

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). You are encouraged to use technology to check all of your hand results. You may use Maple for row reduction without showing individual steps and for matrix inverses. Show matrix multiplication details.

**Print and scan both these sheets to place at the beginning of your test response PDF.**

## pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet stapled on top of your answer sheets as a cover page, with the first test page facing up:

"During this examination, all work has been my own. I have not accessed any of the class web pages or any other sites during the exam. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature: \_\_\_\_\_

Date: \_\_\_\_\_

A coupled system of ODEs representing a 2 mass 3 spring system has the following equations of motion and initial conditions:

$$x_1'' = -3x_1 + 2x_2 + 9\cos(2t), x_2'' = 4x_1 - 5x_2, x_1(0) = 7, x_2(0) = 1, x_1'(0) = 0, x_2'(0) = 0.$$

a) The solution of this IVP has integer coefficients. Verify this using Maple and write down its solutions for the two unknowns. [Use function notation for all variables in Maple:  $x_1(t)$ ,  $x_1'(t)$ , etc. Ask bob if something goes wrong.]

b) Identify the exact and numerical values of the two natural frequencies  $\omega_1 < \omega_2$  which appear in these solutions together with the driving frequency  $\omega_3 = 2$ . What are the exact and numerical values of the corresponding 3 periods  $T_1, T_2, T_3$ ? Give 3 decimal place accuracy. Because the frequency ratios are not all rational numbers, there is no common period.

c) Rewrite this system of DEs and its initial conditions in explicit matrix form  $\vec{x}'' = A\vec{x}' + \vec{F}$  for the vector variable  $\vec{x} = \langle x_1, x_2 \rangle$ , identifying the coefficient matrix  $A$  and the driving vector  $\vec{F}$ .

d) Use Maple to write down its choice of eigenvalues and eigenvectors of  $A$ , ordered so that  $|\lambda_1| < |\lambda_2|$ .

e) By hand showing all (matrix method) steps, find the **smallest integer component** eigenvectors  $\vec{b}_1, \vec{b}_2$  of the coefficient matrix  $A$  produced by the solution algorithm after rescaling of the standard results by positive multiples if necessary, ordered so that the corresponding eigenvalues satisfy  $|\lambda_1| < |\lambda_2|$ , i.e.,  $\omega_1 < \omega_2$ . Write down this matrix  $B = \langle \vec{b}_1 | \vec{b}_2 \rangle$  and use Maple to evaluate its inverse and the diagonalized matrix  $A_B = B^{-1}AB$ .

[Make sure your results agree with Maple's eigenvectors modulo rescaling and/or permutation.]

f) What are the slopes  $m_1, m_2$  of the lines through the origin containing the two eigenvectors (remember, use integer component eigenvectors)? On the grid provided, draw in those two lines, labeling them by their corresponding coordinates  $y_1, y_2$  at the ends in the positive direction (include arrow heads) determined by the

eigenvectors and then indicate by thicker arrows both eigenvectors  $\vec{b}_1, \vec{b}_2$ , labeled by their symbols. Indicate the unit tickmarks marked off along each new axis. Recall  $\vec{x} = B\vec{y}, \vec{y} = B^{-1}\vec{x}$ , where  $\vec{y} = \langle y_1, y_2 \rangle$ . Also label the  $x_1, x_2$  axes.

g) Evaluate  $\vec{y}(0) = B^{-1}\vec{x}(0), B^{-1}\vec{F}(0)$  to find the new components of these two vectors.

h) On the grid provided, draw in the vector  $\vec{x}(0)$  and label this vector. On your graph, draw in exactly the parallelograms parallel to the new coordinate axes which project this vector along those axes and lightly shade it in

- in pencil (pen?). Are the part g) new components numerically consistent with your plot parallelogram? Explain.
- i) Find by hand the general solution of the corresponding decoupled system of DEs  $\vec{y}'' = A_B \vec{y} + B^{-1} \vec{F}$ . First write these equations out in explicit matrix form, then obtain the two equivalent scalar DEs which are its components. Then solve them to find their general solutions using the method of undetermined coefficients. State your general solutions in scalar form and box them:  $y_1(t) = \dots, y_2(t) = \dots$ , identifying the homogeneous and particular parts of each solution:  $y_1 = y_{1h} + y_{1p}, y_2 = y_{2h} + y_{2p}$ .
- j) Then express the general solution for  $\vec{x} = B \vec{y}$  in explicit matrix form (without multiplying matrix factors) **and impose the initial conditions** using matrix methods to solve the linear systems. Write out and box the final scalar solutions:  $x_1(t) = \dots, x_2(t) = \dots$ . Do they agree with Maple's solution from part a)? If not, look for your error. Did you input the equations correctly?
- k) Express the (correct) solution as a sum of the two eigenvector modes and the response mode in the form:  $\vec{x} = y_{1h} \vec{b}_1 + y_{2h} \vec{b}_2 + \cos(2t) \vec{a}_3$  thus identifying the particular solution  $\vec{x}_p$  (last term), the response vector coefficient  $\vec{a}_3$  and the homogeneous solution  $\vec{x}_h$  (first two terms), as well as the final expressions for the two decoupled variables  $y_{1h}$  and  $y_{2h}$ . Is the response term a tandem or accordian mode? Identify the vectors  $\vec{a}_1 = y_{1h}(0) \vec{b}_1, \vec{a}_2 = y_{2h}(0) \vec{b}_2$ . Include these vectors in your plot (together with  $\vec{a}_3$ ) and use them to create the sides of the bounding box enclosing the solution with four endpoints  $\pm \vec{a}_1 \pm \vec{a}_2$ .
- l) Plot the homogeneous solution for large  $t$  to "see" the parallelogram box containing it. Is it consistent with your plot on the grid?

