MAT2705-04/05 20F Quiz 7 Print Name (Last, First) ______________________________|___

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC).

1. $2\,y'' + 12\,y' + 50\,y = 0$, $y(0) = -3$, $y'(0) = 21$ [prime is d/dt , i.e., the independent variable is the time t]

a) Put the DE into standard linear form $y'' + k_0\,y' + \omega_0^{\,2}\,y = 0$ first. Then identify the values of the damping constant $k_0 = 1/\tau_0$, the natural frequency ω_0 , and the numerical value of the quality factor $Q = \omega_0\,\tau_0$.

b) Find the general solution by hand, showing all steps.

c) Find the solution satisfying the initial conditions, showing all steps.

d) Re-express the sinusoidal factor of this solution exactly in phase-shifted cosine form (evaluating the amplitude and phase shift first exactly and then decimally approximate them to 4 decimal places) to obtain the two envelope functions of this decaying oscillation solution. State the two envelope functions. State the phase shift in degrees. What fraction of a cycle ($2\,\pi$) is the phase shift approximately?

e) Make a rough plot of the sinusoidal factor for one half period before and after $t = 0$. Identify the points on the graph where $\omega\,t - \delta = 0$. Is the cosine is shifted left (earlier in time) or right (later in time) on the time line (by a phase angle less than or equal to π)?

f) Make a rough sketch of the plot of your solution and its two envelope functions in a viewing window of width 5 times the characteristic time of the solution exponential factor.

g) Use calculus to determine ***exactly*** the t and y values of the first maximum of the solution funcion $y(t)$ for $t \ge 0$ and their approximate values to 4 decimal places. Locate and label this point on your sketch. Do the numbers you found agree with what your eyes see in the technology plot? [Yes or no, with an explanation would be a good response.]

h) State Maple's solution of the initial value problem.

▶ **solution**

① a) $\frac{1}{2}[2y'' + 12y' + 50y = 0]$

$y'' + 6y' + 25y = 0$

$k_0 = 6$ $\quad \omega_0 = 5$

$\tau_0 = 1/6$ $\quad (T_0 = \frac{2\pi}{5} \approx \quad)$

$Q = \omega_0 \tau_0 = \frac{5}{6} \approx$

b) $y = e^{rt}$: $r^2 + 6r + 25 = 0$

$r = \frac{-6 \pm \sqrt{36 - 4(25)}}{2} = -\frac{6 \pm 8i}{2} = -3 \pm 4i$

$e^{rt} = e^{-3t} e^{\pm 4it} = e^{-3t}(\cos 4t \pm i \sin 4t)$

$\to \{e^{-3t}\cos 4t,\ e^{-3t}\sin 4t\}$ real basis

$\boxed{y = e^{-3t}(c_1 \cos 4t + c_2 \sin 4t)}$ gen soln

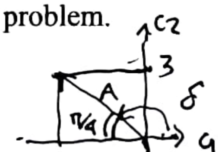
c) $y' = -3e^{-3t}(c_1\cos 4t + c_2 \sin 4t)$
$\quad + e^{-3t}(-4c_1 \sin 4t + 4c_2 \cos 4t)$

$y(0) = c_1 = -3$

$y'(0) = -3c_1 + 4c_2 = 21 \to c_2 = \frac{1}{4}(21 + 3(-3)) = 3$

$\boxed{y = e^{-3t}(-3\cos 4t + 3\sin 4t)}$ IVP soln

d)



$A = 3\sqrt{2}$, $\tan\delta = 3/(-3) = -1$

≈ 4.2426 $\quad \delta = \pi - \arctan 1$

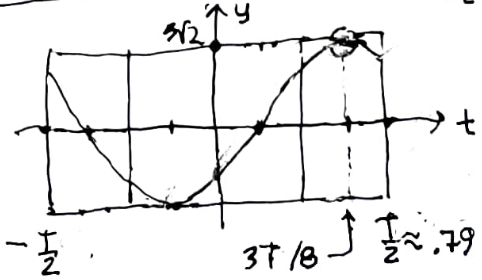
$= 3/4\,\pi = 135°$

$= \frac{3}{8}$ cycle

≈ 2.3561 (easy !)

$\boxed{y = 3\sqrt{2}\,e^{-3t}\cos(4t - 3\pi/4)}$

e)



$\omega = 4$

$T = 2\pi/4$

$= \pi/2$

≈ 1.5708

$\omega t - \delta = 4t - 3\pi/4 = 0 \to t = \frac{3\pi}{16} \approx 0.5890$

cosine shifted right (later in time)

f) $\tau = 1/3 \to 5\tau = 5/3 \approx 1.67$

envelope curves:

$y = \pm 3\sqrt{2}\,e^{-3t}$

(out of space – continued on next sheet)

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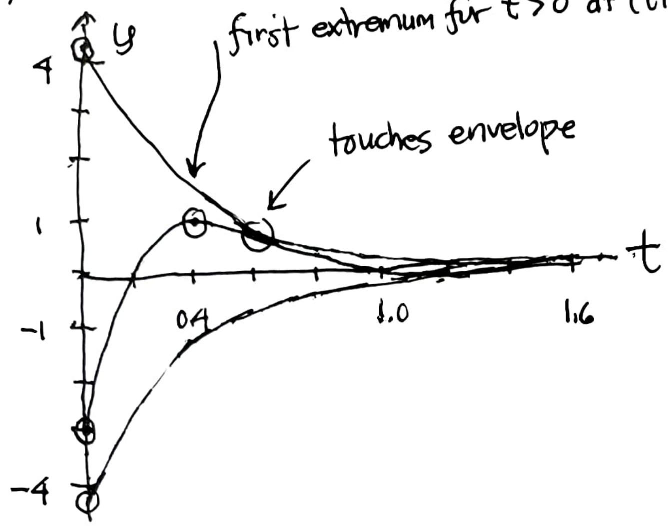
f) continued



h) no surprise.
Maple agrees with the above IVP soln.

g) $y = 3\sqrt{2}e^{-3t}\cos(4t - \frac{3\pi}{4}) = e^{-3t}(-3\cos 4t + 3\sin 4t)$

$y' = -3e^{-3t}(-3\cos 4t + 3\sin 4t)$
$\quad + e^{-3t}(12\sin 4t + 12\cos 4t)$

$= e^{-3t}(\underbrace{(9+12)}_{21}\cos 4t + \underbrace{(-9+12)}_{3}\sin 4t) = 0$

at t_1: $\cos 4t_1 = -\frac{1}{\sqrt{50}}$, $\sin 4t_1 = \frac{7}{\sqrt{50}}$

$\left(\frac{3}{\sqrt{50}} + \frac{21}{\sqrt{50}}\right)e^{-3t_1}$

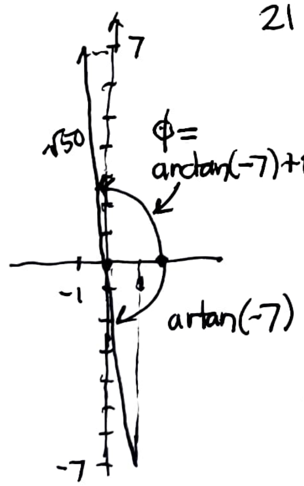
$\frac{24}{5\sqrt{2}} = \frac{12}{5}\sqrt{2}\, e^{-\frac{3}{4}(\pi - \arctan 7)}$

$21\cos 4t + 3\sin 4t = 0$

$\tan 4t = \frac{\sin 4t}{\cos 4t} = -\frac{21}{3} = -7$

$4t = \underbrace{\arctan(-7)}_{\text{4th quadrant}} + \pi$

π pushes to 2nd quadrant for first positive value of $4t$ where this is zero



$\cos\phi = -1/\sqrt{50}$
$\sin\phi = 7/50$

$t_1 \equiv \boxed{t = \frac{1}{4}(\pi - \arctan 7) \approx 0.4282}$

agrees with graph estimate $t \approx 0.4$

$y_1 \equiv y\left(\frac{1}{4}(\pi - \arctan 7)\right) = \frac{12\sqrt{2}}{5}\exp\left(\frac{3}{4}(\arctan 7 - \pi)\right) \approx \boxed{0.9394}$

$= \frac{12\sqrt{2}}{5}\left(\frac{e^{\arctan 7}}{e^{\pi}}\right)^{3/4}$

reading off Maple's evaluation

this also agrees with graph