MAT2705-04/05 20F Quiz 8 Print Name (Last, First) ______________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (**but give decimal approximations for interpretation**). Indicate where technology is used and what type (Maple, GC).

1. $2\,y''(t) + 20\,y'(t) + 1300\,y(t) = 50\cos(25\ t)$, $y(0) = 0$, $y'(0) = 0$ [Maple notation].

a) State Maple's solution of the initial value problem (use function notation $y(t)$).

b) Put the DE into standard linear form (unit leading coefficient). Then identify the values of the damping constant and characteristic time $k_0 = 1/\tau_0$, the natural frequency ω_0 , and the quality factor $Q = \omega_0\,\tau_0$, exactly and numerically. Is this underdamped, critically damped or overdamped?

c) Find the general solution by hand, showing all steps.

d) Find the solution satisfying the initial conditions, showing all steps.

e) Give exact and numerical values of the amplitude and phase shift of the steady state solution (the particular solution!) and re-express this sinusoidal function in phase-shifted cosine form. [Make sure you use a diagram to justify your values.] State what numerical fraction of a cycle (2π) the phase shift is (i.e., evaluate $\delta/2\pi$) as well as its numerical value in degrees, and whether the cosine curve is shifted left (earlier in time) or right (later in time) on the time line (by a phase less than or equal to half a cycle of course). Explain. [You can check by graphing!]

f) Plot the solution and its steady state part together in a window where at the last peak, the pixels of the two curves finally merge so one cannot distinguish the two curves. Make a sketch of what you see, including labeled axes and tickmarks, and label the two curves.

g) Find the two envelope functions of the decaying oscillating transient solution (the homogeneous part of the solution) and in a separate plot without the full solution, plot them together with that transient. Then make a rough hand sketch of what you see for 5 characteristic decay times for the envelope, including labeled axes and tickmarks.

① a) $\frac{1}{2}[2y''+20y'+1300y = 50\cos 25t]$

$y''+\underbrace{10}_{k_0}y' + \underbrace{650}_{\omega_0^2}y = 25\cos 25t$ ($\uparrow\omega = 25 \approx \omega_0$!)

$k_0 = 10$, $\tau_0 = \frac{1}{10} = .1$, $\omega_0 = \sqrt{650} = 5\sqrt{26}$
≈ 25.495 ($\omega \lesssim \omega_0$)

$Q = \omega_0\tau_0 = \frac{5\sqrt{26}}{10} = \frac{\sqrt{26}}{2} \approx 2.55 > \frac{1}{2}$

underdamped

b) hom. soln.:

$y = e^{rt} \to$ DE: $\underbrace{(r^2+10r+650)}\,e^{rt} = 0$

$= 0 \to r = -5 \pm 25i$

MAPLE

$e^{rt} = e^{-5t}e^{\pm 25it} = e^{-5t}(\cos 25t \pm i\sin 25t)$

$\hookrightarrow e^{-5t}\cos 25t,\ e^{-5t}\sin 25t$ basis

$y_h = (c_1\cos 25t + c_2\sin 25t)\,e^{-5t}$

$25\cos 25t$: $r = \pm 25it$, $m = 1$

$e^{rt} = e^{\pm 25it} = \cos 25t \pm i\sin 25t$

b) $(D^2+25^2)(25\cos 25t) = 0$

$(D^2+25^2)(D^2+10D+650)\,y = 0$

no root overlap, gen soln:

$$\boxed{\begin{aligned} y = {} & \underbrace{e^{-5t}(c_1\cos 25t + c_2\sin 25t)}_{y_h} \\ & + \underbrace{c_3\cos 25t + c_4\sin 25t}_{y_p} \end{aligned}}$$ almost

$650\,[y_p = c_3\cos 25t + c_4\sin 25t]$

$10\,[y_p' = -25c_3\sin 25t + 25c_4\cos 25t]$

$1\,[y_p'' = -25^2c_3\cos 25t - 25^2c_4\sin 25t]$

$y_p'' + 10y_p' + 650y_p = [(650-25^2)c_3 + 250c_4]\cos 25t$
$+ [-250c_3 + (650-25^2)c_4]\sin 25t$
$= 25\cos 25t$

$\begin{bmatrix} 25 & 250 \\ -250 & 25 \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 25 \\ 0 \end{bmatrix}$ divide thru by 25 to make easier (not necessary)

$\begin{bmatrix} 1 & 10 \\ -10 & 1 \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

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(1b) continued

$$\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 & 10 \\ -10 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{101}\begin{bmatrix} 1 & -10 \\ 10 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{101}\begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

$$y_p = \frac{1}{101}(\cos 25t + 10 \sin 25t)$$

$$\boxed{y = y_h + y_p = e^{-5t}(c_1 \cos 25t + c_2 \sin 25t) + \frac{1}{101}(\cos 25t + 10 \sin 25t)}$$

gen soln

d) $y' = -5e^{-5t}(c_1 \cos 25t + c_2 \sin 25t)$
$\quad + e^{-5t}(-25c_1 \sin 25t + 25 c_2 \cos 25t)$
$\quad + \frac{1}{101}(-25 \sin 25t + 250 \cos 25t)$

$y(0) = c_1 + \frac{1}{101} = 0 \rightarrow c_1 = -\frac{1}{101}$

$y'(0) = -5c_1 + 25c_2 + \frac{250}{101}$

$c_2 = \frac{1}{25}\left(5c_1 - \frac{250}{101}\right) = \frac{-255}{25 \cdot 101} = -\frac{51}{505}$

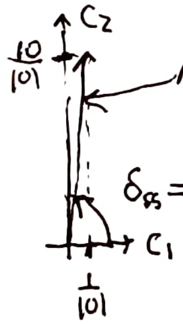
$$\boxed{y = e^{-5t}\left(-\frac{1}{101}\cos 25t - \frac{51}{505}\sin 25t\right) + \frac{1}{101}(\cos 25t + 10 \sin 25t)}$$

IVP soln

y transient

y_{ss}

e)



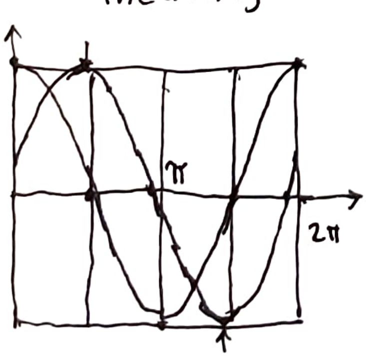
$A_{ss} = \frac{\sqrt{1^2+10^2}}{101} = \frac{\sqrt{101}}{101} = \frac{1}{\sqrt{101}} \approx 0.0995$

$\delta_{ss} = \arctan 10 \approx 1.4711$ (rad)
$\approx 84.3°$
≈ 0.2341 cycles

divide by 2π

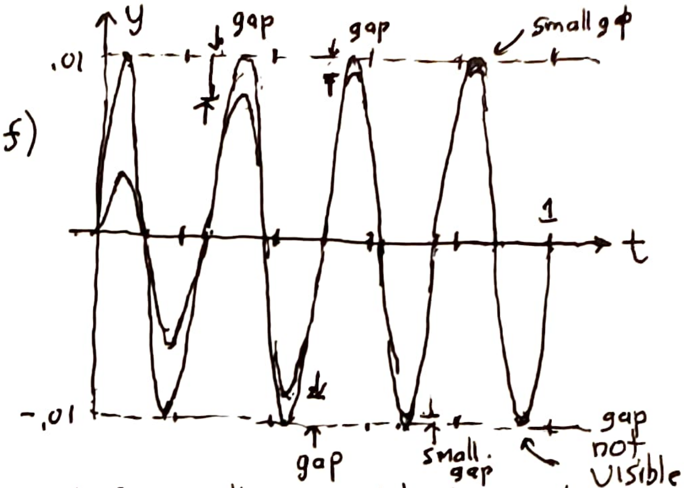
$y_{ss} = A_{ss}\cos(25t - \delta_{ss}) = \frac{1}{\sqrt{101}}\cos(25t - \arctan 10)$

$\delta > 0$ means shifted right on time line by nearly 1/4 cycle, lagging behind the driving cosine function



plot not required but is a check: $6°$ ahead of quarter cycle. nearly at natural frequency so expect $\delta_{ss} \sim \pi/2 \sim 90°$

f)

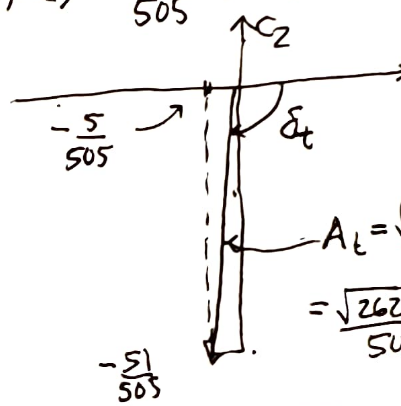


soln is smaller amplitude graph

$\tau = \frac{1}{5} \rightarrow t = 0..5\tau = 1$ viewing window

5 characteristic times of exp decay

g) $\langle c_1, c_2 \rangle = \frac{1}{505}\langle -5, -51 \rangle$ 3rd quad



$A_t = \frac{\sqrt{(-5)^2 + (-51)^2}}{505} = \frac{\sqrt{2626}}{505} \approx 0.1015$

envelope $y = \pm \frac{\sqrt{2626}}{505} e^{-5t}$

$\tau = \frac{1}{5}$

so plot $t = 0..5\tau = 1$