpha\beta} e_\gamma \ , \qquad d\omega^\alpha = -\tfrac{1}{2} C^\alpha{}_{\beta\gamma} \omega^\beta \otimes \omega^\gamma \ . \tag{A.3}$$

A $\binom{p}{q}$-tensor field can be expressed in terms of the frame as

$$S = S^{\alpha\cdots}{}_{\beta\ldots} e_\alpha \otimes \cdots \otimes \omega^\beta \otimes \cdots \ , \qquad S^{\alpha\cdots}{}_{\beta\ldots} = S(\omega^\alpha, \ldots, e_\beta, \ldots) \ . \tag{A.4}$$

It is convenient to adopt the notation

$$\omega^{\alpha_1\ldots\alpha_p} = \omega^{\alpha_1} \wedge \cdots \wedge \omega^{\alpha_p} \tag{A.5}$$

to more compactly represent a p-form, or antisymmetric $\binom{0}{p}$-tensor

$$S = \frac{1}{p!} S_{\alpha_1\ldots\alpha_p} \omega^{\alpha_1\ldots\alpha_p} = S_{|\alpha_1\ldots\alpha_p|} \omega^{\alpha_1\ldots\alpha_p} \ , \tag{A.6}$$

where the vertical bar notation indicates a sum only over strictly increasing sequences of indices. A similar notation $e_{\alpha_1\dots\alpha_p}$ will be used to express p-vector fields, namely antisymmetric $\binom{p}{0}$-tensor fields.

A tensor field whose components are antisymmetric in a subset of p covariant indices may be interpreted as a tensor-valued p-form

$$\begin{aligned} S &= S^{\alpha\dots}{}_{\beta\dots\gamma_1\dots\gamma_p} e_\alpha \otimes \dots \otimes \omega^\beta \otimes \dots \otimes \omega^{\gamma_1} \otimes \dots \otimes \omega^{\gamma_p} \\ &= \frac{1}{p!} S^{\alpha\dots}{}_{\beta\dots\gamma_1\dots\gamma_p} e_\alpha \otimes \dots \otimes \omega^\beta \otimes \dots \otimes \omega^{\gamma_1\dots\gamma_p} \\ &= e_\alpha \otimes \dots \otimes \omega^\beta \otimes \dots \otimes S^{\alpha\dots}{}_{\beta\dots} \ . \end{aligned} \tag{A.7}$$

where

$$S^{\alpha\dots}{}_{\beta\dots} = \frac{1}{p!} S^{\alpha\dots}{}_{\beta\dots\gamma_1\dots\gamma_p} \omega^{\gamma_1\dots\gamma_p} \tag{A.8}$$

are the components of the tensor-valued arguments of the tensor-valued p-form. In most of the literature this collection of p-forms is often used in place of the tensor-valued p-form it represents. One may take the exterior derivative of this set of p-forms, which in turn defines the exterior derivative of the tensor-valued p-form

$$dS = e_\alpha \otimes \dots \otimes \omega^\beta \otimes \dots \otimes dS^{\alpha\dots}{}_{\beta\dots} \ . \tag{A.9}$$

This clearly depends on the choice of frame.

A.3 Linear transformations

A function whose value at each point is a linear transformation of the tangent space may be identified with a $\binom{1}{1}$-tensor field in a natural way, acting on vectors fields by right contraction

$$\begin{aligned} B &= B^\alpha{}_\beta e_\alpha \otimes \omega^\beta \ , \\ B \llcorner X &= B^\alpha{}_\beta X^\beta e_\beta \ . \end{aligned} \tag{A.10}$$

The matrix of components $(B^\alpha{}_\beta)$ then acts on the R^n-vector of components (X^α) by matrix multiplication. Similarly right contraction between $\binom{1}{1}$-tensor fields corresponds to matrix multiplication of their square matrices of components

$$[A \llcorner B]^\alpha{}_\beta = A^\alpha{}_\gamma B^\gamma{}_\beta \ . \tag{A.11}$$

It is convenient to use the abbreviation

$$A^2 = A \llcorner A \ . \tag{A.12}$$

The identity transformation is represented by the unit tensor or Kronecker delta tensor δ

$$\delta = \delta^\alpha{}_\beta e_\alpha \otimes \omega^\beta \ . \tag{A.13}$$

The trace of a $\binom{1}{1}$-tensor corresponds to the trace of the linear transformation

$$\mathrm{Tr}\, A = A^\alpha{}_\alpha \ . \tag{A.14}$$

The abbreviation $\mathrm{Tr}^2\, A = (\mathrm{Tr}\, A)^2$ is convenient. Any such tensor can be decomposed into its pure trace and tracefree parts

$$\begin{aligned} A &= \frac{1}{n}[\mathrm{Tr}\, A]\delta + A^{(\mathrm{TF})} \ , \\ A^{(\mathrm{TF})} &= A - \frac{1}{n}[\mathrm{Tr}\, A]\delta \ . \end{aligned} \tag{A.15}$$

A.4 Change of frame

Given a nondegenerate $\binom{1}{1}$-tensor field A, i.e., such that the determinant $\det(A^{\alpha}{}_{\beta})$ is everywhere nonvanishing, it has a corresponding inverse tensor field A^{-1} whose component matrix is the inverse matrix of the component matrix of A

$$A^{-1} \llcorner A = \delta \ , \qquad A^{-1\,\alpha}{}_{\gamma} A^{\gamma}{}_{\beta} = \delta^{\alpha}{}_{\beta} \ . \tag{A.16}$$

These may be used to define a change of frame by defining new frame vector fields and dual 1-forms by

$$\overline{e}_{\alpha} = A^{-1} \llcorner e_{\alpha} = A^{-1\,\beta}{}_{\alpha} e_{\beta} \ , \qquad \overline{\omega}^{\alpha} = \omega^{\alpha} \llcorner A = A^{\alpha}{}_{\beta} \omega^{\beta} \ . \tag{A.17}$$

Under such a change of frame, the components of a $\binom{p}{q}$-tensor field "transform" under the representation $\rho^{p,q}$ of the general linear group $GL(n,R)$

$$\begin{aligned} \overline{S}^{\alpha\cdots}{}_{\beta\ldots} &= A^{\alpha}{}_{\gamma} \cdots A^{-1\,\delta}{}_{\beta} \cdots S^{\gamma\cdots}{}_{\delta\ldots} \\ &\equiv [\rho^{p,q}(\mathbf{A}) S]^{\alpha\cdots}{}_{\beta\ldots} \ . \end{aligned} \tag{A.18}$$

The derivative of a 1-parameter family of such transformations leads to the associated representation $\sigma^{p,q}$ of the Lie algebra of linear transformations $gl(n,R)$

$$[\sigma^{p,q}(\mathbf{A}) S]^{\alpha\cdots}{}_{\beta\ldots} \equiv A^{\alpha}{}_{\gamma} S^{\gamma\cdots}{}_{\beta\ldots} + \ldots - A^{\gamma}{}_{\beta} S^{\alpha\cdots}{}_{\gamma\ldots} - \ldots \ . \tag{A.19}$$

This linear operator $\sigma^{p,q}$, which is convenient to abbreviate by just σ when the context is clear, appears in the component formula for any differential operator which obeys a product rule with respect to the tensor product. It is convenient to allow the argument of σ either to be a matrix-valued function (depending on the choice of frame) or the corresponding $\binom{1}{1}$-tensor, whose component matrix is then understood to be substituted in its place.

Occasionally tensor densities and oriented tensor densities prove useful. These transform with an additional factor which is a power (called the weight) of the inverse of the determinant of the linear transformation matrix

$$\begin{aligned} \overline{S}^{\alpha\cdots}{}_{\beta\ldots} &= (\det A)^{-W} A^{\alpha}{}_{\gamma} \cdots A^{-1\,\delta}{}_{\beta} \cdots S^{\gamma\cdots}{}_{\delta\ldots} \\ &\equiv [\rho^{p,q}_{W}(\mathbf{A}) S]^{\alpha\cdots}{}_{\beta\ldots} \ . \end{aligned} \tag{A.20}$$

The derivative of a 1-parameter family of such transformations leads to the associated representation $\sigma^{p,q}_{W}$ of the Lie algebra of linear transformations $gl(n,R)$

$$[\sigma^{p,q}_{W}(\mathbf{A}) S]^{\alpha\cdots}{}_{\beta\ldots} \equiv A^{\alpha}{}_{\gamma} S^{\gamma\cdots}{}_{\beta\ldots} + \ldots - A^{\gamma}{}_{\beta} S^{\alpha\cdots}{}_{\gamma\ldots} - \ldots - W A^{\gamma}{}_{\gamma} S^{\alpha\cdots}{}_{\beta\ldots} \ . \tag{A.21}$$

It is difficult to find clear discussions of these objects in modern textbooks. Oriented tensor densities instead transform with an additional sign factor which is the sign of the determinant of the transformation matrix, thus undergoing an additional sign change for an orientation changing transformation with negative determinant.

A.5 Metric

If $(g_{\alpha\beta})$ is a symmetric nondegenerate matrix-valued function of signature s, i.e., can be reduced to a diagonal matrix with N negative diagonal values and $n-N$ positive values, with $s = (n-N) - N = n - 2N$, and if $(g^{\alpha\beta})$ is its inverse, then

$$\mathrm{g} = g_{\alpha\beta}\, \omega^{\alpha} \otimes \omega^{\beta} \ , \qquad \mathrm{g}^{-1} = g^{\alpha\beta} e_{\alpha} \otimes e_{\beta} \tag{A.22}$$

locally defines a pseudo-Riemannian (Riemannian if $N = 0$) metric and its inverse on M, satisfying

$$\mathrm{g}^{-1} \lrcorner \mathrm{g} = \delta \ , \qquad g^{\alpha\gamma} g_{\gamma\beta} = \delta^{\alpha}{}_{\beta} \ . \tag{A.23}$$

Let $g = |\det(g_{\alpha\beta})|$ be the absolute value of the determinant of the matrix of components of the metric in this frame, the sign being $(-1)^N$. For a Lorentz metric, one has $N = 1$ or $N = n-1$. The metric determinant derivative satisfies

$$d \ln g = g^{\alpha\beta} d g_{\alpha\beta} = \mathrm{Tr}[\mathrm{g}^{-1} \lrcorner d\mathrm{g}] \ . \tag{A.24}$$

Given a choice of orientation on M, namely an everywhere nonzero n-form $\mathcal{O}$, then $\{e_\alpha\}$ is an oriented frame if $\mathcal{O}(e_1,\ldots,e_n) > 0$ everywhere. In an oriented frame, the unit volume-form associated with the metric is defined by

$$\eta = g^{1/2}\omega^1 \wedge \cdots \wedge \omega^n \ . \tag{A.25}$$

For a Lorentz metric it is often convenient to use instead the indices $0,\ldots,n-1$ for certain choices of frame, choosing the orientation so that in an oriented frame of this type one has

$$\eta = g^{1/2}\omega^0 \wedge \cdots \wedge \omega^{n-1} \ . \tag{A.26}$$

Since the metric is nondegenerate, it determines an isomorphism between the tangent and cotangent spaces at each point which in index-notation corresponds to "raising" and "lowering" indices. For a vector field X and a 1-form θ one has

$$\begin{aligned} X^\flat &= \mathrm{g} \llcorner X \ , & X_\alpha &= g_{\alpha\beta}X^\beta \ , \\ \theta^\sharp &= \mathrm{g}^{-1} \lrcorner \theta \ , & \theta^\alpha &= g^{\alpha\beta}\theta_\beta \ . \end{aligned} \tag{A.27}$$

As indicated in the introduction, the sharp and flat notation for an arbitrary tensor will be understood to mean the tensor obtained by raising or lowering respectively all of the indices which are not already of the appropriate type.

A.6 Unit volume n-form

The Levita-Civita permutation symbols $\epsilon_{\alpha_1\ldots\alpha_n}$ and $\epsilon^{\alpha_1\cdots\alpha_n}$ are totally antisymmetric with

$$\epsilon_{1\ldots n} = 1 = \epsilon_{1\ldots n} \ , \tag{A.28}$$

so that it vanishes unless $\alpha_1\ldots\alpha_n$ is a permutation of $1\ldots n$, in which case its value is the sign of the permutation. The components of the unit volume n-form are related to these objects by

$$\eta_{\alpha_1\ldots\alpha_n} = g^{1/2}\epsilon_{\alpha_1\ldots\alpha_n} \ , \eta^{\alpha_1\ldots\alpha_n} = (-1)^N g^{-1/2}\epsilon_{\alpha_1\ldots\alpha_n} \ . \tag{A.29}$$

For a Lorentz 4-dimensional spacetime, it is convenient to use the ordered index labels 0,1,2,3 so that one has the Misner, Thorne and Wheeler convention

$$\epsilon_{0123} = 1 = \epsilon^{0123} \tag{A.30}$$

and therefore the unit-oriented 4-form has components

$$\eta_{0123} = g^{1/2} \ , \ \eta^{0123} = -g^{-1/2} \ . \tag{A.31}$$

Others prefer the Ellis convention using the index labels 1,2,3,4, but one finds both sign conventions $\eta_{1234} = g^{1/2}$ and $\eta_{1234} = -g^{1/2}$ in use, making comparisons difficult.

A.7 Connection

A general linear connection or covariant derivative ∇ on M is (locally) determined by its components in a frame. Adopting the "del" convention for the order of the covariant indices, these components are defined by

$$\nabla_{e_\alpha} e_\beta = \Gamma^\gamma{}_{\alpha\beta} e_\gamma \quad \leftrightarrow \quad \nabla_{e_\alpha}\omega^\beta = -\Gamma^\beta{}_{\alpha\gamma}\omega^\gamma \ . \tag{A.32}$$

The covariant derivative ∇S of an arbitrary $\binom{p}{q}$-tensor S is a $\binom{p}{q+1}$-tensor with components

$$\begin{aligned} [\nabla S]^{\alpha\ldots}{}_{\beta\ldots\gamma} &\equiv \nabla_\gamma S^{\alpha\ldots}{}_{\beta\ldots} \equiv S^{\alpha\ldots}{}_{\beta\ldots;\gamma} \\ &= S^{\alpha\ldots}{}_{\beta\ldots,\gamma} + \Gamma^\alpha{}_{\gamma\delta}S^{\delta\ldots}{}_{\beta\ldots} + \cdots - \Gamma^\delta{}_{\gamma\beta}S^{\alpha\ldots}{}_{\delta\ldots} - \cdots \\ &\equiv S^{\alpha\ldots}{}_{\beta\ldots,\gamma} + [\sigma^{p,q}(\boldsymbol{\omega}_\gamma)S]^{\alpha\ldots}{}_{\beta\ldots} \ . \end{aligned} \tag{A.33}$$

For a $\binom{p}{q}$-tensor density of weight W, one need only replace $\sigma^{p,q}(\boldsymbol{\omega}_\gamma)$ by $\sigma^{p,q}_W(\boldsymbol{\omega}_\gamma)$, adding the additional term $-W\Gamma^\delta{}_{\gamma\delta}S^{\alpha\ldots}{}_{\beta\ldots}$.

The $\binom{1}{1}$-tensor-valued connection 1-form for the frame $\{e_\alpha\}$ is defined by

$$\boldsymbol{\omega} = e_\alpha \otimes \boldsymbol{\omega}^\alpha{}_\beta \otimes \omega^\beta = \Gamma^\alpha{}_{\gamma\beta} e_\alpha \otimes \omega^\gamma \otimes \omega^\beta \ , \tag{A.34}$$

but is often treated as a matrix-valued 1-form whose entries are

$$\boldsymbol{\omega}^\alpha{}_\beta = \Gamma^\alpha{}_{\gamma\beta}\omega^\gamma \ . \tag{A.35}$$

The γ component of this matrix of 1-forms is understood to be the argument of the the linear operator σ in the definition of the covariant derivative.

The components of the torsion tensor of the connection are

$$T^\alpha{}_{\beta\gamma} = 2\Gamma^\alpha{}_{[\beta\gamma]} - C^\alpha{}_{\beta\gamma} \tag{A.36}$$

and define a vector-valued torsion 2-form

$$\boldsymbol{\Theta}^\alpha = \tfrac{1}{2} T^\alpha{}_{\beta\gamma}\omega^{\beta\gamma} \ , \qquad \boldsymbol{\Theta} = e_\alpha \otimes \boldsymbol{\Theta}^\alpha \ . \tag{A.37}$$

Of course as tensor fields $\boldsymbol{\Theta} = T$, but the different notation helps distinguish the tensor-valued differential form interpretation from the tensor, and provides a notation for evaluating only the differential form arguments of the tensor. In most of the literature $\boldsymbol{\Theta}$ is treated as an R^n-valued 2-form ($\boldsymbol{\Theta}^\alpha$), as are all other vector-valued forms.

The frame independent definition of the torsion 2-form, with its 2-form arguments evaluated on a pair of vector fields X and Y is

$$\nabla_X Y - \nabla_Y X - [X,Y] = \boldsymbol{\Theta}(X,Y) \ . \tag{A.38}$$

Evaluation of this formula using the frame vectors themselves leads to the above component formula. This may also be expressed equivalently in terms of the exterior derivative

$$\begin{aligned} &d\omega^\alpha + \boldsymbol{\omega}^\alpha{}_\beta \wedge \omega^\beta = \boldsymbol{\Theta}^\alpha \ , \\ &d\boldsymbol{\vartheta} + \boldsymbol{\omega} \llcorner\!\wedge \boldsymbol{\vartheta} = \boldsymbol{\Theta} \ , \end{aligned} \tag{A.39}$$

where the right contraction is between the tensor-valued indices and the wedge is between the differential form indices, and

$$\boldsymbol{\vartheta} = e_\alpha \otimes \boldsymbol{\vartheta}^\alpha = \delta^\alpha{}_\beta e_\alpha \otimes \omega^\beta \tag{A.40}$$

is the identity tensor thought of as a vector-valued 1-form with component 1-forms $\boldsymbol{\vartheta}^\alpha = \omega^\alpha$ equal to the frame 1-forms.

A connection for which the torsion is zero is called torsion-free or symmetric, since in a coordinate frame the connection components are symmetric in the pair of covariant indices

$$T^\alpha{}_{\beta\gamma} = 0 = C^\alpha{}_{\beta\gamma} \quad \to \quad \Gamma^\alpha{}_{[\beta\gamma]} = 0 \ . \tag{A.41}$$

For a general connection, one can always introduce a 1-parameter family of related connections whose torsion is any real multiple of the original torsion by using a multiple of the torsion tensor itself as a difference tensor between the two connections. In particular there is a symmetric connection SYM ∇ with zero torsion and a transposed connection $\tilde{\nabla}$ with opposite torsion

$$\begin{aligned} \Gamma^\alpha{}_{\beta\gamma} &= \Gamma^\alpha{}_{(\beta\gamma)} + \Gamma^\alpha{}_{[\beta\gamma]} = \Gamma^\alpha{}_{(\beta\gamma)} + \tfrac{1}{2}C^\alpha{}_{\beta\gamma} + \tfrac{1}{2}T^\alpha{}_{\beta\gamma} \ , \\ [\mathrm{SYM}\,\Gamma]^\alpha{}_{\beta\gamma} &= \Gamma^\alpha{}_{\beta\gamma} - \tfrac{1}{2}T^\alpha{}_{\beta\gamma} \ = \Gamma^\alpha{}_{(\beta\gamma)} + \tfrac{1}{2}C^\alpha{}_{\beta\gamma} \ , \\ \tilde{\Gamma}^\alpha{}_{\beta\gamma} &= \Gamma^\alpha{}_{\beta\gamma} - T^\alpha{}_{\beta\gamma} \quad = \Gamma^\alpha{}_{(\beta\gamma)} + \tfrac{1}{2}C^\alpha{}_{\beta\gamma} - \tfrac{1}{2}T^\alpha{}_{\beta\gamma} \\ &= \Gamma^\alpha{}_{\gamma\beta} + C^\alpha{}_{\beta\gamma} \ . \end{aligned} \tag{A.42}$$

In a coordinate frame these reduce to simple relationships

$$[\mathrm{SYM}\,\Gamma]^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{(\beta\gamma)} \ , \qquad \tilde{\Gamma}^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\gamma\beta} \ . \tag{A.43}$$

Such triplets of connections play an important role in the geometry of Lie groups.

A.8 Metric connection

A connection is said to be metric with respect to a given metric g if that metric is covariant constant

$$\begin{aligned} 0 = g_{\alpha\beta;\gamma} &= g_{\alpha\beta,\gamma} - g_{\delta\beta}\Gamma^{\delta}{}_{\gamma\alpha} - g_{\alpha\delta}\Gamma^{\delta}{}_{\gamma\beta} \\ &= g_{\alpha\beta,\gamma} - \Gamma_{\alpha\gamma\beta} - \Gamma_{\beta\gamma\alpha} \rightarrow g_{\alpha\beta,\gamma} = 2\Gamma_{(\alpha|\gamma|\beta)} \ , \end{aligned} \tag{A.44}$$

extending the index-shifting convention to the components of the connection. This relation may be inverted using the definition of the torsion to express the components of the connection entirely in terms of the torsion, the structure functions of the frame and the ordinary derivatives of the metric components. Using the notation

$$A_{\{\delta\beta\gamma\}_-} = A_{\delta\beta\gamma} - A_{\beta\gamma\delta} + A_{\gamma\delta\beta} \ . \tag{A.45}$$

and shifting indices on the structure functions with the frame component matrix of the spacetime metric, the connection components can be written

$$\begin{aligned} \Gamma^{\alpha}{}_{\beta\gamma} &= \tfrac{1}{2} g^{\alpha\delta}(g_{\{\delta\beta,\gamma\}_-} + C_{\{\delta\beta\gamma\}_-} + T_{\{\delta\beta\gamma\}_-}) \ , \\ &= \Gamma(\mathrm{g})^{\alpha}{}_{\beta\gamma} + K^{\alpha}{}_{\beta\gamma} \ , \end{aligned} \tag{A.46}$$

where

$$\begin{aligned} \Gamma(\mathrm{g})^{\alpha}{}_{\beta\gamma} &= \tfrac{1}{2} g^{\alpha\delta}(g_{\{\delta\beta,\gamma\}_-} + C_{\{\delta\beta\gamma\}_-}) \ , \\ K^{\alpha}{}_{\beta\gamma} &= \tfrac{1}{2} g^{\alpha\delta} T_{\{\delta\beta\gamma\}_-} \ . \end{aligned} \tag{A.47}$$

$\Gamma(\mathrm{g})^{\alpha}{}_{\beta\gamma}$ are the components of the unique metric connection for which the torsion vanishes, often just called "the metric connection." Its connection components in a coordinate frame reduce to the Christoffel symbols of the metric. $K^{\alpha}{}_{\beta\gamma}$ are the components of the difference tensor between the general metric connection and the unique one associated with the metric. This difference tensor is sometimes referred to as the contorsion tensor.

A.9 Curvature

The components of the curvature tensor of a connection in a frame are defined by

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\delta\beta,\gamma} - \Gamma^{\alpha}{}_{\gamma\beta,\delta} - C^{\epsilon}{}_{\gamma\delta}\Gamma^{\alpha}{}_{\epsilon\beta} + \Gamma^{\alpha}{}_{\gamma\epsilon}\Gamma^{\epsilon}{}_{\delta\beta} - \Gamma^{\alpha}{}_{\delta\epsilon}\Gamma^{\epsilon}{}_{\gamma\beta} \ . \tag{A.48}$$

This tensor is explicitly antisymmetric in its last pair of indices and so defines a $\binom{1}{1}$-tensor-valued 2-form $\boldsymbol{\Omega} = R$

$$\boldsymbol{\Omega} = e_{\alpha} \otimes \omega^{\beta} \otimes \boldsymbol{\Omega}^{\alpha}{}_{\beta} = \tfrac{1}{2} R^{\alpha}{}_{\beta\gamma\delta} e_{\alpha} \otimes \omega^{\beta} \otimes \omega^{\gamma\delta} \ , \tag{A.49}$$

whose tensor-valued components are often thought of as a matrix-valued 2-form whose entries are

$$\boldsymbol{\Omega}^{\alpha}{}_{\beta} = \tfrac{1}{2} R^{\alpha}{}_{\beta\gamma\delta}\omega^{\gamma\delta} \ . \tag{A.50}$$

The $\binom{1}{1}$-tensor values of this 2-form define linear transformations of the tangent and cotangent spaces.

The invariant definition of the curvature tensor of a connection with the 2-form arguments evaluated on a pair of vector fields X and Y, and acting as a linear transformation on the vector field Z, is

$$\{[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\} Z = \boldsymbol{\Omega}(X,Y) \llcorner Z \ . \tag{A.51}$$

When evaluated on the frame vectors themselves, one obtains the above component formula.

This may also be represented alternatively in terms of the connection and curvature forms

$$\begin{aligned} & d\boldsymbol{\omega}^{\alpha}{}_{\beta} + \boldsymbol{\omega}^{\alpha}{}_{\gamma} \wedge \boldsymbol{\omega}^{\gamma}{}_{\beta} = \boldsymbol{\Omega}^{\alpha}{}_{\beta} \ , \\ & d\boldsymbol{\omega} + \boldsymbol{\omega} \llcorner \wedge \boldsymbol{\omega} = \boldsymbol{\Omega} \end{aligned} \tag{A.52}$$

where again the right contraction refers to the adjacent tensor-valued indices and the wedge product to the differential form indices.

The totally covariant Riemann tensor obtained by lowering its first index can be re-expressed in terms of the covariant connection component derivatives using (A.44) to re-express the metric derivatives in terms of the symmetric part of the connection. For a torsion-free metric connection one finds

$$R_{\alpha\beta\gamma\delta} = \Gamma_{\alpha\delta\beta,\gamma} - \Gamma_{\alpha\gamma\beta,\delta} - C^{\epsilon}{}_{\gamma\delta}\Gamma_{\alpha\epsilon\beta} + \Gamma_{\epsilon\delta\alpha}\Gamma^{\epsilon}{}_{\gamma\beta} - \Gamma_{\epsilon\gamma\alpha}\Gamma^{\epsilon}{}_{\delta\beta}\ . \tag{A.53}$$

This in turn may be used to make the second derivatives of the metric coefficients explicit in a coordinate frame ($C^{\alpha}{}_{\beta\gamma} = 0$) leading to the Landau-Lifshitz formula (92.4) from their *Classical Theory of Fields*

$$R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}(g_{\alpha\delta,\beta\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\gamma,\beta\delta} - g_{\beta\delta,\alpha\gamma}) + \Gamma_{\epsilon\delta\alpha}\Gamma^{\epsilon}{}_{\gamma\beta} - \Gamma_{\epsilon\gamma\alpha}\Gamma^{\epsilon}{}_{\delta\beta}\ . \tag{A.54}$$

A.10 Total covariant derivative

The total covariant derivative of a tensor field along a parametrized curve generalizes the ordinary derivative of a function along such a curve. It is called the intrinsic or absolute derivative in the older literature [Synge and Schild 1949, Møller 1952] and occurs as a special case of an induced connection over a map in the modern literature [Sachs and Wu 1977, Bishop and Goldberg 1968]. The terminology "total covariant derivative" as used in this text may be a result of teaching too many calculus classes while thinking about relativity.

Given a parametrized curve $c(\lambda)$, with corresponding tangent vector $c'(\lambda)$, one can covariantly differentiate any tensor $S(\lambda)$ defined along the curve by correcting the ordinary derivative by the representation of the connection 1-form matrix evaluated on the tangent vector

$$DS^{\alpha\cdots}{}_{\beta\ldots}/d\lambda = dS^{\alpha\cdots}{}_{\beta\ldots}/d\lambda + [\sigma(\boldsymbol{\omega}(c'(\lambda)))S(\lambda)]^{\alpha\cdots}{}_{\beta\ldots}\ . \tag{A.55}$$

The correction terms come from the product rule and the total covariant derivatives of the frame vectors and dual 1-forms along the curve

$$D[e_\alpha \circ c(\lambda)]/d\lambda = \boldsymbol{\omega}(c'(\lambda))^{\beta}{}_{\alpha} e_\beta\ , \qquad D[\omega^\alpha \circ c(\lambda)]/d\lambda = -\boldsymbol{\omega}(c'(\lambda))^{\alpha}{}_{\beta}\omega^\beta\ . \tag{A.56}$$

For a tensor field S, the total covariant derivative of its restriction to the curve $S(\lambda) = S \circ c(\lambda)$ agrees with the covariant derivative of S along the tangent vector at each point of the curve

$$[\nabla_{c'(\lambda)} S] \circ c(\lambda) = DS(\lambda)/d\lambda\ . \tag{A.57}$$

It is convenient to follow the sloppy convention of not distinguishing between the two types of derivatives and suppressing the parametrization variable, writing simply

$$DS/d\lambda = \nabla_{c'} S\ . \tag{A.58}$$

Given a local coordinate system, a parametrized curve is represented by the composed functions $c^\alpha(\lambda) = x^\alpha \circ c(\lambda)$, the derivatives of which provide the coordinate frame components of its tangent vector $[c'(\lambda)]^\alpha = dc^\alpha(\lambda)/d\lambda = c^{\alpha\prime}(\lambda)$, often sloppily represented by the symbols $x^\alpha(\lambda)$ and $dx^\alpha(\lambda)/d\lambda$ respectively. For a vector field $X(\lambda)$ defined along the curve, the coordinate frame expression for its total covariant derivative then takes the form

$$DX^\alpha(\lambda)/d\lambda = dX^\alpha(\lambda)/d\lambda + \Gamma^{\alpha}{}_{\beta\gamma} \circ c(\lambda) c^{\beta\prime}(\lambda) X^\gamma(\lambda)\ . \tag{A.59}$$

For the tangent vector itself, this becomes

$$[Dc'(\lambda)]^\alpha = d^2c^\alpha(\lambda)/d\lambda + \Gamma^{\alpha}{}_{\beta\gamma} \circ c(\lambda) dc^\beta(\lambda)/d\lambda dc^\gamma(\lambda)/d\lambda\ . \tag{A.60}$$

Under a change of parametrization $\lambda = f(\overline{\lambda})$ of the curve, the total covariant derivative changes according to the familiar chain rule

$$DS/d\overline{\lambda} = (d\lambda/d\overline{\lambda})\, DS/d\lambda\ . \tag{A.61}$$

A.11 Parallel transport and geodesics

A tensor field $S(\lambda)$ defined along a parametrized curve $c(\lambda)$ is said to be parallel transported (sometimes "propagated") along the curve if its total covariant derivative is identically zero

$$DS(\lambda)/d\lambda = 0 \ . \tag{A.62}$$

A parametrized curve whose tangent vector is parallel transported along the curve is called a geodesic

$$Dc'(\lambda)/d\lambda = 0 \ . \tag{A.63}$$

When expressed in a coordinate system, these are second order ordinary differential equations for the values of the coordinate functions along the parametrized curve.

A.12 Generalized Kronecker deltas

The following identities define the generalized Kronecker deltas and relate them to the Levi-Civita epsilon and to the unit volume n-form η when a metric is available

$$\begin{aligned}
\delta^{\alpha_1\ldots\alpha_n}{}_{\beta_1\ldots\beta_n} &= \epsilon^{\alpha_1\ldots\alpha_n}\epsilon_{\beta_1\ldots\beta_n} = (-1)^N \eta^{\alpha_1\ldots\alpha_n}\eta_{\beta_1\ldots\beta_n} \ , \\
\delta^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_p} &= \frac{1}{(n-p)!}\delta^{\alpha_1\ldots\alpha_p\gamma_{p+1}\ldots\gamma_n}{}_{\beta_1\ldots\beta_p\gamma_{p+1}\ldots\gamma_n} \\
&= (-1)^N\frac{1}{(n-p)!}\eta^{\alpha_1\ldots\alpha_p\gamma_{p+1}\ldots\gamma_n}\eta_{\beta_1\ldots\beta_p\gamma_{p+1}\ldots\gamma_n} \\
&= (-1)^{N+p(n-p)}\frac{1}{(n-p)!}\eta^{\alpha_1\ldots\alpha_p\gamma_{p+1}\ldots\gamma_n}\eta_{\beta_1\ldots\beta_p\gamma_{p+1}\ldots\gamma_n} \ .
\end{aligned} \tag{A.64}$$

Alternative definitions are instead

$$\begin{aligned}
\delta^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_p} = \begin{vmatrix} \delta^{\alpha_1}{}_{\beta_1} & \cdots & \delta^{\alpha_1}{}_{\beta_p} \\ \vdots & & \vdots \\ \delta^{\alpha_p}{}_{\beta_1} & \cdots & \delta^{\alpha_p}{}_{\beta_p} \end{vmatrix} &= p!\,\delta^{\alpha_1}{}_{[\beta_1}\cdots\delta^{\alpha_p}{}_{\beta_p]} \\
&= p\,\delta^{\alpha_1}{}_{[\beta_1}\delta^{\alpha_2\ldots\alpha_p}{}_{\beta_2\ldots\beta_p]} \\
&= \sum_{i=1}^{p}(-1)^{i-1}\delta^{\alpha_i}{}_{\beta_1}\delta^{\alpha_1\ldots\hat{\alpha}_i\ldots\alpha_p}{}_{\beta_2\ldots\qquad\beta_p} \ ,
\end{aligned} \tag{A.65}$$

where here the hatted index notation implies the removal of the index from the indicated position.

A.13 Symmetrization/antisymmetrization

Given an object with only covariant or contravariant indices, or a subset of only covariant or contravariant indices from those of a mixed object, only can always project out the purely symmetric and purely antisymmetric parts. For example for a $\binom{0}{p}$-tensor field one has

$$\begin{aligned}
[\mathrm{ALT}\,S]_{\alpha_1\ldots\alpha_p} &= S_{[\alpha_1\ldots\alpha_p]} = \frac{1}{p!}\delta^{\beta_1\ldots\beta_p}{}_{\alpha_1\ldots\alpha_p}S_{\beta_1\ldots\beta_p} \ , \\
[\mathrm{SYM}\,S]_{\alpha_1\ldots\alpha_p} &= S_{(\alpha_1\ldots\alpha_p)} = \frac{1}{p!}\sum_\sigma S_{\sigma(\alpha_1)\ldots\sigma(\alpha_p)} \ .
\end{aligned} \tag{A.66}$$

A.14 Exterior product

The exterior or wedge product of p 1-forms like the frame 1-forms is defined antisymmetrizing without the factorial factor

$$\omega^{\alpha_1\ldots\alpha_p} = \omega^{\alpha_1} \wedge \cdots \wedge \omega^{\alpha_p} = p!\, \omega^{[\alpha_1} \otimes \cdots \otimes \omega^{\alpha_p]} = \delta^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_p} \omega^{\beta_1} \otimes \cdots \otimes \omega^{\beta_p} \ . \tag{A.67}$$

With this convention the components of an antisymmetric tensor are themselves the expansion coefficients in the basis $\{\omega^{|\alpha_1\ldots\alpha_p|}\}$.

This leads to a correction factor for the exterior product of p-forms with $p > 1$. For a p-form S and a q-form T, one has

$$\begin{aligned} [S \wedge T]_{\alpha_1\ldots\alpha_{p+q}} &= \frac{(p+q)!}{p!q!} S_{[\alpha_1\ldots\alpha_p} T_{\alpha_{p+1}\ldots\alpha_{p+q}]} \\ &= \delta^{\beta_1\ldots\beta_p\gamma_1\ldots\gamma_q}{}_{\alpha_1 \quad \ldots \quad \alpha_{p+q}} S_{|\beta_1\ldots\beta_p|} T_{|\gamma_1\ldots\gamma_p|} \ . \end{aligned} \tag{A.68}$$

One has the identity

$$S \wedge T = (-1)^{pq}\, T \wedge S \ . \tag{A.69}$$

A.15 Hodge star duality operation

The Hodge star duality operation is defined for a p-form S to be the $(n-p)$ form ${}^*S$ with components

$$[{}^*S]_{\alpha_{p+1}\ldots\alpha_n} = \frac{1}{p!} S_{\alpha_1\ldots\alpha_p} \eta^{\alpha_1\ldots\alpha_p}{}_{\alpha_{p+1}\ldots\alpha_n} \tag{A.70}$$

following the right dual convention of Misner, Thorne and Wheeler in which the contraction of $S^\sharp$ with η has η on the right.

By introducing the $(n-p)$-forms

$$\eta^{\alpha_1\ldots\alpha_p} = {}^*\omega^{\alpha_1\ldots\alpha_p} = \frac{1}{(n-p)!} \eta^{\alpha_1\ldots\alpha_p}{}_{\alpha_{p+1}\ldots\alpha_n} \omega^{\alpha_{p+1}\ldots\alpha_n} \ , \tag{A.71}$$

one can re-express this in the following way

$$\begin{aligned} {}^*S &= \frac{1}{p!} S_{\alpha_1\ldots\alpha_p} \eta^{\alpha_1\ldots\alpha_p} \\ &= \frac{1}{p!} S^\sharp \overset{p}{\lrcorner} \eta \ , \end{aligned} \tag{A.72}$$

where the generalized left contraction of p indices indicates the contraction of the last p contravariant indices of the left factor with the first p covariant indices of the right factor in order. Note the extreme cases

$${}^*1 = \eta \ , \qquad {}^*\eta = (-1)^N \ . \tag{A.73}$$

Using the above definitions and identities one establishes the identity

$${}^{**}S = (-1)^{N+p(n-p)} S \tag{A.74}$$

for a p-form S, where the term N in the sign arises from the signature and the second from the interchange of the group of indices. For a 4-dimensional Lorentzian manifold and a 3-dimensional Riemannian manifold respectively, this reduces to

$$\begin{aligned} &n = 4,\ N = 1: \qquad {}^{**}S = (-1)^{p-1} S \qquad \text{(spacetime)} \\ &n = 3,\ N = 0: \qquad {}^{**}S = S \qquad \text{(space)} \ . \end{aligned} \tag{A.75}$$

This is a special case of the more general identity for a p-form S and a q-form T with $q \geq p$

$${}^*(S \wedge {}^*T) = (-1)^{N+(n-q)(q-p)} \frac{1}{p!} S^\sharp \overset{p}{\lrcorner} T \ . \tag{A.76}$$

With $p = 0$ and $S = 1$ this reduces to the previous identity. With $p = q$ one instead has

$$^*(S \wedge {}^*T) = (-1)^N \, \frac{1}{p!} S^\sharp \overset{p}{\lrcorner} T \ , \tag{A.77}$$

or since $^*\eta = (-1)^N$, removing the duality operation from each side leads to

$$S \wedge {}^*T = \frac{1}{p!} S^\sharp \overset{p}{\lrcorner} T \, \eta = \langle S, T \rangle \, \eta \ , \tag{A.78}$$

where

$$\langle S, T \rangle = \frac{1}{p!} S^{\alpha_1 \ldots \alpha_p} T_{\alpha_1 \ldots \alpha_p} \tag{A.79}$$

defines a natural local (pointwise) inner product between p-forms.

A.16 Complex Duality Operation

For a 4-dimensional Lorentz spacetime, 2-forms ("bivectors") and 4-forms which are symmetric under exchange of the first and second index pairs (like the Riemann and Weyl tensors) play an important role. With $N = 1$ and $p = 2$, one has $^{*\,*}S = -S$ for all 2-forms. The minus sign comes from the fact that raising all the indices of the unit volume 4-form in an orthonormal frame changes the sign of the ordered index component as in (A.29). By using the square root of the negative metric determinant rather than of its absolute value in the definition of the volume form, the extra sign in (A.29) is eliminated

$$E_{\alpha_1 \ldots \alpha_4} = (-g)^{1/2} \epsilon_{\alpha_1 \ldots \alpha_4} = i \eta_{\alpha_1 \ldots \alpha_n} \ , \ E^{\alpha_1 \ldots \alpha_4} = (-g)^{-1/2} \epsilon^{\alpha_1 \ldots \alpha_4} = -i \eta^{\alpha_1 \ldots \alpha_4} \ . \tag{A.80}$$

Using this purely imaginary 4-form instead of η for the duality operation on 2-form index pairs defines the "hook" duality operation with symbol $\breve{\ }$, apparently used by Veblen and von Neumann, and later Taub.

This leads to $\breve{\ } S = i\,{}^*S$ on 2-forms which therefore satisfy $\breve{\ }\breve{\ } S = S$, so that one can find eigentensors of the operation with eigenvalues ± 1, called self-dual (+) and anti-self-dual (−) respectively. Self-duality then translates back to $i\,{}^*S = S$ or $^*S = -iS$.

A.17 Exterior derivative

The exterior derivative is defined to obey the following identity for a p-form S and a q-form T

$$d(S \wedge T) = dS \wedge T + (-1)^p S \wedge dT \ , \tag{A.81}$$

and to reduce to the ordinary differential of a 0-form or function

$$df = \partial_\alpha f \, \omega^\alpha \ . \tag{A.82}$$

For a p-form

$$S = \frac{1}{p!} S_{\alpha_1 \ldots \alpha_p} \omega^{\alpha_1 \ldots \alpha_p} \ , \tag{A.83}$$

this identity yields

$$dS = \frac{1}{p!} dS_{\alpha_1 \ldots \alpha_p} \wedge \omega^{\alpha_1 \ldots \alpha_p} + \frac{1}{p!} S_{\alpha_1 \ldots \alpha_p} d\omega^{\alpha_1 \ldots \alpha_p} = \frac{1}{(p+1)!} [dS]_{\alpha_1 \ldots \alpha_{p+1}} \omega^{\alpha_1 \ldots \alpha_{p+1}} \ , \tag{A.84}$$

while for the basis p-forms it yields

$$d\omega^{\alpha_1 \ldots \alpha_p} = -\tfrac{1}{2} \sum_{i=1}^{p} (-1)^{i-1} C^{\alpha_i}{}_{\beta\gamma} \omega^{\alpha_1 \ldots \beta\gamma \ldots \alpha_p} \ . \tag{A.85}$$

Putting these together yields the formula

$$[dS]_{\alpha_1 \ldots \alpha_{p+1}} = (p+1)(\partial_{[\alpha_1} S_{\alpha_2 \ldots \alpha_{p+1}]} - \tfrac{1}{2} p C^\beta{}_{[\alpha_1 \alpha_2} S_{|\beta| \alpha_3 \ldots \hat{\alpha}_i \ldots \alpha_{p+1}]}) \ , \tag{A.86}$$

where the vertical bar delimiters exclude the indicated index from the antisymmetrization. Only the first term remains in a coordinate frame.

This result may be re-expressed in terms of a general connection as

$$[dS]_{\alpha_1\ldots\alpha_{p+1}} = (p+1)(\nabla_{[\alpha_1}S_{\alpha_2\ldots\alpha_{p+1}]} + \tfrac{1}{2}pT^{\gamma}{}_{[\alpha_1\alpha_2}S_{|\gamma|\alpha_3\ldots\hat{\alpha}_i\ldots\alpha_p]}) \ . \tag{A.87}$$

For a symmetric connection only the first term remains yielding a formula which arises by substituting a covariant derivative for the ordinary derivative in the coordinate frame formula, i.e., the "comma to semicolon" rule.

The exterior derivative may be expressed in a frame-independent way by evaluating its vector field arguments

$$dS(X_1,\ldots,X_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i-1}X_iS(X_1,\ldots,\hat{X}_j,\ldots,X_{p+1}) + \sum_{i<j}(-1)^{i+j}S([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{p+1}) \tag{A.88}$$

where the hat notation here means that the corresponding arguments are omitted. For a function and 1-form respectively this reduces to the result

$$\begin{aligned} df(X) &= Xf \ , \\ dS(X,Y) &= X\,S(Y) - Y\,S(X) - S([X,Y]) \ . \end{aligned} \tag{A.89}$$

A.18 Differential form divergence operator

For a $(p-1)$form S and a p-form T, the identities

$$\begin{aligned} d(S\wedge {}^*T) &= dS\wedge {}^*T + (-1)^{p-1}S\wedge d\,{}^*T \\ &= \langle dS,T\rangle\eta - (-1)^{N+p+(n-p+1)(p-1)}\langle S,\,{}^*d\,{}^*T\rangle\,\eta \\ &= \langle dS,T\rangle\eta - \langle S,\delta T\rangle\,\eta \end{aligned} \tag{A.90}$$

define the adjoint operator δ for a p-form

$$\delta T = (-1)^{N+1+n(p-1)}\,{}^*d\,{}^*T \ . \tag{A.91}$$

For an appropriate space of p-forms for which the integral of the above exact n-form vanishes, this is the adjoint operator with respect to the global inner product defined by the integral of the local inner product over the manifold

$$\langle\!\langle dS,T\rangle\!\rangle = \int\langle dS,T\rangle\eta = \langle\!\langle S,\delta T\rangle\!\rangle \ . \tag{A.92}$$

This has the interesting special cases

$$\begin{aligned} &n=4,\,N=1: \qquad \delta T = {}^*d\,{}^*T \qquad \text{(spacetime)} \\ &n=3,\,N=0: \qquad \delta T = (-1)^p\,{}^*d\,{}^*T \qquad \text{(space)} \end{aligned} \tag{A.93}$$

for a 4-dimensional Lorentzian manifold and a 3-dimensional Riemannian manifold respectively.

For a metric connection ∇, it is easy to obtain a more familiar expression using the covariant derivative expression for the exterior derivative

$$[\nabla_{\alpha_1}S_{\alpha_2\ldots\alpha_p}]T^{\alpha_1\ldots\alpha_p}\eta = \nabla_{\alpha_1}[S_{\alpha_2\ldots\alpha_p}T^{\alpha_1\ldots\alpha_p}]\eta - S_{\alpha_2\ldots\alpha_p}[\nabla_{\alpha_1}T^{\alpha_1\ldots\alpha_p}]\eta \ . \tag{A.94}$$

Identifying this equation with the above identity, the divergence term corresponding to the exact term above, leads one to the simple result that δ is just the negative of the covariant divergence operator

$$[\delta T]_{\alpha_2\ldots\alpha_p} = -\nabla^{\alpha}T_{\alpha\alpha_2\ldots\alpha_p} = -[\operatorname{div}T]_{\alpha_2\ldots\alpha_p} \ . \tag{A.95}$$

The covariant divergence term and exact term are related by the following identity for a 1-form X

$$\delta X\,\eta = {}^*\delta X = (-1)^{N+1}\,{}^*{}^*d\,{}^*X = -d\,{}^*X \tag{A.96}$$

which can also be written in the form

$$d(X^\sharp \lrcorner\,\eta) = [\operatorname{div}X]\,\eta \ . \tag{A.97}$$

A.19 De Rham Laplacian

Again for the appropriate space of differential forms, one has the self-adjoint operator $\Delta_{(\mathrm{dR})}$ defined by

$$\Delta_{(\mathrm{dR})} = \delta d + d\delta \tag{A.98}$$

which is called the de Rham Laplacian, differing from the covariant Laplacian Δ

$$[\Delta S]_{\alpha_1 \ldots \alpha_p} = -\nabla_\alpha \nabla^\alpha S_{\alpha_1 \ldots \alpha_p} \tag{A.99}$$

by curvature terms which arise from the Ricci identity. For the linear connection associated with a metric it is given by

$$\begin{aligned}
[\Delta_{(\mathrm{dR})} S]_{\alpha_1 \ldots \alpha_p} &= [\Delta S]_{\alpha_1 \ldots \alpha_p} + \sum_{i=1}^{p} R^{\beta}{}_{\alpha_i} S_{\alpha_1 \ldots \beta \ldots \alpha_p} \\
&\quad - \sum_{i \neq j=1}^{p} R^{\beta}{}_{\alpha_i}{}^{\gamma}{}_{\alpha_j} S_{\alpha_1 \ldots \beta \ldots \gamma \ldots \alpha_p} \\
&= [\Delta S]_{\alpha_1 \ldots \alpha_p} + \sum_{i=1}^{p} (-1)^i R^{\beta}{}_{\alpha_i} S_{\beta \alpha_2 \ldots \hat{\alpha}_i \ldots \alpha_p} \\
&\quad + \sum_{i \neq j=1}^{p} (-1)^{i+j} R^{\beta}{}_{\alpha_i}{}^{\gamma}{}_{\alpha_j} S_{\beta\gamma\alpha_1 \ldots \hat{\alpha}_i \ldots \hat{\alpha}_j \ldots \alpha_p} \ ,
\end{aligned} \tag{A.100}$$

where $R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta}$ is the symmetric Ricci tensor defined below.

For an $n = 4$ Lorentzian spacetime (signature $- + ++$) and a 1-form A with $F = dA$, and $dF = 0$, $\delta F = -4\pi J$, one immediately finds

$$\begin{aligned}
\Delta_{(\mathrm{dR})} A - d\delta A &= -4\pi J \ , \\
\Delta_{(\mathrm{dR})} F &= -4\pi dJ \ .
\end{aligned} \tag{A.101}$$

A.20 Covariant exterior derivative

The *covariant exterior derivative* operator D is defined for tensor-valued differential forms [Schouten 1954, Misner, Thorne and Wheeler 1973] as a derivative operator which acts as the exterior derivative on the form indices and as the covariant derivative on the tensor-valued indices (wedging the extra covariant index into the form indices). This operator may be defined by the following formula

$$\begin{aligned}
S &= e_\alpha \otimes \cdots \otimes \omega^\beta \otimes \cdots \otimes S^{\alpha \ldots}{}_{\beta \ldots} \ , \\
DS &= e_\alpha \otimes \cdots \otimes \omega^\beta \otimes \cdots \otimes dS^{\alpha \ldots}{}_{\beta \ldots} + [\nabla \wedge][e_\alpha \otimes \cdots \otimes \omega^\beta \otimes \cdots] \otimes S^{\alpha \ldots}{}_{\beta \ldots} \ ,
\end{aligned} \tag{A.102}$$

where the notation $\nabla \wedge$ in the final formula indicates a wedge product between the additional covariant index of the covariant derivative and the coefficient differential forms. For an ordinary differential form, D reduces to d, while for an ordinary tensor, it reduces to ∇.

With the sloppy conventional notation

$$[DS]^{\alpha \ldots}{}_{\beta \ldots} \to DS^{\alpha \ldots}{}_{\beta \ldots} \ , \tag{A.103}$$

the formula can be written in the more conventional form

$$DS^{\alpha \ldots}{}_{\beta \ldots} = dS^{\alpha \ldots}{}_{\beta \ldots} + [\sigma(\boldsymbol{\omega}) \wedge S]^{\alpha \ldots}{}_{\beta \ldots} \tag{A.104}$$

or suppressing indices entirely as

$$DS = dS + \sigma(\boldsymbol{\omega}) \wedge S \ . \tag{A.105}$$

With this compact notation one easily derives the identity

$$D^2 S = \sigma(\boldsymbol{\Omega}) \wedge S \tag{A.106}$$

which generalizes the identity $d^2 = 0$ for ordinary differential forms.

The component formula for the covariant exterior derivative includes torsion terms to compensate for the noncovariant derivatives in the exterior derivative part

$$\begin{aligned}[DS^{\alpha\cdots}{}_{\beta\cdots}]_{\alpha_1\ldots\alpha_{p+1}} = (p+1)(\nabla_{[\alpha_1} S^{\alpha\cdots}{}_{|\beta\ldots|\,\alpha_2\ldots\alpha_{p+1}]} \\ + \tfrac{1}{2} p T^{\gamma}{}_{[\alpha_1\alpha_2} S^{\alpha\cdots}{}_{|\beta\ldots\gamma|[\alpha_2\ldots\hat{\alpha}_i\ldots\alpha_{p+1}]}) \ .\end{aligned} \tag{A.107}$$

where the vertical bar delimiters exclude the indicated indices from the antisymmetrization. For a symmetric connection only the first term remains, yielding a formula which can be obtained by the "comma-to-semicolon" transformation of the familiar coordinate frame formula for the exterior derivative.

Note that the definition of the torsion 2-form is very simple in terms of this notation. It is just the covariant exterior derivative of the identity tensor

$$\boldsymbol{\Theta} = D\boldsymbol{\vartheta} \ . \tag{A.108}$$

A metric may be considered as a tensor-valued 0-form whose covariant exterior derivative is

$$Dg_{\alpha\beta} = dg_{\alpha\beta} - g_{\gamma\beta}\boldsymbol{\omega}^{\gamma}{}_{\alpha} - g_{\alpha\gamma}\boldsymbol{\omega}^{\gamma}{}_{\beta} = dg_{\alpha\beta} - 2\boldsymbol{\omega}_{(\alpha\beta)} \ . \tag{A.109}$$

For a metric connection and a frame in which the metric components are constants, one has

$$0 = Dg_{\alpha\beta} = dg_{\alpha\beta} \quad \rightarrow \quad \boldsymbol{\omega}_{(\alpha\beta)} = 0 \ , \tag{A.110}$$

i.e., the connection 1-form is antisymmetric in its tensor-valued indices. This is true for orthonormal frames.

A.21 Ricci identities

The Ricci identity for a $\binom{p}{q}$-tensor field S is

$$[\nabla_\gamma, \nabla_\delta] S^{\alpha\cdots}{}_{\beta\cdots} = 2S^{\alpha\cdots}{}_{\beta\ldots;[\delta\gamma]} = [\sigma(\boldsymbol{\Omega}_{\gamma\delta})S]^{\alpha\cdots}{}_{\beta\cdots} - T^{\epsilon}{}_{\gamma\delta}\nabla_\epsilon S^{\alpha\cdots}{}_{\beta\cdots} \ . \tag{A.111}$$

Interpreting S as a $\binom{p}{q}$-tensor valued 0-form, this becomes a special case of the identity $D^2 S = \sigma(\boldsymbol{\Omega})S$.

For a metric and a metric connection one has

$$D^2 g_{\alpha\beta} = g_{\alpha\beta;[\delta\gamma]}\omega^{\gamma\delta} = -g_{\gamma\beta}\boldsymbol{\Omega}^{\gamma}{}_{\alpha} - g_{\alpha\gamma}\boldsymbol{\Omega}^{\gamma}{}_{\beta} = -2\boldsymbol{\Omega}_{(\alpha\beta)} \ . \tag{A.112}$$

Thus for a metric connection, the curvature 2-form is also antisymmetric in its tensor-valued indices

$$D^2 g_{\alpha\beta} = 0 \quad \rightarrow \quad \boldsymbol{\Omega}_{(\alpha\beta)} = 0 \quad \text{or} \quad R_{(\alpha\beta)\gamma\delta} = 0 \ . \tag{A.113}$$

A.22 Bianchi identities of the first and second kind

The Bianchi identity of the first kind re-expresses the covariant exterior derivative of the torsion 2-form and is a direct consequence of the previous identity for the vector-valued 1-form $\boldsymbol{\vartheta}$

$$D\boldsymbol{\Theta}^{\alpha} = D^2\boldsymbol{\vartheta}^{\alpha} = \boldsymbol{\Omega}^{\alpha}{}_{\beta} \wedge \boldsymbol{\vartheta}^{\beta} = \tfrac{1}{2} R^{\alpha}{}_{[\beta\gamma\delta]}\omega^{\beta\gamma\delta} \ . \tag{A.114}$$

For a symmetric connection this reduces to the familiar identity

$$3R^{\alpha}{}_{[\beta\gamma\delta]} = R^{\alpha}{}_{\beta\gamma\delta} + R^{\alpha}{}_{\gamma\delta\beta} + R^{\alpha}{}_{\delta\beta\gamma} = 0 \ . \tag{A.115}$$

The Bianchi identity of the second kind re-expresses the covariant exterior derivative of the curvature 2-form

$$\begin{aligned} D\boldsymbol{\Omega} = 0 \qquad \text{or} \\ 3R^{\alpha}{}_{\beta[\gamma\delta;\epsilon]} = R^{\alpha}{}_{\beta\gamma\delta;\epsilon} + R^{\alpha}{}_{\beta\delta\epsilon;\gamma} + R^{\alpha}{}_{\beta\epsilon\gamma;\delta} = 0 \ .\end{aligned} \tag{A.116}$$

For a symmetric metric connection, namely the unique such connection associated with a given metric, the cyclic Bianchi identity of the first kind (for zero torsion) together with the antisymmetry of the first index pair (covariant constant metric) leads to the pair interchange symmetry for the Riemann curvature tensor of the metric. Summing the four cyclic permutations of the Bianchi identity of the first kind

$$\begin{aligned} 0 &= 3(R_{\alpha[\beta\gamma\delta]} + R_{\beta[\gamma\delta\alpha]} + R_{\gamma[\delta\alpha\beta]} + R_{\delta[\alpha\beta\gamma]}) = \dots \\ &= 2(R_{\alpha\gamma\beta\delta} - R_{\beta\delta\alpha\gamma}) \ , \end{aligned} \tag{A.117}$$

leads to twelve terms, eight of which cancel in pairs due to the antisymmetry of the first index pair while the remaining four collapse to two terms. Relabeling the result yields the pair interchange symmetry

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \ . \tag{A.118}$$

A.23 Ricci Tensor and Scalar Curvature

For a metric connection the natural contraction of the curvature tensor defines the Ricci curvature

$$R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta} \tag{A.119}$$

which is a symmetric tensor due to the Bianchi identities of the first kind. Its trace defines the scalar curvature

$$R = R^{\alpha}{}_{\alpha} = R^{\gamma\alpha}{}_{\gamma\alpha} \ . \tag{A.120}$$

A.24 Contracted Bianchi Identities of the Second Kind and the Weyl Tensor

The Riemann curvature tensor may be expressed in terms of the Weyl curvature tensor, the Ricci tensor, and the curvature scalar as follows [Eisenhart 1966] (which may be taken as a definition of the Weyl curvature tensor)

$$\begin{aligned} R^{\alpha\beta}{}_{\gamma\delta} &= C^{\alpha\beta}{}_{\gamma\delta} + 4/(n-2) R^{[\alpha}{}_{[\gamma}\delta^{\beta]}{}_{\delta]} - R/[(n-1)(n-2)]\delta^{\alpha\beta}_{\gamma\delta} \\ &= C^{\alpha\beta}{}_{\gamma\delta} + 4/(n-2) Q^{[\alpha}{}_{[\gamma}\delta^{\beta]}{}_{\delta]} + R/[n(n-1)]\delta^{\alpha\beta}_{\gamma\delta} \end{aligned} \tag{A.121}$$

where the second equality expresses it instead using the tracefree part of the Ricci tensor

$$Q^{\alpha}{}_{\beta} = R^{\alpha}{}_{\beta} - 1/n\, \delta^{\alpha}{}_{\beta} \ , \tag{A.122}$$

while the Weyl tensor is itself tracefree $C^{\alpha}{}_{\beta\alpha\delta} = 0$.

The once contracted Bianchi identity of the second kind (using the tracefree property of the Weyl tensor) reduces to

$$\begin{aligned} 0 &= 3R^{\alpha\beta}{}_{[\gamma\delta;\epsilon]}\delta^{\gamma}{}_{\alpha} \\ &= R^{\alpha\beta}{}_{\delta\epsilon;\alpha} + 2R^{\beta}{}_{[\delta;\epsilon]} \\ &= C^{\alpha\beta}{}_{\delta\epsilon;\alpha} + (n-3)/(n-2)\, \mathcal{R}^{\beta}{}_{\delta\epsilon} \ , \end{aligned} \tag{A.123}$$

where

$$\mathcal{R}^{\beta}{}_{\delta\epsilon} = 2(R^{\beta}{}_{[\delta} - R/[2(n-1)]\delta^{\beta}{}_{[\delta})_{;\epsilon]} \ , \tag{A.124}$$

thus expressing the divergence of the Weyl tensor for $n > 3$ in terms of a vector-valued 2-form, the Cotton tensor [Cotton 1899], which is the covariant exterior derivative of a linear combination of the Ricci tensor and the scalar curvature seen as a vector valued 1-form.

The twice contracted Bianchi identity of the second kind

$$\begin{aligned} 0 &= R^{\alpha\beta}{}_{[\gamma\delta;\epsilon]}\delta^{\gamma}{}_{\alpha}\delta^{\delta}{}_{\beta} \\ &= -2G^{\delta}{}_{\epsilon;\delta} \ , \end{aligned} \tag{A.125}$$

which states that the Einstein tensor is divergence-free, immediately leads to the tracefree property of the Cotton tensor

$$\mathcal{R}^{\beta}{}_{\alpha\beta} = -G^{\beta}{}_{\alpha;\beta} = 0 \ , \tag{A.126}$$

whose totally antisymmetric part is easily seen to be zero using the symmetry of the Ricci and metric tensors

$$3R_{[\alpha\beta\gamma]} = R_{\alpha\beta\gamma} + R_{\beta\gamma\alpha} + R_{\gamma\alpha\beta} = 0 \ . \tag{A.127}$$

The divergence of the Cotton-tensor is also zero

$$R^{\beta}{}_{\delta\epsilon;\beta} = 2G^{\beta}{}_{[\delta;|\beta|;\epsilon]} = 0 \ . \tag{A.128}$$

For $n = 3$ dimensions where the Weyl tensor vanishes identically, one can take the dual of the covariant index pair of the Cotton tensor to obtain a two-index tensor, the Cotton-York tensor [York 1971]

$$C^{\alpha\beta} = R^{\alpha}{}_{\gamma\delta}\eta^{\beta\gamma\delta} = (R^{\alpha}{}_{\gamma} - 1/4R\delta^{\alpha}{}_{\gamma})_{;\delta}\eta^{\beta\gamma\delta} \ , \tag{A.129}$$

which can then be decomposed into its symmetric and antisymmetric parts, the latter of which is the spatial dual of a vector

$$\begin{aligned} C^{\alpha\beta} &= C^{(\alpha\beta)} + C^{[\alpha\beta]} = Y^{(\alpha\beta)} + \eta^{\alpha\beta\gamma}A_{\gamma} \\ R^{\alpha}{}_{\beta\gamma} &= C^{\alpha\delta}\eta_{\delta\beta\gamma} = \eta_{\beta\gamma\delta}Y^{\alpha\beta} + A_{\delta}\delta^{\delta\alpha}_{\beta\gamma} \ . \end{aligned} \tag{A.130}$$

The explicitly symmetric York tensor and the vector (which vanishes by the twice contracted Bianchi identity) are

$$\begin{aligned} y^{\alpha\beta} &= \eta^{\gamma\delta(\alpha}R^{\beta)}{}_{\gamma;\delta} = -[\mathrm{Scurl}(Ricci)]^{\alpha\beta} = -[\mathrm{Scurl}(Ricci^{(\mathrm{TF})})]^{\alpha\beta} = -[\mathrm{Scurl}(Einstein)]^{\alpha\beta} \ , \\ A_{\beta} &= R^{\alpha}{}_{\beta\alpha} = G^{\alpha}{}_{\beta;\alpha} = 0 \ , \end{aligned} \tag{A.131}$$

The subsequent equalities for the York tensor y hold since the symmetrized covariant exterior derivative "Scurl", see section (2.6)) of a symmetric 2-tensor interpreted as a vector-valued 1-form annihilates the pure trace part (and automatically yields a tracefree result, making the York tensor tracefree since the Ricci tensor is symmetric).

The York tensor is automatically divergence-free as a consequence of the same property for the Cotton tensor.

A.25 Conformal Transformations in 3 Dimensions

Introduce the notation $U_a = \nabla_a U$ for a scalar U, which satisfies

$$\nabla_a U_b = \nabla_a \nabla_b U = \nabla_b \nabla_a U = \nabla_b U_a \tag{A.132}$$

or equivalently $\eta^{abc}\nabla_b U_c = \hat{0}$. Then the standard conformal transformation laws for the metric and the curvature tensors are

$$\tilde{g}_{ab} = e^{2U}g_{ab} \ , \tilde{g}^{ab} = e^{-2U}g^{ab} \ , \tilde{g}^{1/6} = e^{U}g^{1/6} \ , \tag{A.133}$$

$$\tilde{\eta}_{abc} = e^{3U}\eta_{abc} \ , \ \tilde{\eta}^{abc} = e^{-3U}\eta^{abc} \ , \tag{A.134}$$

$$\tilde{\nabla}_d X^b{}_c = \nabla_d X^b{}_c + C^b{}_{de}X^e{}_c - C^e{}_{dc}X^b{}_e \ , \tag{A.135}$$

$$C^c{}_{ab} = 2\delta^c{}_{(a}U_{b)} - g_{ab}U^c \ , \ C^c{}_{[ab]} = \hat{0} \ , \tag{A.136}$$

$$\tilde{R}_{ac} = R_{ac} - \nabla_a U_c + U_a U_c + g_{ac}(-\nabla^e U_e - U^e U_e) \ , \tag{A.137}$$

$$\tilde{R} = e^{-2U}[R - 4\nabla^e U_e - 2U^e U_e], \tag{A.138}$$

$$\tilde{G}_{ac} = G_{ac} - \nabla_a U_c + U_a U_c + g_{ac}\nabla^e U_e \ , \tag{A.139}$$

$$\tilde{R}^a{}_c = e^{-2U}\tilde{R}_{ac} \ , \ \tilde{\nabla}_d R^a{}_c = e^{-2U}(\nabla_d - 2U_d)\tilde{R}_{ac} \ . \tag{A.140}$$

The York tensor then satisfies

$$\begin{aligned} g^{-5/6}\tilde{g}^{5/6}\tilde{y}^{ea} &= g^{-5/6}\tilde{g}^{5/6}\tilde{\eta}^{edc}\tilde{\nabla}_d\tilde{R}^a{}_c = \eta^{edc}\tilde{(}\nabla_d - 2U_d)\tilde{R}^a{}_c \\ &= y^{ea} + [\mathrm{Scurl}(-\nabla U + U \otimes U) + U \times (\nabla U - R)]^{ea} \\ &= y^{ea} - [\mathrm{Scurl}(\nabla U) + U \times R]^{ea} + [\mathrm{Scurl}(U \otimes U) + U \times \nabla U]^{ea} \ . \end{aligned} \tag{A.141}$$

The first pair of terms in square brackets is identically zero for a gradient $U_a = \nabla_a U$ by the relevant Ricci identity, while the second is identically zero for a gradient just by symmetry of the second derivatives alone, leading to the conformal invariance of the Cotton-York contravariant tensor-density $Y^{ea} = g^{5/6}y^{ea}$ (see exercise 21.22 of Misner, Thorne and Wheeler [1973]). It must therefore be zero for any metric which is conformally flat.

A.26 $n = 3$ Structure Functions and Orthonormal Frame Connection Components

In any frame in $n = 3$ dimensions, the structure constant functions are antisymmetric in the lower indices, so a 2-index object can be defined by taking the spatial dual on those indices, then decomposing the result into its symmetric and antisymmetric parts, the latter of which can be expressed as the dual of a 1-index object

$$\begin{aligned} C^{\alpha\delta} &= C^{\alpha}{}_{\beta\gamma}\eta^{\delta\beta\gamma} \\ &= C^{(\alpha\delta)} + C^{[\alpha\delta]} \\ &= m^{(\alpha\delta)} + \eta^{\alpha\delta\gamma}a_{\gamma} \ , \\ C^{\alpha}{}_{\beta\gamma} &= \eta_{\beta\gamma\delta}m^{\beta\delta} + a_{\delta}\delta^{\delta\alpha}_{\beta\gamma} \ , \end{aligned} \tag{A.142}$$

where

$$m^{\alpha\delta} = C^{(\alpha}{}_{\beta\gamma}\eta^{\delta)\beta\gamma} \ , \qquad a_{\gamma} = \tfrac{1}{2}C^{\alpha}{}_{\gamma\alpha} \ . \tag{A.143}$$

The mixed object $C^{\alpha}{}_{\beta}$ may be interpreted as a "vector"-valued 1-form in the sense of an R^3-valued 1-form. This notation was introduced by Ellis and MacCallum (1969) after the decomposition was described by Behr (1968, 2002).

In any frame for which the metric components are constants (an orthonormal frame in general or an invariant metric on a Lie group, for example), the antisymmetry of the matrix-valued connection 1-form

$$0 = \omega_{(\alpha\beta)} = \Gamma_{(\alpha|\gamma|\beta)}\omega \tag{A.144}$$

means that the connection components are antisymmetric in their outer indices, so one may take their spatial dual to define a matrix-valued scalar which may then be interpreted as a "vector"-valued 1-form

$$\Gamma^{\beta}{}_{\delta} = \Gamma^{\alpha\beta\gamma}\eta_{\alpha\gamma\delta} \ , \tag{A.145}$$

which in turn can be decomposed into its symmetric and antisymmetric parts and pure trace and tracefree parts

$$\Gamma^{\beta\delta} = \Gamma^{(\beta\delta)} + \Gamma^{[\beta\delta]} \ . \tag{A.146}$$

The relationship between these two "vector"-valued 1-forms in an orthonormal frame follows from the specialization $\partial_{\alpha}g_{\beta\gamma} = 0$ of the general formula for the metric connection to the following result

$$\Gamma^{\alpha}{}_{\beta\gamma} = \tfrac{1}{2}(C^{\alpha}{}_{\beta\gamma} + C_{\beta}{}^{\alpha}{}_{\gamma} + C_{\gamma}{}^{\alpha}{}_{\beta}) = \tfrac{1}{2}C^{\alpha}{}_{\beta\gamma} + C_{(\beta}{}^{\alpha}{}_{\gamma)} \equiv \tfrac{1}{2}C^{\alpha}{}_{\beta\gamma} + K^{\alpha}{}_{\beta\gamma} \ , \tag{A.147}$$

where

$$K_{\alpha\beta\gamma} = -\tfrac{1}{2}\pounds_{e_{\alpha}}g_{\beta\gamma} \ , \tag{A.148}$$

namely

$$\begin{aligned} \Gamma^{\beta}{}_{\delta} &= C^{\beta}{}_{\delta} - \tfrac{1}{2}C^{\gamma}{}_{\gamma}\delta^{\beta}{}_{\delta} \\ &= (m^{\beta}{}_{\delta} - \tfrac{1}{2}m^{\gamma}{}_{\gamma}\delta^{\beta}{}_{\delta}) + \eta^{\beta}{}_{\delta}{}^{\gamma}a_{\gamma} \ . \end{aligned} \tag{A.149}$$

The double dual of the Riemann tensor yields a symmetric tensor, namely the sign-reversed Einstein tensor

$$\begin{aligned} [{}^*R^*]^{\alpha}{}_{\beta} &= \tfrac{1}{4}\eta^{\alpha\gamma\delta}R_{\gamma\delta}{}^{\mu\nu}\eta_{\mu\nu\beta} = \tfrac{1}{4}\delta^{\alpha\gamma\delta}_{\beta\mu\nu}R_{\gamma\delta}{}^{\mu\nu} = -G^{\alpha}{}_{\beta} \ , \\ R^{\alpha\beta}{}_{\gamma\delta} &= -\eta^{\alpha\beta\mu}\eta_{\gamma\delta\nu}G^{\nu}{}_{\mu} \ . \end{aligned} \tag{A.150}$$

For a metric with constant frame components, the Einstein tensor then reduces to

$$\begin{aligned} \mathbf{G} = (G^{\alpha}{}_{\beta}) &= (-\eta_{\beta}{}^{\gamma\delta}\partial_{\gamma}\Gamma^{\alpha}{}_{\delta} + \Gamma^{\alpha}{}_{\epsilon}C^{\epsilon}{}_{\beta} - \tfrac{1}{2}\eta^{\alpha}{}_{\mu\nu}\Gamma^{\mu}{}_{\gamma}\Gamma^{\nu}{}_{\delta}\eta^{\gamma\delta}{}_{\beta}) \\ &= 2\mathbf{m}^2 - \mathbf{m}\,\mathrm{Tr}\,\mathbf{m} - \tfrac{1}{2}(\mathrm{Tr}\,\mathbf{m}^2 - \tfrac{1}{2}\,\mathrm{Tr}^2\,\mathbf{m} - 6a_f a^f)\mathbf{I} - 2\mathbf{a}\mathbf{a}^T - 2a^f\mathbf{K}_f \ , \end{aligned} \tag{A.151}$$

using a more efficient matrix notation $\mathbf{m} = (m^{\alpha}{}_{\beta})$, $\mathbf{a} = (a^{\alpha})$, $\mathbf{a}^T = (a_{\alpha})$, $\mathbf{K}_{\alpha} = (K_{\alpha}{}^{\beta}{}_{\gamma})$, $\mathbf{I} = (\delta^{\alpha}{}_{\beta})$. Similarly

$$\begin{aligned} \mathbf{R} = (R^{\alpha}{}_{\beta}) &= (-\eta_{\beta}{}^{\gamma\delta}\partial_{\gamma}\Gamma^{\alpha}{}_{\delta} + \Gamma^{\alpha}{}_{\epsilon}C^{\epsilon}{}_{\beta} - \tfrac{1}{2}\eta^{\alpha}{}_{\mu\nu}\Gamma^{\mu}{}_{\gamma}\Gamma^{\nu}{}_{\delta}\eta^{\gamma\delta}{}_{\beta}) \\ &= 2\mathbf{m}^2 - \mathbf{m}\,\mathrm{Tr}\,\mathbf{m} - (\mathrm{Tr}\,\mathbf{m}^2 - \tfrac{1}{2}\,\mathrm{Tr}^2\,\mathbf{m})\mathbf{I} - 2\mathbf{a}\mathbf{a}^T - 2a^f\mathbf{K}_f \ , \\ \mathrm{Tr}\,\mathbf{R} &= -(\mathrm{Tr}\,\mathbf{m}^2 - \tfrac{1}{2}\,\mathrm{Tr}^2\,\mathbf{m}) - 6a_f a^f \ . \end{aligned} \tag{A.152}$$

A.27 Lie derivative

The Lie derivative along a vector field X is defined so that it is the ordinary derivative along X of a function f and the Lie bracket by X of another vector field Y,

$$\pounds_X f = Xf \ , \qquad \pounds_X Y = [X,Y] \ , \tag{A.153}$$

and it is then extended to all other tensor fields so that it obeys the obvious product rule for the evaluation of a tensor field on all of its arguments (reducing it to a function). Solving this product rule for the Lie derivative of the tensor alone yields

$$\begin{aligned}
&[\pounds_X S](\theta_{(1)},\ldots,\theta_{(p)},X^{(1)},\ldots X^{(q)}) \\
&\qquad = \pounds_X [S(\theta_{(1)},\ldots,\theta_{(p)},X^{(1)},\ldots X^{(q)})] \\
&\qquad\quad - S(\pounds_X \theta_{(1)},\ldots,\theta_{(p)},X^{(1)},\ldots X^{(q)}) - \ldots \\
&\qquad\quad - S(\theta_{(1)},\ldots,\theta_{(p)},\pounds_X X^{(1)},\ldots X^{(q)}) - \ldots \ .
\end{aligned} \tag{A.154}$$

The Lie derivative is a linear operator which also acts on differential forms (sends p-forms into p-forms) and obeys the usual product rules with respect to the tensor and exterior products (no extra signs).

This last equation immediately leads to a formula for the components of the Lie derivative of a $\binom{p}{q}$-tensor field in a frame by evaluating it on the frame vectors and 1-forms

$$\begin{aligned}
\pounds_X S^{\alpha\cdots}{}_{\beta\ldots} &\equiv [\pounds_X S]^{\alpha\cdots}{}_{\beta\ldots} \\
&= S^{\alpha\cdots}{}_{\beta\ldots,\gamma} X^\gamma + [\sigma^{p,q}(\mathbf{L}(X,e))S]^{\alpha\cdots}{}_{\beta\ldots} \ ,
\end{aligned} \tag{A.155}$$

where the component matrix which is the argument of $\sigma^{p,q}$ is given by

$$\begin{aligned}
\mathbf{L}(X,e)^\alpha{}_\beta = [\pounds_X e_\beta]^\alpha &= -[\pounds_X \omega^\alpha]_\beta \\
&= -X^\alpha{}_{,\beta} + C^\alpha{}_{\gamma\beta} X^\gamma \\
&= -X^\alpha{}_{;\beta} + \tilde{\Gamma}^\alpha{}_{\gamma\beta} X^\gamma \ ,
\end{aligned} \tag{A.156}$$

and $\sigma^{p,q}$ is the associated representation $\sigma^{p,q}$ of the Lie algebra of linear transformations $gl(n,R)$ introduced in section A.4. The last equality holds when a linear connection ∇ is available and enables the Lie derivative to be expressed in terms of the covariant derivative ∇X and the transposed covariant derivative $\tilde{\nabla} S$

$$\pounds_X S^{\alpha\cdots}{}_{\beta\ldots} = \tilde{\nabla}_X S^{\alpha\cdots}{}_{\beta\ldots} - [\sigma^{p,q}(\nabla X)S]^{\alpha\cdots}{}_{\beta\ldots} \ . \tag{A.157}$$

For a symmetric connection, this reduces to

$$\pounds_X S^{\alpha\cdots}{}_{\beta\ldots} = S^{\alpha\cdots}{}_{\beta\ldots;\gamma} X^\gamma - [\sigma^{p,q}(\nabla X)S]^{\alpha\cdots}{}_{\beta\ldots} \ , \tag{A.158}$$

which is the formula which results from the substitution of a semicolon for the comma in the formula for the coordinate frame components of the Lie derivative.

For a differential form S, the Lie derivative can be expressed in terms of the exterior derivative and the contraction operation

$$\pounds_X S = X \lrcorner dS + d(X \lrcorner S) \ . \tag{A.159}$$

The geometric significance of the Lie derivative comes from its relation to the generator of a 1-parameter family of diffeomorphisms. If X_λ denotes the flow of a vector field, then each tensor field S may be dragged along by the flow for each value of λ to define a 1-parameter family of tensor fields, indicated by the abbreviation

$$S_\lambda = X_\lambda S \ , \qquad S_0 = S \ . \tag{A.160}$$

Then the parameter derivative of this family at $\lambda = 0$ defines the Lie derivative reversed in sign

$$d/d\lambda\, S_\lambda|_{\lambda=0} = -\pounds_X S \ . \tag{A.161}$$

For a $\binom{p}{q}$-tensor density field of weight W the formula for the components of the Lie derivative has one extra term compared to a $\binom{p}{q}$-tensor field

$$
\begin{aligned}
\pounds_X S^{\alpha\cdots}{}_{\beta\cdots} &\equiv [\pounds_X S]^{\alpha\cdots}{}_{\beta\cdots} \\
&= S^{\alpha\cdots}{}_{\beta\cdots,\gamma} X^\gamma + [\sigma^{p,q}_W(\mathbf{L}(X,e))S]^{\alpha\cdots}{}_{\beta\cdots} \\
&= S^{\alpha\cdots}{}_{\beta\cdots,\gamma} X^\gamma + [\sigma^{p,q}(\mathbf{L}(X,e))S]^{\alpha\cdots}{}_{\beta\cdots} - W\mathbf{L}(X,e)^\gamma{}_\gamma S^{\alpha\cdots}{}_{\beta\cdots}\ .
\end{aligned}
\tag{A.162}
$$

A.1 final page