Understanding Spacetime Splittings and Their Relationships or Gravitoelectromagnetism: the User Manual

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Abstract

A historical overview is given of the various approaches to splitting spacetime into space-plus-time. These may all be described within a common framework—gravitoelectromagnetism—based on test observer families.

This is the preface and introduction to a long document. The main text and references are available at:
http://www.homepage.villanova.edu/robert.jantzen/gem/
References also available at:
http://arXiv.org/abs/gr-qc/0010070
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Preface

I was a sophomore in John Wheeler's modern physics course at Princeton University in 1972 when the proofs for some chapters of *Gravitation* began showing up. This book and its authors (Misner, Thorne and Wheeler) have helped to shape generations of relativists including myself and have done much to establish the use of more modern mathematical notation and style as commonplace in the field and in neighboring areas. In particular it is the one textbook which presents the $3+1$ approach to general relativity to a wide audience.

However, an alternative "$1+3$" approach to the splitting of spacetime into space plus time also existed and was most readily found in the text *The Classical Theory of Fields* by Landau and Lifshitz, but few versed in the $3+1$ school took the time to understand this alternative in terms of the same beautiful language that made the $3+1$ approach such a powerful tool in gravitational physics. The culprit is of course quantum gravity which was always lurking in the background as one of the prime motivations for using the $3+1$ approach. In astrophysical applications the superiority of either approach is not obvious and it seems clear that both approaches together can reveal complementary features of a physical problem, shedding more insight than either one alone. Indeed the 1+3 approach underlies much of what is done in the post-Newtonian approximation of general relativity, although it is not generally recognized, and provides the variable decomposition for the stationary exact solution industry, while in the cosmological context it appears that gauge invariant perturbation theory for Friedmann-Robertson-Walker cosmological models may even be simpler in this approach.

Unfortunately the $1+3$ approach, which apparently flourished in the fifties before the $3+1$ school took over, suffered a near fatal public relations blackout in the intervening decades preceding the nineties. This has left many of us who have entered the field since the seventies almost "completely unaware" of much of this formalism beyond recognition of it from limited exposure to it in *The Classical Theory of Fields* or in the closely related congruence approach systematized for applications in cosmology by Ehlers and more widely known through the work of Hawking and Ellis. When Paolo Carini as a student of Remo Ruffini in Rome in 1989 got me interested in trying to explain what was really underlying the peculiar coordinate decomposition of the electromagnetic field found in an exercise in the Landau-Lifshitz text and its relationship to the $3+1$ and $1+3$ approaches, we were drawn into an examination of the very foundations of the splitting formalisms independent of electromagnetism. We were also "completely unaware" of the work in the fifties by Møller, Zel'manov, and Cattaneo on the $1+3$ point of view. Driven by our ignorance of these matters and the lack of a clear discussion of their mathematical structure in the literature, we embarked on a study in which we learned a great deal not only about the larger framework in which these questions fit together, but also about the work that had been done and effectively buried in the past. Later joined by Donato Bini, we went on to examine some new aspects of the application of splitting formalisms to questions in general relativity. It seems not only valuable to share with the larger community what we have understood in a notation that easily allows comparison of the different approaches but also appropriate that some of the historical roots of the subject be recalled and not forgotten. Meanwhile during the decade of the nineties the $3+1$ approach using the more modern point of view of congruences has enjoyed increasing visibility through many articles by Ellis and collaborators, and the need for understanding the relationships among various approaches has only increased.

In this spirit the following monograph has grown to its present form. Certainly it is not meant to be exhaustive, especially where the $3+1$ approach is concerned given the enormous amount of existing literature which deals with that case alone. Rather it is meant to reveal more clearly the relationships between the various approaches in an effort to break down the artificial barriers which divide them. If this attempt is only partially successful, our efforts will have been well spent.

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bob jantzen

1 Introduction

1.1 Motivation: Local Special Relativity plus Rotating Coordinates

Most of us know special relativity pretty well and are quite happy switching back and forth between the spacetime picture of 4-vector algebra and the space plus time picture of events occurring in space as time elapses, using 3-vector algebra. We have no difficulty using Lorentz transformations to transform 3-dimensional quantities from those measured by one inertial observer to another. We also have no problem extending our splitting algebra to spacetime derivative operators in inertial coordinates yielding the space plus time equivalent of time derivatives and spatial derivative operators. After all, these are the derivatives we began with before learning special relativity. Figure 1.1 suggestively compares these two pictures.

Since old habits die hard, there is a strong incentive to push this habit into general relativity. This helps us interpret spacetime information in a curved spacetime using the same intuition that we have about classical 3-dimensional physics. The catch is that one no longer has a privileged class of "global inertial frames" as in special relativity which effectively allows a single inertial observer in flat spacetime to set up a preferred class of global inertial coordinate systems that may be used to interpret observations at all other points in spacetime. Essentially, one inertial observer in flat spacetime uniquely determines a family of inertial observers filling the spacetime with no relative velocities, and the "observations" of the original observer of an event not on his own worldline are understood to be those of the companion observer who is present.

The only option allowing us to continue to make a spacetime splitting without relying on flatness is to give up the global splitting associated with a single preferred observer and settle for the local splitting of each member of a family of observers filling the spacetime and in arbitrary motion in the absence of any preference. Such a splitting takes place in the tangent space to each event in spacetime, describing the locally Minkowskian neighborhood of each observer in the family. If we agree to split each such tangent space based on the 4-velocity of the observer at a given event in the same way that we split flat spacetime globally based on the worldline of a single inertial observer, then all of our familiarity with special relativity can be transferred to general relativity in a rather straightforward way. The only difference is that what we did globally before, with the splitting at every spacetime point determined by the splitting at a single spacetime point, namely any point on the worldline of a chosen inertial observer, must be abandoned in favor of doing the same thing independently at each spacetime point, modulo continuity/differentiability conditions.

There is one catch. Being creatures of habit, we like global splittings, so even though in general there is no preferred way of doing a global splitting, we can just do it arbitrarily, though clearly matters may be simplified if such a splitting can be adapted to any special structure that may exist in a particular spacetime. One then has to reconcile the local observer splittings with such a

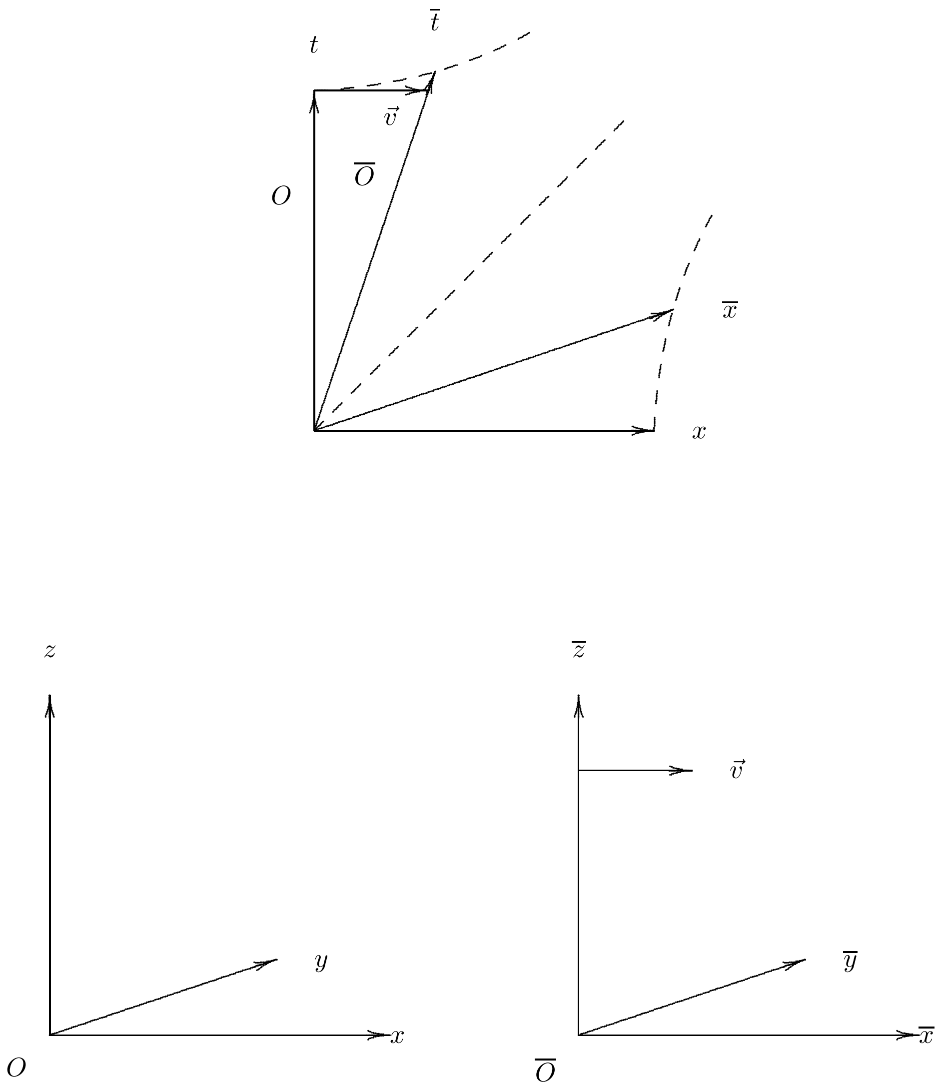


Figure 1: Spacetime and Space plus Time. Although spacetime is the arena where calculations are simpler, we always interpret them through our space plus time worldview, which depends on the choice of inertial observer.

global splitting. This too is not anything particularly deep, and it involves using the linear algebra of nonorthogonal bases independently at each spacetime point (since such a global splitting will in general be nonorthogonal) to represent the orthogonal splitting of the local observers on spacetime.

The new complication in general relativity is that in general one must deal with a family of so called "test observers" in arbitrary motion, and this introduces the well known effects that accompany noninertial (i.e., accelerated) observers even in nonrelativistic physics. However, in the classical example of a "rigidly rotating" family of noninertial observers in nonrelativistic physics, one still has a global correlation between the different members of the family of observers since one has a global (though not physical) Cartesian coordinate system which rotates with the passage of time. In the extension to general relativity one must consider such effects locally at each spacetime point.

The effects of rigid rotation are well known and familiar. There is the centrifugal force that a body feels in the rotating frame even if it is not moving with respect to that frame due to the frame's acceleration and the Coriolis force that a body feels if it is in motion with respect to that frame due to the frame's rotation. Quantitatively in a system rotating with constant angular velocity $\vec{\Omega}$, there is a force on an otherwise free body

$$\ddot{\vec{x}} = \vec{F}/m = \vec{g} + \vec{v} \times \vec{H} \ , \tag{1}$$

where the "gravitoelectric" force (per unit mass)

$$\vec{g} = -\vec{A} = -\vec{\Omega} \times \vec{V} = -\vec{\Omega} \times (\vec{\Omega} \times \vec{x}) \tag{2}$$

is the negative of the acceleration field $\vec{A}$ of the rotating observers, whose velocity field is $V = \vec{\Omega} \times \vec{x}$. Similarly the "gravitomagnetic" force is the cross product of the body's velocity $\vec{v} = \dot{\vec{x}}$ in the rotating system with the "gravitomagnetic" vector field $\vec{H} = 2\vec{\Omega} = \operatorname{curl} \vec{V}$, which is twice the angular velocity vector of the rotating frame and equals the local vorticity of the velocity field.

The "gravitoelectromagnetic" terminology due to Thorne is in direct analogy with the Lorentz force of electromagnetism in a nonrotating system

$$(m/q)\ddot{\vec{x}} = \vec{F}/q = \vec{E} + \vec{v} \times \vec{B} \ . \tag{3}$$

Note that the "gravitomagnetic" vector field $\vec{H} = \operatorname{curl} \vec{V}$ admits a vector potential $\vec{V}$ in the same way that the magnetic field $\vec{B} = \operatorname{curl} \vec{A}$ locally admits a vector potential. Similarly the "gravitoelectric" vector field

$$\vec{g} = -\operatorname{grad}[-\tfrac{1}{2}\vec{V} \cdot \vec{V}] \tag{4}$$

admits a scalar potential in this time-independent case just like the conservative electric field $\vec{E}$ in electrostatics.

The velocity field of the rotating observers and minus half its length squared (once multiplied by the mass m) serve as vector and scalar potentials for the noninertial forces we often call "fictitious" forces in classical mechanics, and they are directly interpretable in terms of kinematical properties of the velocity field of the family of noninertial observers used to describe the motion of the body being studied. These forces vanish as soon as we require the members of the family of observers to be inertial.

In a curved spacetime such global frames are not immediately available, so one must analyse the situation in the local rest space of each observer in the family of test observers used to describe the physics in 3-dimensional form. The kinematical properties of the 4-velocity field of these observers in spacetime, with some extra complication, directly generalize the above problem, leading to the introduction of "gravitoelectromagnetic" forces which enter into the force equation when expressed in terms of a family of noninertial observers. Rather than doing a global comparison of their motion with respect to a global inertial frame which does not exist in curved spacetime, it must be a local comparison at each point of spacetime with a suitable inertial observer having the same 4-velocity and a set of nonrotating spatial axes. Of course there are new features in curved spacetime which have no analogy in electromagnetism and this has to do with the spatial metric which describes the relative distances between nearby test observers as well as the different proper times that different observers use at the same event in spacetime, both of which depend on the gravitational field. These features, as well as the potentials for the gravitoelectromagnetic vector forces, are contained in the spacetime metric. Taking into account the local proper time complicates slightly the above analogy which relies on a global proper time function on flat spacetime.

Thus we need relativistic definitions of acceleration, gradient, curl, time derivative, etc. This together with the details of the way in which an observer in spacetime measures quantities at a given event will enable us to push our present knowledge about special relativity and noninertial motion to the case of general relativity. It helps explain the so called ADM or three-plus-one approach to general relativity as well as the slightly different Landau-Lifshitz approach and shows their similarities and differences, both of which involve the structure of a family of test observers together with a global time function on spacetime. It also shows how both relate to the Ehlers-Hawking-Ellis splitting approach, which is based only on a family of test observers with no global time function assumed to be available.

1.2 Why bother?

You might say, why invest a lot of time into understanding the details of this way of looking at all the different splitting approaches used in general relativity? After all, relativity physics liberated us from the prison of 3-dimensional language into the arena of spacetime where the true nature of kinematics and dynamics became much simpler to understand. Why try to climb back into the cage of 3-dimensional physics? Wasn't centuries of solitary confinement enough?

Well, many spacetimes and idealized problems we use in understanding gravitational theory practically beg us to do this. A rotating black hole spacetime, for example, has two very different privileged families of test observers, one of which is suited to the ADM picture and the other to the Landau-Lifshitz picture. The one we use depends on the question we want to study. Both turn out to be useful. The Ehlers-Hawking-Ellis picture provides the means to relate these two different pictures to each other, which is important if they both turn out to be needed, as they in fact do. The failure to investigate this "relativity of spacetime splitting formalisms" has kept most of us from having better intuition not only about black holes, or even the relativistic picture of the nonrelativistic problem of rotating coordinate systems (which continues to confuse people even now), but other interesting rotating spacetimes like the Gödel universe.

Many of us are somewhat familiar with (or at least aware of) the initial value problem for the ADM splitting of spacetime based on a family of spacelike hypersurfaces, but the corresponding problem for the splitting of Landau and Lifshitz based on a timelike congruence is almost unknown. In fact the exact solution industry for stationary spacetimes studies precisely this problem without the difficulties that breaking the stationarity symmetry introduces, since only the initial value problem remains of the Einstein equations in that case. The choice of metric variables made in studying this problem is adapted to the Landau-Lifshitz splitting, though few stop to think about its geometric significance or realize it is a decomposition parallel to the ADM splitting.

Even if none of these special spacetimes interests you, if you are interested in any post-Newtonian calculations of more realistic, say isolated self-gravitating systems, then understanding the splitting game helps to make a little more sense out of what is universally done in that field. In recent years many references to "the gravitomagnetic field" have sprung up, but since no one took the time to unambiguously define just what this field was, controversy has blossomed between different schools of thought about just what a "real gravitomagnetic field" is. People can agree on a naive definition of the field in this stationary weak-field limit but for strong fields many different definitions are possible depending on the choices one makes for the way in which observers make measurements. All of these different choices can be fit into a general framework and related to each other.

Of course the big difficulty in general relativity has always been the limited analytical computations that can be done exactly. An entire industry has grown up around the approximation schemes for general relativity which in practice involves rather complicated details. Certainly the naive things one can do for relatively simple exact solutions in terms of interpreting them in 3-dimensional form cannot be extended easily to the more realistic calculations done in post-Newtonian approximations to general relativity. However, one can interpret features of the approximate analysis in terms of the exact spacetimes which it is meant to approximate. Unfortunately many aspects of this exact geometry being approximated seem to be lost in the details of the approximation scheme itself. The language of the present discussion can help put these details into some perspective.

The splitting of the gravitational field is closely related to the splitting of the electromagnetic field, though historically reversed in direction, since the individual electric and magnetic fields of classical Newtonian physics were unified into a single spacetime field by special relativity, while the spacetime metric of general relativity only later gave birth to the electric- and magnetic-like gravitational fields accompanied by the spatial metric. The four-dimensional form of Maxwell's equations in a curved spacetime is very elegant and powerful, perhaps because it is independent of any particular observers or local coordinates. However, in many practical applications, spacetime is endowed at least locally with either a preferred congruence of integral curves of a timelike vector field or a preferred slicing by a family of spacelike hypersurfaces or both, and it is convenient to decompose the electromagnetic field in some way using this additional structure. Of course the electric and magnetic fields measured by an observer in spacetime are well defined and easily expressed in an orthonormal frame adapted to the observer's local rest space, but often coordinate systems and nonorthonormal frames prove more convenient for studying the field equations. In this context one must reinterpret the computational quantities which are naturally introduced in terms of some family of test observers.

When the present work began, a clear discussion of the various approaches to formulating Maxwell's equations in terms of three-dimensional quantities and their relationship to each other did not exist. Moreover, the analogous discussion for the gravitational field itself was even more conspicuously absent in the literature. It therefore seemed useful to carefully develop the mathematics of the splitting formalisms in general relativity which provide the foundation for the subsequent splittings of the electromagnetic field as well as other matter fields on spacetime. In so doing one obtains a precise description of spatial gravitational force fields in the different points of view and of their interrelationships, as well as a clear exposition of the similarities between those nonlinear fields and the linear electric and magnetic fields. In view of the evolution of terminology which has taken place, it seems natural to refer to this analogy by the name of gravitoelectromagnetism.

The analogy between the linear theory of electromagnetism and the linearized theory of general relativity was noted by Einstein even before arriving at his final formulation of the Einstein equations [Verbin and Nielsen, 2004], who compared the geodesic equations with the Lorentz force law. This analogy for the final theory was then soon spelled out explicitly by Thirring [1918]. Although the analogy in practice is usually considered for the linearized gravitational field, the implications and limitations of this analogy are best seen in the fully nonlinear context of general relativity.

Of course as previously noted, an obvious question to ask is "Why bother to split spacetime at all?" Certainly the idea of a four-dimensional spacetime and its local Lorentzian geometry has been an important advance of this century. However, our intuition and experience are decidedly three-dimensional in character, and splittings of spacetime into space plus time allow us to interface better with the four-dimensional information, even when a splitting does not occur naturally. When it does, it can considerably simplify the presentation and interpretation of both the gravitational field and whatever matter fields are present. This is not to say that space-plus-time splittings are always useful. Sometimes $2+2$-splittings are important, examples of which occur in spherical symmetry or in the null initial value problem. In many other instances splittings are actually unproductive, obscuring the spacetime structure of a problem. However, when splittings are useful, they are worth doing carefully.

Many different points of view may be taken in splitting spacetime into space plus time but their interrelationships are rarely considered. This is a useful thing to do since our intuition is based on the standard splitting of flat spacetime, but the three-dimensional quantities which serve as a foundation for this intuition often find themselves associated with distinct points of view in the context of a general splitting of an arbitrary spacetime. The result is that no single point of view captures all aspects of our three-dimensional intuition, and the particular application really should determine the choice which is most appropriate. However, no common mathematical framework and no common notation yet exist to enable one to easily switch point of view or compare different points of view. Most relativists are consequently prisoners of the language of one particular choice. The goal of the present work is to establish such a common mathematical framework to help break down the barriers which exist between different schools of relativists who have settled upon a single choice of point of view.

Unfortunately no text on general relativity can spare the space to do justice to this idea of a relativity of splitting formalisms, so we learn general relativity from one school or another but rarely appreciate more than one approach in our working lives even if we are relativists. The present text attempts to provide both a universal language and the detailed formulas which describe the relatively straightforward but systematic analysis of these various approaches.

1.3 Starting vocabulary

An overriding necessity in this enterprise is a careful definition of the vocabulary to be employed since the standard labels which occur in these discussions do not have universally accepted interpretations. All of the various splitting points of view can be nicely classified and will be assigned labels according to a neutral scheme independent of any particular surnames, thus sidestepping the issue of who "owns" which ideas. Each splitting point of view is based on two fundamental concepts: *measurement* and *evolution*, the realization of which differ for each of the possible choices. Before sketching the history of this topic, it is useful to establish some of the basic terminology to be used in what follows. Of course in order to encompass all of the approaches one finds in the literature in a simple scheme, one must be a bit loose about exactly what mathematical details characterize each of the basic categories into which these approaches will be divided.

The notions of time and space are complementary since a "time line" represents "time elapsing at a point fixed in space" while a "time hypersurface" represents "space at a moment of time". These two different notions of time, the first which focuses on measuring time at a single point of space and the second which is associated with some kind of synchronization of times at different points of space, will be assigned the labels "time" and "space" respectively. These divide the splitting points of view into two categories, those in which a local time direction is fundamental, and those in which a nonlocal correlation of local times, i.e., space is fundamental. In the time category, the "time lines" must be timelike in order to represent a local time direction at each event in spacetime, while in the space category, the "time hypersurfaces" or "spaces" must be spacelike in order to be associated with a moment of time in the usual sense of a Riemannian space (alternatively, in order that orthogonality define a local time direction). Given this division, one may consider a partial splitting or a full splitting depending on whether any additional structure is assumed. Table 1.1 establishes this general classification of points of view and the terminology that will be used to describe it.

Given no additional structure, one has only a partial splitting of spacetime, splitting off either the time or the space alone. In the first case, to be called the "*congruence point of view*," one has only a timelike congruence at one's disposal, with a unit timelike tangent vector field u. Spacetime will be assumed to be time-oriented as well as oriented, so one may assume that u is future-pointing. It may then be interpreted as the 4-velocity of a family of test observers whose worldlines are the curves of the congruence, and it determines the *local time direction* at each point of spacetime. The orthogonal complement of this local time direction in the tangent space is the *local rest space* LRS_u of the test observer at that event. It is exactly this structure that one needs for the *measurement process* which will be the same for the full and partial splittings in a given category in Table 1. The orthogonal decomposition of the tensor algebra induced by this decomposition of the tangent space at each event will define the measurement process, modulo a final step in which projection along the local time direction is replaced by contraction with u, yielding a collection of "*spatial tensor fields*" of different rank for each spacetime tensor field that is split.

In general the splitting of the tangent spaces does not extend to the spacetime manifold. Only in the special case that the rotation of u vanishes does such an extension exist and one has a family of orthogonal spacelike hypersurfaces which slice the spacetime, leading to a full splitting in this category to be discussed below. The second case in the category of partial splittings of spacetime, that of a spacelike slicing of spacetime with no additional structure, is essentially equivalent to the special case of a nonrotating congruence since every spacelike slicing admits a family of timelike orthogonal trajectories. These are the integral curves of the (rotation-free) unit normal vector field n to the slicing, which may be assumed to be future-pointing. The accompanying point of view, for the sake of completeness, might be called the "*hypersurface point of view*". Its measurement process is associated with the normal congruence, taking $u = n$ as the 4-velocity of the family of test observers who do the measuring. The local rest spaces of this family are integrable and coincide with the subspaces of the tangent space which are tangent to the slicing.

A full splitting of spacetime at the manifold level requires both a *slicing* of the spacetime and a congruence, to be referred to as a "*threading*" of the spacetime, together with a compatibility condition that the two families be everywhere transversal. Such a structure will be called a "*nonlinear reference*

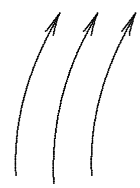
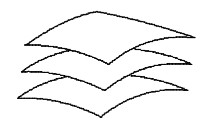
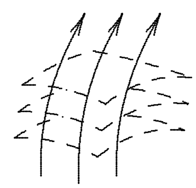
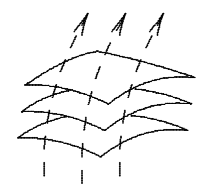
	time: 1 (“single-observer” time)	space: 3 (“moment of time”)
PARTIAL SPLITTING u or LRS_u GE, GM fields	time: 1  (3) congruence p.o.v.	space: 3  (4) hypersurface p.o.v.
FULL SPLITTING parametrized nonlinear reference frame: $\{t, x^a\}$ GE, GM fields and potentials	time + space: 1 + 3  (2) threading p.o.v.	space + time: 3 + 1  (1) slicing p.o.v.
TIME gauge	timelike observers $\leftrightarrow$ threading	spacelike local rest spaces (timelike normal observers) $\leftrightarrow$ slicing
SPACE gauge	arbitrary synchronization of observer times $\leftrightarrow$ slicing	arbitrary identification of “points of space” $\leftrightarrow$ threading

Table 1: A characterization of the different points of view (p.o.v.) that may be adopted in splitting spacetime. Solid lines in diagrams imply the use of the appropriate causality condition while dashed lines indicate that no causality condition is assumed. The hypersurface p.o.v. is essentially equivalent to the vorticity-free congruence p.o.v. The reference p.o.v. corresponds to a full splitting in which no causality assumptions are made.

frame" in order to distinguish it first from the terms "reference frame," "frame of reference," "reference system" and "system of reference" that one finds in the literature, second from the connotation of "frame" in the context of a linear frame of vector fields, and third from related terminology which occurs in the discussion of globally constant frames in flat spacetime. A "*parametrized nonlinear reference frame*" will consist of a nonlinear reference frame together with a choice of parametrization of the family of slices. Such a parametrization defines a specific "time function" t on the spacetime which in turn provides an obvious parametrization for each curve in the threading congruence.

In the category of full splittings, the distinguishing criterion is the causality condition imposed on the nonlinear reference frame. In the "*slicing point of view*" the slicing is assumed to be spacelike, but no assumption is made about the causality properties of the threading, which serves only as a way of identifying the points on different slices. In the "*threading point of view*," the threading is assumed to be timelike, but no assumption is made about the causality properties of the slicing, which serves only to synchronize in some arbitrary fashion points on different curves in the congruence. If both causality conditions hold, then both points of view hold and one can transform between them. On the other hand it can also be useful in the case that at least one of the two causality conditions holds to not take advantage of that condition and exploit only the structure of the nonlinear reference frame that does not depend on it. This leads to the "*reference point of view*," whose measurement process is associated with the nonorthogonal decomposition of the tangent space into the direct sum of a 1-dimensional subspace tangent to the threading and a three-dimensional subspace tangent to the slicing. One can always relate either the slicing or threading points of view to this acausal approach, which is the way in which they are usually represented in a local coordinate system adapted to the nonlinear reference frame.

The partial splittings may be related to the full splittings in different ways. In the threading point of view one may define a (future-pointing) unit timelike tangent vector field m along the threading congruence, while in the slicing point of view one has the (future-pointing) timelike unit normal vector field n. By making the respective choices $o = m$ and $o = n$ ("o" for "observer") of the 4-velocity of a privileged family of test observers in these two points of view, the identification $u = o$ relates each of them to a corresponding congruence point of view described above, defining for each a measurement process. When both the slicing and threading points of view hold, then a unique boost in each tangent space relates the two timelike unit vectors m and n and this may be extended to a transformation of the measurement process. In the special case of an *orthogonal nonlinear reference frame* (one for which both the causality conditions hold and the slicing and threading are everywhere orthogonal), then $m = n$ and the two points of view coincide.

Evolution is defined first by a choice of a 1-parameter group of diffeomorphisms of the spacetime into itself which in some sense advances into the future (either its orbits are timelike or it pushes certain spacelike hypersurfaces into their future), and second by a choice of transport along its orbits for the spatial fields of the given point of view. For a partial splitting only one congruence is available and it is timelike. In the absense of additional structure one can take u or n respectively in the congruence or hypersurface points of view as the generator of such a group, and choose either spatially-projected ("*spatial*") Lie transport ("noncovariant" but integrable) or spatially-projected parallel transport ("covariant" but in general nonintegrable) along this congruence. The latter transport of spatial fields coincides with Fermi-Walker transport which defines locally nonrotating axes along a worldline. Each of these choices may be extended to the full splitting in its category but it is the spatial Lie transport along the threading congruence which defines the evolution relative to the nonlinear reference frame, since fields which are "rigidly" attached to this frame do not evolve with this choice. However, unlike Fermi-Walker transport, spatial Lie transport is in general incompatible with orthonormal frames. A compromise between the two kinds of transport leads to *co-rotating Fermi-Walker transport*, which is the closest one can get to attaching an orthonormal frame to the nonlinear reference frame.

1.4 Historical background

Armed with this initial vocabulary, the historical background may be sketched in a way that puts the different formalisms into some perspective. The slicing and threading points of view today are introduced to most of us through two leading textbooks, respectively *Gravitation* by Misner, Thorne, and Wheeler [1973] and *The Classical Theory of Fields* by Landau and Lifshitz [1975], each of which carefully avoids mention of the "competing" point of view. Both points of view can be traced back to the early forties when the first edition of the Landau-Lifshitz text [1941] introduced the threading point of view splitting of the spacetime metric and, in the stationary case, of the spacetime connection to yield spatial gravitational forces, as still described in their last edition. Soon after, Lichnerowicz [1944] introduced the beginnings of the slicing point of view with an article discussing the initial value problem in an orthogonal nonlinear reference frame. His later book on general relativity and electromagnetism [Lichnerowicz 1955] curiously enough makes use of the threading split, but in actual applications uses an orthogonal nonlinear reference frame in which the two points of view agree. The threading point of view apparently dominated during the fifties when much interest was focused on the equations of motion for test particles. Møller discussed a parametrization-dependent definition of spatial gravitational forces for a general spacetime in the first edition of his text *The Theory of Relativity* [Møller 1952] at the beginning of the decade. This was then refined to a parametrization-independent splitting later in the decade by Zel'manov [1956, 1959] in the Soviet Union and then independently by Cattaneo [1958,1959a,b,c] in Italy. Unfortunately most of the few references to these works that do appear in the literature cite papers written in Russian, Italian or French, so one must dig to find English versions. The most accessible discussion of much of this material is the second edition of Møller's text [1972] which describes it in detail and also contrasts it with his original splitting. However, for some reason this text itself is not very prominent among the relativity texts one usually encounters for those of us who have entered the field since the sixties, perhaps because of its old-fashioned viewpoint. Of these authors, only Zel'manov [1956] discussed the splitting of Einstein's equations in the general case.

Meanwhile the slicing point of view was further developed during the fifties by Choquet-Bruhat [1956] and Dirac [1959]. Choquet-Bruhat extended Lichnerowicz's initial value discussion to the general case, identifying but not naming the lapse and shift variables of the slicing point of view using orthonormal frame techniques. Dirac recognized the significance of the slicing point of view metric decomposition for the Hamiltonian dynamics of general relativity and its relation to his theory of constrained Hamiltonian systems. This was then refined in a series of papers at the turn of the decade by Arnowit, Deser and Misner who used the Hamiltonian formulation permitted by the slicing point of view to study the true degrees of freedom of the gravitational field, culminating in an often cited review article [Arnowit, Deser and Misner 1962]. This ushered in the new era of domination of the splitting scene by the slicing point of view, pushed by the problem of quantum gravity where Hamiltonian techniques have played a rather important role in an endless quest that has not yet met success. The notation of Arnowit, Deser and Misner, soon labeled by Wheeler's lapse and shift terminology [Wheeler 1964] and later effectively propagated by the text of Misner, Thorne and Wheeler [1973], has found widespread acceptance. The slicing point of view is also commonly referred to as the "$3+1$" or ADM formalism. A number of useful reviews of various aspects of this formalism exist, among them being articles by York [1979], Isenberg and Nester [1980], Fischer and Marsden [1978, 1979] and Gotay et al [1991]. The term "$1+3$" formalism with its obvious change in emphasis has been suggested as an alternative label for the threading point of view.

Although Cattaneo stopped his analysis of the threading point of view at the connection level, Cattaneo-Gasperini [1961, 1963] and Ferrarese [1963, 1965] continued it to the curvature level, studying the splitting of the spacetime curvature and of Einstein's equations, and the various definitions of spatial curvature that are possible. This previously unexplored area of differential geometry dealing with a degenerate but nonintegrable connection, namely the spatial covariant derivative or transverse covariant derivative which occurs in the threading and congruence points of view, has never been fully understood in the context of the initial value problem for those points of view. This problem, which has been discussed by Stachel [1980] and Ferrarese [1987,1988, 1989], is of a completely different character than in the slicing point of view where it is rather well understood, and its resolution is still an open problem. The approach of Cattaneo and Ferrarese to the threading point of view was reformulated by Massa [1974a,b,c] and used to discuss gyroscope precession [Massa and Zordan, 1975]. This latter problem, more than any other, has focused attention on the effects of spatial gravitational fields. More recently Perjes [1988] and Abramowicz [1988, 1990] have considered variations of Møller's parametrization-dependent threading approach.

In the slicing point of view the natural extension of the splitting of the metric to the splitting of the connection and the discussion of spatial gravitational forces has only recently been considered, perhaps forgotten in the emphasis on the initial value problem and Hamiltonian dynamics. The roots of this discussion can be traced back to the original threading point of view work, although this link is not made apparent in citations. At the crossover point between the popularity of the threading and slicing points of view, Forward [1961] described the analogy between electromagnetism and linearized general relativity using the reference point of view, in a slight variation of Møller's formalism. In the late seventies this article then inspired a reference point of view discussion of the PPN formalism by Braginsky, Caves and Thorne [1977] who introduced an "electric-type" gravitational field and a "magnetic-type" gravitational field, the latter of which became the "gravitomagnetic" field of the eighties in a discussion of linearized general relativity by Braginsky, Polnarev and Thorne [1984]. In linearized gravity, in the usual weak field slow motion discussions, the corresponding spatial gravitational fields defined in the threading, slicing and reference points of view are very closely related and agree to the lowest order, although not to full post-Newtonian order. Curiously enough, it is always the threading fields which are used in post-Newtonian discussions even when the slicing point of view is advocated before linearization. The introduction of the terminology "gravitoelectric" field and the first discussion of slicing spatial gravitational forces finally appeared in the text *Black Holes: The Membrane Paradigm* by Thorne et al [1986], but only in the case of a stationary gravitational field in their treatment of black hole spacetimes. This may be extended in an obvious way to general spacetimes in a notation which shows the close relationship to the threading point of view as will be described below.

The congruence point of view is briefly introduced in an article by Hawking [1966] and at great length in a pair of articles by Ellis [1971, 1973] at the beginning of the seventies (later updated by Ellis and van Elst [1998]), all based on earlier unifying work of Ehlers [1961] and of Kundt and Trümper [1962] unavailable in English until the Ehlers article alone finally appeared in tranlation over three decades later [1994]. Completing the congruence vector field u to an orthonormal frame leads to the explicit orthonormal frame approach of Estabrook and Wahlquist [1964], who in a note added in proof in their article thank Pirani for calling their attention to Cattaneo's work and cite articles in French; they were not the first or the last to have been as they say "completely unaware of this work." (Cattaneo himself appears to have been completely unaware of Zel'manov's work, while the present authors were completely unaware of both when this project was begun.) The hypersurface point of view is described in the article by Zel'manov [1973] in the early seventies in terms of a nonlinear reference frame, but some work is required to decipher his notation. Ehlers [1961] and Ellis [1971] also treat the hypersurface point of view as a special case of the congruence point of view.

These various splittings of spacetime are particularly interesting in the case of electromagnetism, where all of our intuition is tied to individual electric and magnetic fields, and astrophysical applications can be aided by allowing this intuition to find expression in the context of a splitting. Early work by Ruffini and collaborators [Hanni and Ruffini 1973, Hanni and Ruffini 1975, Ruffini and Wilson 1975, Damour and Ruffini 1975, Hanni 1977, Damour et al 1978, Ruffini 1978] followed up in more detail by Damour [1978, 1982] revealed the utility of introducing the concept of electric and magnetic fields in studying black hole systems. This was discussed later in great detail from the slicing point of view by Thorne and Macdonald [1982], who summarize the history of the different splittings in general and as applied to Maxwell's equations, and by Thorne et al [1986] for application to black hole systems.

Maxwell's equations may be expressed in the congruence point of view as done by Ellis [1973], in the threading point of view as done by Benvenuti [1960] in an application of Cattaneo's formalism unfortunately appearing only in Italian, and in the slicing point of view as described by Misner, Thorne and Wheeler [1973] (for the correct vector potential splitting, see Isenberg and Nester [1980], for example). The reference point of view splitting is much older, dating back to the beginnings of general relativity in an article by Tamm [1924], as noted by Skrotsky [1957] and Plebanski [1960]. This appears in a peculiar mix of the reference and threading points of view in an exercise in the Landau and Lifshitz text [1975]. The mystery of this latter approach has caused its share of confusion about how one should define electric and magnetic fields in applications in general relativity. Hanni [1977] has given the complementary version which mixes the reference and slicing points of view. It is rather interesting to compare each of these numerous splittings of Maxwell's equations with the others. Maxwell's equations from the slicing point of view, first considered by Misner and Wheeler [1957] using the language of differential forms, are discussed in great detail by Thorne and Macdonald [1982] and by Thorne et al [1986].

An enormous language barrier exists at present between the slicing and threading points of view, preventing those versed in the formalism and notation of one from easily penetrating the other or understanding how the two are related. This is exactly the problem the present exposition hopes to address, namely the lack of a common mathematical framework to discuss both approaches on an equal footing. Both of these splitting formalisms can be developed in a completely parallel way as complementary aspects of a single geometrical structure imposed on spacetime (the nonlinear reference frame), aspects which in a close way are related by the same duality that links contravariant and covariant fields on the spacetime manifold. Furthermore, each of these approaches has important ties with the congruence point of view which invariantly describes the geometry of the observer congruence and with the reference point of view which links these discussions to adapted coordinate systems in practice. The style of the threading approach as usually presented is somewhat more cumbersome than the slicing one, so it will be recast in the slicing style, generalizing from adapted coordinate systems to adapted local frames. In the special case of an orthogonal slicing and threading of a spacetime, the two descriptions will then coincide.

This standardization of ideas and notation can also prove useful in approximation techniques, including the recent axiomatization of the idea of a Newtonian limit of general relativity based on a family of spacetime splittings as discussed by Ehlers [1989], Lottermoser [1989], and Schmidt [Ehlers, Schmidt and Lottermoser 1990]. The relatively unknown geometry of the spatial connection of the congruence and threading points of view is also currently of interest in the reformulation of gauge-invariant perturbation theory for Friedmann-Robertson-Walker spacetimes by Ellis and coworkers [Ellis and Bruni 1989, Ellis, Hwang, and Bruni 1989]. Each of these limits concern special cases of the problem of perturbation theory for a general spacetime from the somewhat unfamiliar alternative points of view to be discussed in this monograph.

Chapter 2 describes the congruence point of view, which is later used to define the measurement process for the slicing and threading points of view. Chapter 3 studies the splitting geometry associated with a nonlinear reference frame and discusses both the slicing and threading points of view and their relationship to the congruence point of view and the reference point of view. Chapter 4 discusses electromagnetism in detail from each of the points of view while Chapter 5 considers the very important case of stationary spacetimes and the Sagnac effect and synchronization questions. The special case of flat spacetime in rotating coordinates clarifies the relationship of the spatial gravitational forces to the centrifugal and Coriolis forces. Chapter 6 discusses the weak field, slow motion limit and almost Friedmann-Robertson-Walker spacetimes.

Although this text contains many formulas, only a few simple ideas applied in a methodical way are behind much of the detail. Orthogonal decomposition, by itself or represented in the context of a nonorthogonal decomposition, is at the heart of the algebra of respectively partial and full splittings of spacetime. Orthogonal projection of differential operators in the same spirit then provides the differential tools necessary to extend the splitting algebra to include derivatives.

1.5 Orthogonalization in the Lorentzian plane

The heart of linking spacetime splittings to coordinate systems is a simple idea, but one which receives little attention in our academic preparation: the use of nonorthogonal coordinate systems. It is very useful to look at the Euclidean and Lorentzian 2-planes E^2 and M^2 to recall first a case for which our geometric intuition holds and then see how it differs in the spacetime arena.

1.5.1 The Euclidean example: general linear coordinates

Given any two nonzero linearly independent vectors (X, Y) in R^2 with its usual Euclidean structure, then expressing the position vector $r = xX + yY$ defines its coordinates (x, y), which may be thought of as functions on R^2. Then the self-dot product of the position vector defines the quadratic distance formula

$$\vec{r} \cdot \vec{r} = Ax^2 + Bxy + Cy^2 \ , \tag{5}$$

where

$$A = X \cdot X \ , B = X \cdot Y \ , C = Y \cdot Y \ . \tag{6}$$

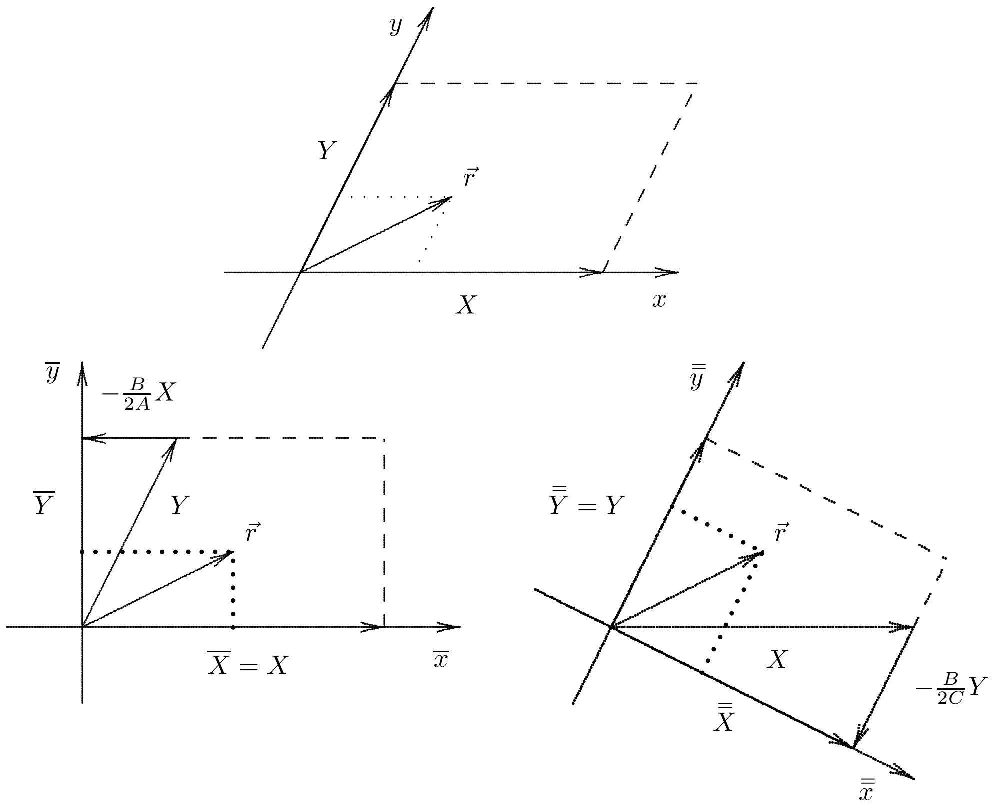


Figure 2: General linear coordinates on E^2 and their two orthogonalizations. The dashed lines indicate the unit coordinate rectangle and the dotted lines the projections of a typical vector along the coordinate axes.

Assume this is positive-definite: $B^2 - 4AC < 0, A > 0, C > 0$. Figure 1.5.1 illustrates the geometry.

There are two choices for completing the square on the quadratic form:

$$\begin{aligned} r \cdot r &= A(x + \frac{B}{2A}y)^2 + (C - \frac{B^2}{4A})y^2 \qquad &&= (A - \frac{B^2}{2C})x^2 + C(y + \frac{B}{2C}x)^2 \ , \\ &= A(\overline{x})^2 + (C - \frac{B^2}{4A})(\overline{y})^2 &&= (A - \frac{B^2}{4C})(\overline{\overline{x}})^2 + C(\overline{\overline{y}})^2 \ , \end{aligned} \tag{7}$$

where new adapted coordinates are defined by

$$\begin{aligned} &\overline{x} = x + \frac{B}{2A}y \ , \quad && x = \overline{x} - \frac{B}{2A}y \ , \quad && \overline{\overline{x}} = x \ , && x = \overline{\overline{x}} \ , \\ &\overline{y} = y \ , && y = \overline{y} \ , && \overline{\overline{y}} = y + \frac{B}{2C}x \ , \quad && y = \overline{\overline{y}} - \frac{B}{2C}\overline{\overline{x}} \ , \end{aligned} \tag{8}$$

The new coordinate systems correspond to the new bases

$$r = xX + yY = \overline{x}\overline{X} + \overline{y}\overline{Y} = \overline{\overline{x}}\overline{\overline{X}} + \overline{\overline{y}}\overline{\overline{Y}} \tag{9}$$

which transform in a complimentary way

$$\begin{array}{llll} \overline{X} = X \ , & X = \overline{X} \ , & \overline{\overline{X}} = X - \frac{B}{2C}Y \ , & X = \overline{\overline{X}} + \frac{B}{2C}\overline{\overline{Y}} \ , \\ \overline{Y} = Y - \frac{B}{2A}X \ , & Y = \overline{Y} + \frac{B}{2A}\overline{X} \ , & \overline{\overline{Y}} = Y \ , & Y = \overline{\overline{Y}} \ , \end{array} \tag{10}$$

In each case one basis vector is retained and the additional new basis vector is obtained by orthogonal projection of the other, a process complimentary to completing the square on the corresponding coordinates. Figure 1.5.1 illustrates these projections and the new unit coordinate rectangles.

1.5.2 The Lorentzian case: general linear coordinates

The Lorentzian case of two-dimensional Minkowski space M^2 is similar but our geometric intuition no longer holds.

Given a pair of vectors (X, T) in M^2, one spacelike, the other timelike, then expressing the position vector $r = xX + tT$ defines its coordinates (x, t), which may be thought of as functions on M^2. Then the self-dot product of the position vector defines the quadratic distance formula

$$\vec{r} \cdot \vec{r} = Ax^2 + Bxt + Ct^2 \ , \tag{11}$$

where

$$A = X \cdot X \ , B = X \cdot T \ , C = T \cdot T \ . \tag{12}$$

Assume this is a Lorentzian inner product: $B^2 - 4AC > 0, A > 0, C < 0$. Figure 1.5.2 illustrates the geometry.

Again there are two choices for completing the square on the quadratic form:

$$\begin{aligned} r \cdot r &= A(x + \frac{B}{2A}t)^2 + (C - \frac{B^2}{4A})t^2 && = (A - \frac{B^2}{2C})x^2 + C(t + \frac{B}{2C}x)^2 \ , \\ &= A(\overline{x})^2 + (C - \frac{B^2}{4A})(\overline{t})^2 && = (A - \frac{B^2}{4C})(\overline{\overline{x}})^2 + C(\overline{\overline{t}}\,)^2 \ , \end{aligned} \tag{13}$$

where new adapted coordinates are defined by

$$\begin{array}{llll} \overline{x} = x + \frac{B}{2A}t \ , & x = \overline{x} - \frac{B}{2A}t \ , & \overline{\overline{x}} = x \ , & x = \overline{\overline{x}} \ , \\ \overline{t} = t \ , & t = \overline{t} \ , & \overline{\overline{t}} = t + \frac{B}{2C}x \ , & t = \overline{\overline{t}} - \frac{B}{2C}\overline{\overline{x}} \ , \end{array} \tag{14}$$

The new coordinate systems correspond to the new bases

$$r = xX + tT = \overline{x}\overline{X} + \overline{t}\overline{T} = \overline{\overline{x}}\overline{\overline{X}} + \overline{\overline{t}}\overline{\overline{T}} \tag{15}$$

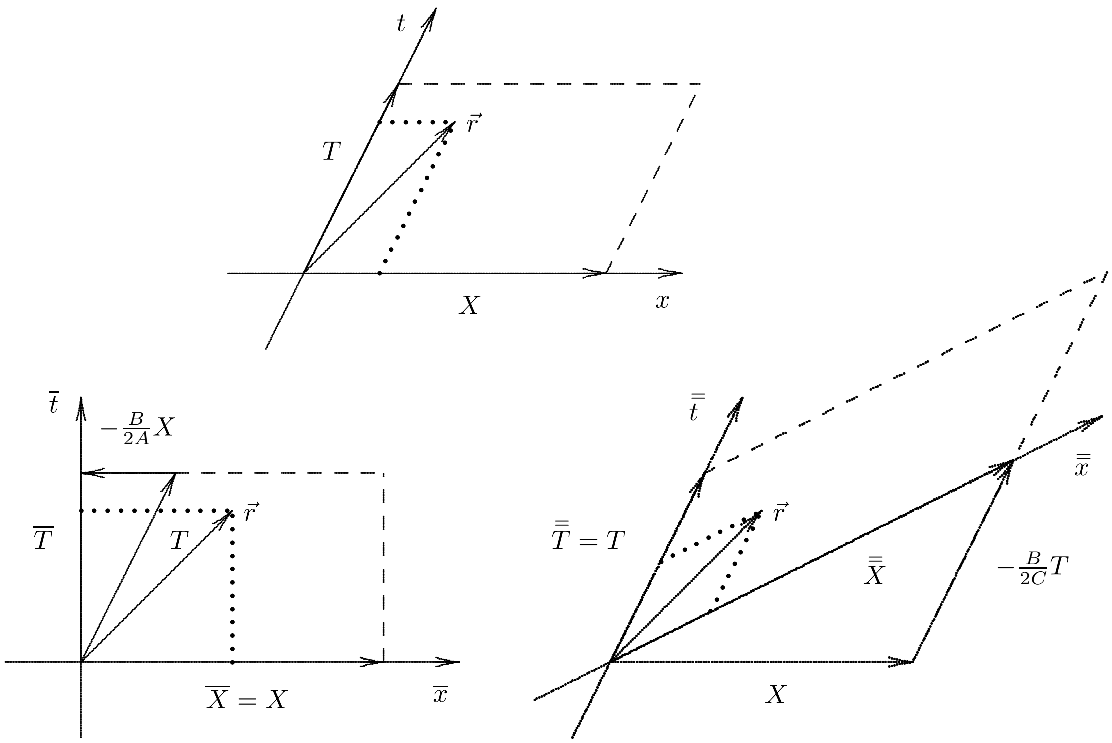


Figure 3: General linear coordinates on M^2 and their two orthogonalizations. The dashed lines indicate the unit coordinate rectangle and the dotted lines the projections of a typical vector along the coordinate axes.

which transform in a complimentary way

$$
\begin{array}{llll}
\overline{X} = X \ , & X = \overline{X} \ , & \bar{\bar{X}} = X - \dfrac{B}{2C}T \ , & X = \bar{\bar{X}} + \dfrac{B}{2C}\bar{\bar{T}} \ , \\
\overline{T} = T - \dfrac{B}{2A}X \ , & T = \overline{T} + \dfrac{B}{2A}\overline{X} \ , & \bar{\bar{T}} = T \ , & T = \bar{\bar{T}} \ ,
\end{array}
\tag{16}
$$

In each case one basis vector is retained and the additional new basis vector is obtained by orthogonal projection of the other, a process complimentary to completing the square on the corresponding coordinates. Figure 1.5.2 illustrates these projections and the new unit coordinate rectangles. The first orthogonalization remains the same, but the second orthogonalization changes due to the change in sign of C.

This whole discussion may be transferred to a tangent space where the starting basis vectors are two frame vectors and the coordinate functions correspond exactly to the dual 1-forms. If one pictures a 1-form σ geometrically in the tangent space by associating with it the two subspaces $\sigma(\vec{r}) = 0, \sigma(\vec{r}) = 1$, then the parallel sides of the unit coordinate rectangle serve this purpose for the two dual 1-forms. Note that the dual 1-form parallel sides for one basis vector are parallel to the other basis vector so that the natural evaluation gives 0. If we assume that the original basis of the tangent space is a coordinate frame $X = \partial_x, T = \partial_t$, then the dual 1-forms are dx and dt. The figure shows that the first orthogonalization does not change dt, while the second one does not change dx.

The quadratic form is then the line element

$$ds^2 = g_{xx}\, dx^2 + 2g_{tx}\, dt\, dx + g_{tt}\, dt^2 \ , \tag{17}$$

and the space orthogonalization is

$$\begin{aligned} ds^2 &= g_{xx}\, (dx + \frac{g_{tx}}{g_{xx}}\, dt)^2 + (g_{tt} - \frac{g_{tx}^2}{g_{xx}})\, dt^2 \\ &= g_{xx}\, (dx + N^x\, dt)^2 - N^2\, dt^2 \ , \end{aligned} \tag{18}$$

while the time orthogonalization is

$$\begin{aligned} ds^2 &= (g_{xx} - \frac{g_{tx}^2}{g_{tt}})\, dx^2 + g_{tt}\, (dt + \frac{g_{tx}^2}{g_{tt}}\, dx)^2 \\ &= \gamma_{xx}\, dx^2 - M^2\, (dt + M_x\, dx)^2 \ . \end{aligned} \tag{19}$$

These two distinct completions of the square lead immediately to a suggestive lapse function and shift vector notation for each case which describes the geometry of the associated projections in the context of the original coordinate frame

$$\begin{aligned} &\overline{X} = \partial_x \ , && \overline{X} \cdot \overline{X} = g_{xx} \ , \\ &\overline{T} = \partial_t + N^x \partial_x \ , && \overline{T} \cdot \overline{T} = -N^2 \ , \\ \\ &\overline{\overline{X}} = \partial_x + M_x \partial_t \ , && \overline{\overline{X}} \cdot \overline{\overline{X}} = \gamma_{xx} \ , \\ &\overline{\overline{T}} = \partial_t \ , && \overline{\overline{T}} \cdot \overline{\overline{T}} = -M^2 \ , \end{aligned} \tag{20}$$

1.6 Notation and conventions

The conventions of Misner, Thorne and Wheeler [1973] will be followed unless stated otherwise or unless an ambiguity arises. Lower case Greek indices will take the values 0,1,2,3 and lower case Latin indices the values 1,2,3. The signature of the spacetime metric will be (−+++). A convenient mix of index and index-free notation will be used in our discussion in order to bridge the gap between those who feel comfortable without indices and those who do not. This leads to a problem when in the index notation the kernel symbol of an object is used to denote its trace, determinant, norm or other scalar property or if the same kernel symbol is used for two different objects. In these cases obvious distinguishing marks must be added to the index-free symbols which in our applications will turn out not to be too cumbersome. For example, “index shifting” with the spacetime metric associates a single kernel symbol with many different objects. The "index lowering" and "index raising" maps associated with the spacetime metric will be denoted by $\flat$ and $\sharp$ respectively, and the symbols $S^\flat$ and $S^\sharp$ will refer to the fully covariant and fully contravariant forms respectively of a given tensor field S. This gives a unique symbol for one-index objects if we identify the symbol alone with the index position which best characterizes the properties of the object or for two-index objects if we agree to use the kernel symbol alone for the mixed object.

The spacetime metric tensor itself ${}^{(4)}g_{\alpha\beta}$ will be denoted by ${}^{(4)}\mathrm{g}$ in an index-free notation (and the inverse or contravariant metric tensor ${}^{(4)}g^{\alpha\beta}$ by ${}^{(4)}\mathrm{g}^{-1} = {}^{(4)}\mathrm{g}^\sharp$) so that the symbol ${}^{(4)}g \equiv |\det({}^{(4)}g_{\alpha\beta})|$ can denote (the absolute value of) its determinant as is customary in the index notation. The prefix "${}^{(4)}$" will distinguish certain four-dimensional objects from related three-dimensional objects for which it is convenient to use the same kernel symbol.

Similarly the lapse function N and the shift vector field N^α of the slicing formalism lead to the need for using $\vec{N}$ to indicate the shift in an index-free notation. The same problem will exist with the lapse function M and shift 1-form M_α to be introduced below for the threading point of view, so the 1-form will be distinguished by the index-free symbol $\overset{=}{M}$. The two parallel lines over the kernel symbol recall the geometrical interpretation of a covector in terms of a family of parallel planes in its related vector space in the same way that the arrow oversymbol recalls the geometrical interpretation of a vector. These symbols interact with those for index-shifting in an obvious way, with $\overset{=}{N} \equiv \vec{N}^\flat$ and $\vec{M} \equiv \overset{=}{M}{}^\sharp$.

One also needs an index-free notation for contraction of tensors. A generalized vector contraction notation will prove useful. Let the left contraction $S \,\lrcorner\, T$ denote the tensor product of the two tensors S and T with a contraction between the rightmost contravariant index of S with the leftmost covariant index of T (i.e., $S^{\cdots\alpha}{}_{\cdots} T^{\cdots}{}_{\alpha\ldots}$), and the right contraction $S \,\llcorner\, T$ the tensor product with a contraction between the leftmost contravariant index of T with the rightmost covariant index of S (i.e., $S^{\cdots}{}_{\ldots\alpha} T^{\alpha\cdots}{}_{\cdots}$), assuming in each case that such indices exist. Each of these contractions may themselves be generalized to an ordered contraction of sets of p adjacent indices, indicated by $\overset{p}{\lrcorner}$ and $\overset{p}{\llcorner}$ respectively, reducing to the previous contractions when $p = 1$.

Finally the spacetime or portion of spacetime under discussion will always be assumed to be orientable and time-orientable and local coordinates and frames will be assumed to be compatible with these orientations when appropriate. The unit oriented volume 4-form ${}^{(4)}\eta$ is defined by the component expression in an oriented frame ${}^{(4)}\eta_{\alpha\beta\delta\gamma} = g^{1/2}\epsilon_{\alpha\beta\delta\gamma}$ with $\epsilon_{0123} = 1$, so that in an (oriented) orthonormal frame one has $\eta_{0123} = 1 = -\eta^{0123}$. The dual of a p-form will then be represented in index form as a left contraction with ${}^{(4)}\eta$ as in Misner, Thorne and Wheeler. For a 2-form F the order does not matter and the dual is

$$ {}^*F_{\alpha\beta} = \tfrac{1}{2} F_{\gamma\delta}\, {}^{(4)}\eta^{\gamma\delta}{}_{\alpha\beta} \ . \tag{21} $$

One must be careful when comparing formulas involving ${}^{(4)}\eta$ with the convention which uses the ordered indices 1,2,3,4 as in Hawking and Ellis [1973] rather than 0,1,2,3. This problem (among others) affects formulas for the vorticity of a timelike congruence.

The one exception to the conventions of Misner, Thorne and Wheeler made in the present text is a reversal of the order of the covariant indices on the symbol for the components of the connection in a spacetime frame $\{e_\alpha\}$ following Hawking and Ellis [1977]

$$ {}^{(4)}\nabla_{e_\alpha} e_\beta = {}^{(4)}\Gamma^\gamma{}_{\alpha\beta} e_\gamma = e_{\beta;\alpha} \ . \tag{22} $$

This convention might be called the "del" convention in contrast with the "semicolon" convention in which the order of the covariant indices instead follows the semicolon notation. Both the comma and subscripted partial notation will be used to indicate the frame derivatives of functions

$$ e_\alpha f = \partial_\alpha f = f_{,\alpha} \ . \tag{23} $$

One notational conflict between the old fashioned component notation and modern frame notation is the convention which doesn't distinguish between the derivatives of components and the components of derivatives. For example, $\pounds_X Y^\alpha = [X,Y]^\alpha$ conventionally denotes the components of the Lie derivative of the vector field Y and not the Lie derivative of its components, which are instead represented by $XY^\alpha = Y^\alpha{}_{,\beta} X^\beta$. On the other hand in the frame notation it is natural to write for the product rule

$$ \pounds_X (Y^\alpha e_\alpha) = (\pounds_X Y^\alpha) e_\alpha + Y^\alpha (\pounds_X e_\alpha) = [\pounds_X Y]^\alpha e_\alpha \ , \tag{24} $$

and here the symbol $\pounds_X Y^\alpha = XY^\alpha$ instead really does stand for the Lie derivative of the components. The context should always make clear which meaning is intended, and if it doesn't, an explicit comment will.

The notation introduced here may be found to be somewhat cumbersome in a particular application by someone already familiar with that particular application and accustomed to a more streamlined choice of symbols. However, the point is not to develop a notation that is the simplest to use in a specific application, but one which is capable of describing unambiguously all possible applications and their relationships. One may always later adopt an abbreviated notation for specific calculations by elimination of some of the qualifying marks on the kernel symbols.

Given this introduction to the notational philosophy to be employed in the text, one can find in Appendix A a list of actual formulas from differential geometry that will be required. All discussion will be local in character. The region of four-dimensional spacetime where the discussion is valid will be designated by ${}^{(4)}M$, and this region will be referred to simply as spacetime. A good example to keep in mind is the exterior of the event horizon in a black hole spacetime, for example, where the Boyer-Lindquist coordinates are valid. The threading point of view is valid outside the ergosphere (the time coordinate lines are are timelike), while the slicing point of view is valid outside the event horizon (the time coordinate hypersurfaces are spacelike).