MAT2500-01/03 21S Quiz 7 Print Name (Last, First) ______________________

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC). Only use technology to CHECK hand calculations, not substitute for them.

1. Consider the function $f(x, y) = x\,y\,e^{-x^2-y^2}$.

a) Make a plot of this function to see explicitly what it looks like and what kind of local extrema it has. How many local minima and local maxima occur and in what quadrants? What is a reasonable value of L for the window $x=-L..L, y=-L..L$ which shows the interesting part of the graph before its pixels seem to merge with the x-y plane?

b) Find the critical points of this function and its values there.

c) Classify these critical points with the second derivative test. [Remember to check both 2nd partials along the axes first.]

d) Write a parametrized equation for the normal line to the graph of f at the point (1,1) and find the point where it hits the x-y plane.

e) Evaluate the directional derivative of f at the point (1,1) in the direction towards the origin.

▶ solution

① a) $f(x,y) = x y e^{-x^2-y^2} \rightarrow$ $\boxed{x=-3..3,\ y=-3..3 \text{ seems to work and one sees clearly 2 local max's in quads 1\&3 and 2 local min's in quads 2\&4.}}$

b) $f_x = y e^{-x^2-y^2} + xy\,e^{-x^2-y^2}(-2x)$

$= y(1-2x^2)e^{-x^2-y^2} = 0 \longrightarrow y=0 \text{ or } x = \pm\frac{1}{\sqrt{2}}$

$f_y = x e^{-x^2-y^2} + xy\,e^{-x^2-y^2}(-2y)$

$= x(1-2y^2)e^{-x^2-y^2} = 0 \longrightarrow x=0 \text{ or } y = \pm\frac{1}{\sqrt{2}}$

$\boxed{\text{critical points: } (0,0),\ \underbrace{(\pm\tfrac{1}{\sqrt{2}}, \pm\tfrac{1}{\sqrt{2}})}_{\text{all 4 combinations}}}$ ←

(note f is an odd function of x & y and of switching x & y)

$\boxed{f(0,0) = 0,\ f(\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{2}}) = f(-\tfrac{1}{\sqrt{2}},-\tfrac{1}{\sqrt{2}}) = \tfrac{1}{2}e^{-2(\frac{1}{2})} = \tfrac{1}{2}e^{-1} \quad f(\tfrac{1}{\sqrt{2}},-\tfrac{1}{\sqrt{2}}) = f(-\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{2}}) = -\tfrac{1}{2}e^{-1}}$

c) $f_{xx} = [y(-4x) + y(1-2x^2)\cancel{e}(-2x)]e^{-x^2-y^2}$

$= [-4xy - 2xy(1-2x^2)]e^{-x^2-y^2} = -2xy[1-2x^2+2]e^{-x^2-y^2} = -2xy(3-2x^2)e^{-x^2-y^2}$

$= -2xy(3-2y^2)e^{-x^2-y^2}$

$f_{yy} = \ldots$

$f_{xy} = \frac{\partial}{\partial y}\left(y(1-2x^2)e^{-x^2-y^2}\right) = \left((1-2x^2) + y(1-2x^2)(-2y)\right)e^{-x^2-y^2}$

$= (1-2x^2)(1-2y^2)e^{-x^2-y^2}$

	(x,y)	(0,0)	$(1/\sqrt{2}, 1/\sqrt{2})$	$(-1/\sqrt{2}, -1/\sqrt{2})$	$(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
f_{xx}	$-2xy(3-2x^2)e^{\cdots}$	0	$-2/2\,(2) < 0$ ⌢	ditto	$2>0$ ⌣	ditto
f_{yy}	$-2xy(3-2y^2)e^{\cdots}$	0	$-2<0$ ⌢	ditto	$2>0$ ⌣	ditto
f_{xy}	$(1-2x^2)(1-2y^2)e^{\cdots}$	$e^{\cdots}>0$	0	ditto	0	ditto
$f_{xx}f_{yy}-f_{xy}^2$		<0	>0	ditto	>0	ditto
		saddle	local max	local max	local min	local min.

d) $\vec{\nabla} f(x,y) = \langle y(1-2x^2), x(1-2y^2) \rangle e^{-x^2-y^2}$

$\vec{\nabla} f(1,1) = \langle 1(-1), 1(-1) \rangle e^{-2} = \langle -1,-1 \rangle e^{-2}$

$z = L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$= e^{-2} + (-e^{-2})(x-1) + (-e^{-2})(y-1)$

$= e^{-2} + e^{-2} + e^{-2} - e^{-2}x - e^{-2}y$

$\underbrace{+e^{-2}x + e^{-2}y + z}_{} = 3e^{-2}$

$\vec{n} = \langle e^{-2}, e^{-2}, 1 \rangle$ upwards normal

OR:

$F(x,y,z) = z - xy\, e^{-x^2-y^2} = 0$

$\vec{\nabla} F(x,y,z) = \langle -f_x, -f_y, 1 \rangle = \langle -y(1-2x^2), -x(1-2y^2) \rangle e^{-x^2-y^2}$

$\vec{\nabla} F(1,1,e^{-2}) = \langle e^{-2}, e^{-2}, 1 \rangle = \vec{n}$

$\vec{r} = \vec{r_0} + t\vec{n} = \langle 1, 1, e^{-2} \rangle + t\langle e^{-2}, e^{-2}, 1 \rangle$

normal line

$$\boxed{\langle x,y,z \rangle = \langle 1 + te^{-2}, 1 + te^{-2}, e^{-2} + t \rangle}$$

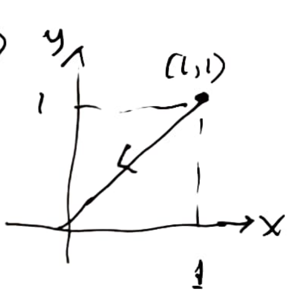
$z = 0 \to t = -e^{-2}$

$x = 1 - e^{-4}$

$y = 1 - e^{-4}$

$$\boxed{\text{hits point } (1-e^{-4}, 1-e^{-4}, 0)}$$

e)



$\vec{u} = \langle -2,-1 \rangle$, $\hat{u} = -\frac{\langle 1,1 \rangle}{\sqrt{2}}$

$\vec{\nabla} f(1,1) = \langle 1(-1), -1 \rangle e^{-2}$

$\boxed{D_{\hat{u}} f(1,1) =} \; \hat{u} \cdot \vec{\nabla} f(1,1) = -\frac{\langle 1,1 \rangle}{\sqrt{2}} \cdot \langle -1,-1 \rangle e^{-2}$

$= \frac{2}{\sqrt{2}} e^{-2} = \boxed{\sqrt{2}\, e^{-2}}$