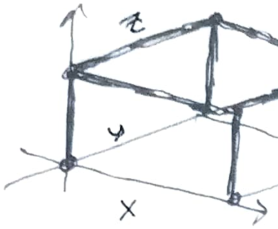
MAT2500-01/02 20S Take Home Test 2 Answers (1)

(1)



$x > 0, y > 0, z > 0$

← 8 edges of "frame"

$P = \underbrace{2x+2y}_{\text{top}} + \underbrace{4z}_{\text{legs}} = 12$ (constraint)

eliminate z

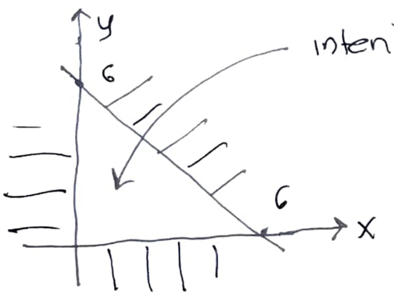
maximize $V = xyz$

$z = \frac{12 - 2(x+y)}{4} = 3 - \frac{1}{2}(x+y) > 0$

$\hookrightarrow x + y < 6$

$V = xy(3 - \frac{1}{2}(x+y))$

$$\boxed{\begin{array}{l} V(x,y) = 3xy - \frac{1}{2}x^2y - \frac{1}{2}xy^2 \\ \qquad x > 0,\ y > 0,\ x+y < 6 \end{array}}$$



interior is allowed region

$V = 0$ on boundary

$V > 0$ inside

$\therefore$ single local max must be global max

critical points:

$\frac{\partial V}{\partial x} = 3y - xy - \frac{1}{2}y^2 = y(3 - x - \frac{1}{2}y) = 0$

$\frac{\partial V}{\partial y} = 3x - \frac{1}{2}x^2 - xy = x(3 - \frac{1}{2}x - y) = 0$

cannot $= 0$

$x + \frac{1}{2}y = 3 \rightarrow y = 2(3-x)$

$\frac{1}{2}x + y = 3 \longrightarrow \frac{1}{2}x + 2(3-x) = 3$

$6 - \frac{3}{2}x = 3 \rightarrow x = 2$

$\rightarrow y = 2(3-2) = 2$

$z = 3 - \frac{1}{2}(2+2) = 1$

$(x,y,z) = (2,2,1) \quad V = 2 \cdot 2 \cdot 1 = 4$

$$\boxed{\text{The maximum volume box has a square base of 2 ft by 2 ft and a height 1 ft, with volume 4 ft}^3}$$

local max:

at (2,2):

$\frac{\partial^2 V}{\partial x^2} = -y \longrightarrow -2 < 0$

$\frac{\partial^2 V}{\partial y^2} = -x \longrightarrow -2 < 0$

} local max

$\frac{\partial^2 V}{\partial x \partial y} = 3 - x - y \rightarrow -1$

$\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 V}{\partial y^2} - \left(\frac{\partial^2 V}{\partial x \partial y}\right)^2 = (-2)(-2) - (-1)^2$

$= 3 > 0$ ✓

local max in all directions!

MAT2500-01/02 20S Test 2 ~~Take~~home Answers (2)

2 a) $z = \frac{1}{2}(1+x^2-y^2) = f(x,y)$

$(x,y) = (\frac{1}{2},\frac{1}{4}) \to z = f(\frac{1}{2},\frac{1}{4}) = \dots = \frac{19}{32} \approx 0.594$

$P(\frac{1}{2},\frac{1}{4},19/32)$

$F(x,y,z) = f(x,y) - z = \frac{1}{2}(1+x^2-y^2) - z$

$\vec{\nabla}F(x,y,z) = \langle x, -y, -1\rangle$ downward normal (upward normal ok too)

$\vec{\nabla}F(\frac{1}{2},\frac{1}{4},\frac{19}{32}) = \langle \frac{1}{2}, -\frac{1}{4}, -1\rangle = \frac{1}{4}\underbrace{\langle 2,-1,-4\rangle}_{\equiv \vec{n}}$

$|\vec{n}| = \sqrt{4+1+16} = \sqrt{21}$

$\hat{n} = \frac{1}{\sqrt{21}}\langle 2,-1,4\rangle$

$\vec{r}_0 = \langle \frac{1}{2},\frac{1}{4},\frac{19}{32}\rangle$

$\vec{r} = \vec{r}_0 + t\hat{n}$

$$\boxed{\langle x,y,z\rangle = \langle \frac{1}{2},\frac{1}{4},\frac{19}{32}\rangle + \frac{t}{\sqrt{21}}\langle 2,-1,-4\rangle = \langle \frac{1}{2}+\frac{2t}{\sqrt{21}}, \frac{1}{4}-\frac{t}{\sqrt{21}}, \underbrace{\frac{19}{32}-\frac{4t}{\sqrt{21}}}\rangle}$$ normal line

$z=0$ floor

$t = \frac{19}{32}\frac{\sqrt{21}}{4} = \boxed{\frac{19\sqrt{21}}{128}} \approx 0.680$ distance from light to spot center on floor

$x = \frac{1}{2} + \frac{2}{\cancel{\sqrt{21}}}\frac{19}{128}\cancel{\sqrt{21}} = \frac{1}{2} + \frac{19}{64} = \frac{32+19}{64} = \frac{51}{64}$

$y = \frac{1}{4} - \frac{1}{\cancel{\sqrt{21}}}\frac{19\cancel{\sqrt{21}}}{128} = \frac{1}{4} - \frac{19}{128} = \frac{1}{128}(32-19) = \frac{13}{128}$

$\boxed{(\frac{51}{64},\frac{13}{128})}$ is the point on the floor ($z=0$!) where the center of the spot is located

$d = \sqrt{(\frac{1}{2}-\frac{51}{64})^2 + (\frac{1}{4}-\frac{13}{128})^2} = \boxed{\frac{19\sqrt{5}}{128} \approx 0.33191 \approx 0.3319}$

b) $f(x,y) = \frac{1}{2}(1+x^2-y^2)$

$\vec{\nabla}f(x,y) = \langle x, -y\rangle$

$\vec{\nabla}f(\frac{1}{2},\frac{1}{4}) = \langle \frac{1}{2}, \frac{-1}{4}\rangle = \frac{1}{4}\langle 2,-1\rangle$

$\hat{\nabla}f(\frac{1}{2},\frac{1}{4}) = \frac{\langle 2,-1\rangle}{\sqrt{5}}$. increases fastest.

$\boxed{\hat{u} = -\frac{\langle 2,-1\rangle}{\sqrt{5}}}$ decreases fastest

c) $\hat{u} = \langle 0,-1\rangle$

$D_{\hat{u}}f(\frac{1}{2},\frac{1}{4}) = \hat{u}\cdot\vec{\nabla}f(\frac{1}{2},\frac{1}{4}) = \langle 0,-1\rangle \cdot \frac{1}{4}\langle 2,-1\rangle$

$= \frac{1}{4} = 0.25$ rate of change in this direction

c) (continued)

If $\Delta s = .02$ in this direction:

$\Delta z = (0.25)(0.02) = \boxed{0.005}$

$\boxed{\text{increases}}$ by this amount

(to $0.594 + 0.005 = 0.599$)

confirmed by 3d plot, roof rises to ridge over x axis!

3 a) $S = 0.1091\, w^{0.425} h^{0.725}$

$dS = \frac{\partial S}{\partial w}dw + \frac{\partial S}{\partial h}dh$

$= 0.1091(0.425\, w^{0.425-1} h^{0.725} dw + 0.725\, w^{0.425} h^{0.725-1} dh)$

$$\frac{dS}{S} = \frac{\cancel{0.1091}(0.425\, w^{0.425} h^{0.725}\frac{dw}{w} + 0.725\, w^{0.425} h^{0.725}\frac{dh}{h})}{\cancel{0.1091}\, w^{0.425} h^{0.725}}$$

$= (0.425\frac{dw}{w} + 0.725\frac{dh}{h})$

$|\frac{dw}{w}| \le 0.02$, $|\frac{dh}{h}| \le 0.02$ (2% error bars)

triangle inequality:

$|\frac{dS}{S}| \le 0.425|\frac{dw}{w}| + 0.725|\frac{dh}{h}|$

$\le 0.425(0.02) + 0.725(0.02)$

$= (0.425+0.725)(0.02)$

$= (1.150)(0.02)$

$= 0.0230 \approx \boxed{2.3\% \text{ maximum error}}$

b) $S(160,72) = 20.948 = S_i$

$S(\underbrace{0.98}160, \underbrace{0.98}72) = 20.469 = S_f$

decrease 2%

$\frac{S_f - S_i}{S_i} = -0.02297$ ✓ just inside the ~~predicted~~ error bar!

$+0.02300$

$.00003$